Coupling domains through absorbing boundary conditions

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1 Governing equations for the coupled system

Consider a 1D grid with N elements, $N+1$ nodes numbered 1 to $N+1$. After standard discretization, for instance with FEM+SUPG a system of equations of the form

$$\begin{aligned}
&\mathbf{F}_1(\mathbf{U}_1,\mathbf{U}_2)=0\\
&\mathbf{F}_2(\mathbf{U}_1,\mathbf{U}_2,\mathbf{U}_3)=0\\
&\vdots\\
&\mathbf{F}_{N+1}(\mathbf{U}_N,\mathbf{U}_{N+1})=0
\end{aligned} \tag{1}$$

It has been assumed that the equation at node i involves only the node states at nodes $i-1$, i, $i+1$, as is true for first order FDM and FEM discretization methods. Also a steady system of equations is assumed. This represents $(N+1)\times n_{\rm dof}$ equations in the $(N+1)\times n_{\rm dof}$ unknowns $\{\mathbf{U}_1,\mathbf{U}_2,\ldots,\mathbf{U}_{N+1}\}$. It is assumed that this non-linear system of equations has a unique solution. Equations at the boundary nodes may include some mixture of Dirichlet and Neumann boundary conditions.

2 Splitting in subdomains

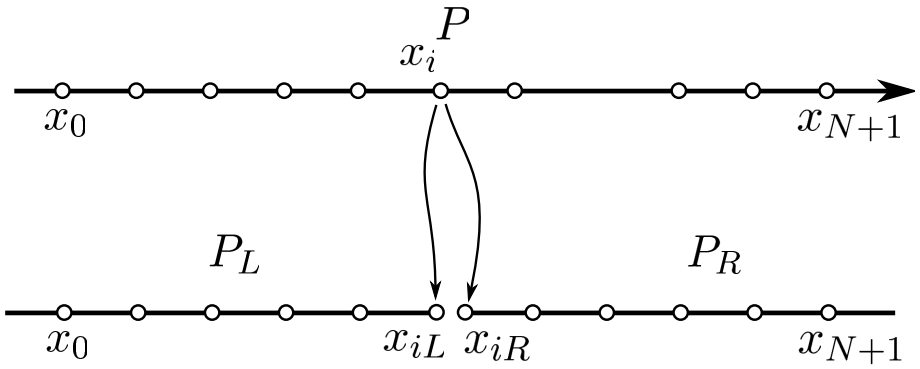

Figure 1:

It is desired to "*split*" this system of equations at a certain internal node i, so that the domain $\Omega_L = [x_1, x_{N+1}]$ is split in $\Omega_1 = [x_1, x_i]$, $\Omega_R = [x_i, x_{N+1}]$. Node i now splits in iL and iR for the left and right domains. The following items must be provided

- Appropriate boundary conditions at iL (iR) for Ω_L (Ω_R). These conditions must ensure that each subdomain problem can be solved independently, and the problem is well posed.
- An iterative approach must guarantee that the solutions at each boundary converge to the solution of the coupled system

$$\begin{aligned}&\mathbf{U}^k_{iL},\mathbf{U}^k_{iR}\rightarrow\mathbf{U}_i\\&\mathbf{U}^k_j\rightarrow\mathbf{U}_j\end{aligned}\tag{2}$$

  for $k\rightarrow\infty$.

It is assumed that the equation at node i can be separated in its right and left contributions

$$\mathbf{F}_i(\mathbf{U}_{i-1},\mathbf{U}_i,\mathbf{U}_{i+1})=\mathbf{F}_{iL}(\mathbf{U}_{i-1},\mathbf{U}_{iL})+\mathbf{F}_{iR}(\mathbf{U}_{iR},\mathbf{U}_{i+1})=0\tag{3}$$

One possibility is at each step to solve the left system with Dirichlet boundary conditions at iL, and compute the flux at this boundary. Then this flux is passed to the right domain as a Neumann boundary condition and the solution at the right domain is computed. This in turn produces a value at the boundary $\mathbf{U}_{iR}$ which is passed as a Dirichlet boundary condition at the left boundary, and so on.

All the combinations of Dirichlet and Neumann boundary conditions are possible, DD,DN,ND and NN.

3 A NN splitting

For instance a NN approach is as follows

$$
\begin{aligned}
P_L &\begin{cases}
\mathbf{F}_1(\mathbf{U}_1^{k+1}, \mathbf{U}_2^{k+1}) = 0 \\
\mathbf{F}_2(\mathbf{U}_1^{k+1}, \mathbf{U}_2^{k+1}, \mathbf{U}_3^{k+1}) = 0 \\
\vdots \\
\mathbf{F}_{iL}(\mathbf{U}_{i-1}^{k+1}, \mathbf{U}_{iL}^{k+1}) + \mathbf{R}^k = 0
\end{cases} \\
P_R &\begin{cases}
\mathbf{F}_{iR}(\mathbf{U}_{iR}^{k+1}, \mathbf{U}_{i+1}^{k+1}) - \mathbf{R}^k = 0 \\
\mathbf{F}_{i+1}(\mathbf{U}_{iR}^{k+1}, \mathbf{U}_{i+1}^{k+1}, \mathbf{U}_{i+2}^{k+1}) = 0 \\
\vdots \\
\mathbf{F}_{N+1}(\mathbf{U}_N^{k+1}, \mathbf{U}_{N+1}^{k+1}) = 0
\end{cases} \\
\mathbf{R}^{k+1} &= \mathbf{R}^k + \omega(\mathbf{U}_{iL}^{k+1} - \mathbf{U}_{iR}^{k+1})
\end{aligned}
\tag{4}
$$

Note that

- $\mathbf{R}^k$ are *"reactions"*, i.e. terms that represent the fluxes that come from the opposite domain or from a Dirichlet boundary condition.
- Both subdomain problems P_L, P_R can be solved independently at each time step. They only need $\mathbf{R}^k$ which is evaluated in the previous step.
- The new value $\mathbf{R}^{k+1}$ with an increment proportional to the imbalance of $\mathbf{U}$ at the boundary with a relaxation factor ω.

It can be shown that if such a scheme converges, then it converges to the solution of the original problems. Effectively, letting $k \to \infty$, the last equation reduces to $\mathbf{U}_{iL} = \mathbf{U}_{iR}$. Replacing this in the iL and iR equations, and summing them, the reaction $\mathbf{R}$ is eliminated and the original equation (3) is recovered.

A key problem is to determine the best combinations of boundary conditions that makes the fixed point algorithm to converge as fast as possible. In the context of the solution of hyperbolic systems of equations some combinations of boundary conditions can be much better than others.

4 Neumann-Dirichlet scheme for right going characteristics

For instance if all characteristic go right (i.e. the sense of propagation is positive), then it is natural to use Dirichlet boundary conditions on the right domain, and Neumann on the left.

$$
\begin{aligned}
P_L &\begin{cases} \mathbf{F}_1(\mathbf{U}_1^{k+1}, \mathbf{U}_2^{k+1}) = 0 \\ \mathbf{F}_2(\mathbf{U}_1^{k+1}, \mathbf{U}_2^{k+1}, \mathbf{U}_3^{k+1}) = 0 \\ \vdots \\ \mathbf{F}_{iL}(\mathbf{U}_{i-1}^{k+1}, \mathbf{U}_{iL}^{k+1}) + \mathbf{R}^k = 0 \end{cases} \\
P_R &\begin{cases} \mathbf{U}_{iR}^{k+1} = \mathbf{U}_{iL}^{k} \\ \mathbf{F}_{i+1}(\mathbf{U}_{iR}^{k+1}, \mathbf{U}_{i+1}^{k+1}, \mathbf{U}_{i+2}^{k+1}) = 0 \\ \vdots \\ \mathbf{F}_{N+1}(\mathbf{U}_{N}^{k+1}, \mathbf{U}_{N+1}^{k+1}) = 0 \end{cases} \\
\mathbf{R}^{k+1} &= \mathbf{F}_{iR}(\mathbf{U}_{iR}^{k+1}, \mathbf{U}_{i+1}^{k+1})
\end{aligned}
\tag{5}
$$

The counting of eqs/unknowns for the P_L problem is the same as for a standard problem with Neumann b.c. at the iL-th node. There are $i \times n_{\text{dof}}$ eqs. in the $i \times n_{\text{dof}}$ unknowns $\{\mathbf{U}_1, \ldots, \mathbf{U}_{iL}\}$ ($\mathbf{R}^k$ is known at this stage, since it comes from the previous stage $k-1$). On the other hand, the counting for $\mathbf{P}_L$ is similar to a problem with Dirichlet b.c. at iR. Represents $(N-i+1) \times n_{\text{dof}}$ equations on the $\{\mathbf{U}_{i+1}, \ldots, \mathbf{U}_{N+1}\}$ unknowns ($\mathbf{U}_{iR}^{k+1}$ is given by the Dirichlet boundary condition).

5 Dirichlet-Neumann scheme for left going characteristics

If all eigenvalues are left-going, then the appropriate scheme would be

$$
\begin{aligned}
P_L &\begin{cases} \mathbf{F}_1(\mathbf{U}_1^{k+1}, \mathbf{U}_2^{k+1}) = 0 \\ \mathbf{F}_2(\mathbf{U}_1^{k+1}, \mathbf{U}_2^{k+1}, \mathbf{U}_3^{k+1}) = 0 \\ \vdots \\ \mathbf{U}_{iL}^{k+1} = \mathbf{U}_{iR}^{k} \end{cases} \\
P_R &\begin{cases} \mathbf{F}_{iR}(\mathbf{U}_{iR}^{k+1}, \mathbf{U}_{i+1}^{k+1}) - \mathbf{R}^k = 0 \\ \mathbf{F}_{i+1}(\mathbf{U}_{iR}^{k+1}, \mathbf{U}_{i+1}^{k+1}, \mathbf{U}_{i+2}^{k+1}) = 0 \\ \vdots \\ \mathbf{F}_{N+1}(\mathbf{U}_{N}^{k+1}, \mathbf{U}_{N+1}^{k+1}) = 0 \end{cases} \\
\mathbf{R}^{k+1} &= \mathbf{F}_{iL}(\mathbf{U}_{i-1}^{k+1}, \mathbf{U}_{iL}^{k+1})
\end{aligned}
\tag{6}
$$

6 Characteristic based split for general systems

$$
\begin{aligned}
P_L &\begin{cases}
\mathbf{F}_1(\mathbf{U}_1^{k+1}, \mathbf{U}_2^{k+1}) = 0 \\
\mathbf{F}_2(\mathbf{U}_1^{k+1}, \mathbf{U}_2^{k+1}, \mathbf{U}_3^{k+1}) = 0 \\
\vdots \\
\mathbf{F}_{iL}(\mathbf{U}_{i-1}^{k+1}, \mathbf{U}_{iL}^{k+1}) + \mathbf{\Pi}^+\mathbf{R}^k + \mathbf{\Pi}^-\mathbf{R}^{k+1} = 0 \\
\mathbf{\Pi}^-(\mathbf{U}_{iL}^{k+1} - \mathbf{U}_{iR}^{k}) = 0
\end{cases} \\
P_R &\begin{cases}
\mathbf{F}_{iR}(\mathbf{U}_{iR}^{k+1}, \mathbf{U}_{i+1}^{k+1}) - \mathbf{\Pi}^+\mathbf{R}^{k+1} - \mathbf{\Pi}^-\mathbf{R}^k = 0 \\
\mathbf{\Pi}^+(\mathbf{U}_{iL}^{k} - \mathbf{U}_{iR}^{k+1}) = 0 \\
\mathbf{F}_{i+1}(\mathbf{U}_{iR}^{k+1}, \mathbf{U}_{i+1}^{k+1}, \mathbf{U}_{i+2}^{k+1}) = 0 \\
\vdots \\
\mathbf{F}_{N+1}(\mathbf{U}_{N}^{k+1}, \mathbf{U}_{N+1}^{k+1}) = 0
\end{cases}
\end{aligned}
\tag{7}
$$

Note that after each iteration P_L provides the left going part of the new reactions $\mathbf{\Pi}^-\mathbf{R}^{k+1}$, whereas P_R provides the left going part. All together the whole reaction vector $\mathbf{R}^{k+1}$ is obtained.

Counting of eqs/unknowns is a little more complicated in this case. There are $in_{\text{dof}} + n_-$ equations and unknowns for P_L, where n_- are the number of left-going characteristics. The n_- eqs come from the last row of eqs., since $\mathbf{\Pi}^-$ has rank n_-. The n_- additional unknowns come form the term $\mathbf{\Pi}^-\mathbf{R}^{k+1}$, since the term $\mathbf{\Pi}^+\mathbf{R}^k$ is known. For instance we could rewrite

$$
\mathbf{\Pi}^-\mathbf{R}^k = \mathbf{V}^-\mathbf{r}^- \tag{8}
$$

Where $\mathbf{V}^-$ is a $n_{\text{dof}} \times n_-$ matrix with the n_- linearly independent columns of $\mathbf{\Pi}^-$, and $\mathbf{r}^-$ has dimension $n_- \times 1$.

Again, if the system converges it can be shown that it converges to the solution of the coupled problem. Effectively, if the scheme converges the following system is obtained

$$
\begin{aligned}
P_L &\begin{cases}
\mathbf{F}_1(\mathbf{U}_1, \mathbf{U}_2) = 0 \\
\vdots \\
\mathbf{F}_{iL}(\mathbf{U}_{i-1}, \mathbf{U}_{iL}) + \mathbf{\Pi}^+\mathbf{R} + \mathbf{\Pi}^-\mathbf{R} = 0 \\
\mathbf{\Pi}^-(\mathbf{U}_{iL} - \mathbf{U}_{iR}) = 0
\end{cases} \\
P_R &\begin{cases}
\mathbf{F}_{iR}(\mathbf{U}_{iR}, \mathbf{U}_{i+1}) - \mathbf{\Pi}^+\mathbf{R} - \mathbf{\Pi}^-\mathbf{R} = 0 \\
\mathbf{\Pi}^+(\mathbf{U}_{iL} - \mathbf{U}_{iR}) = 0 \\
\mathbf{F}_{i+1}(\mathbf{U}_{iR}, \mathbf{U}_{i+1}, \mathbf{U}_{i+2}) = 0 \\
\vdots \\
\mathbf{F}_{N+1}(\mathbf{U}_{N}, \mathbf{U}_{N+1}) = 0
\end{cases}
\end{aligned}
\tag{9}
$$

Summing up the restrictions,

$$
\mathbf{\Pi}^-(\mathbf{U}_{iL} - \mathbf{U}_{iR}) + \mathbf{\Pi}^+(\mathbf{U}_{iL} - \mathbf{U}_{iR}) = (\mathbf{U}_{iL} - \mathbf{U}_{iR}) = 0 \tag{10}
$$

implies $\mathbf{U}_{iL} = \mathbf{U}_{iR}$. And again, summing up the iL-th and iR-th equation the original i-th equation is recovered. The advantage of this scheme is that for each subdomain the system of equations obtained is equivalent to an absorbing boundary condition.

7 Using penalization

Note that both restrictions can be summed up

$$\mathbf{\Pi}^-(\mathbf{U}_{iL}^{k+1} - \mathbf{U}_{iR}^{k}) + \mathbf{\Pi}^+(\mathbf{U}_{iL}^{k} - \mathbf{U}_{iR}^{k+1}) = 0 \tag{11}$$

since each part can be recovered by multiplying at right by $\mathbf{\Pi}^-$ or $\mathbf{\Pi}^+$. For instance if it is multiplied by $\mathbf{\Pi}^+$, the equation

$$\mathbf{\Pi}^+(\mathbf{U}_{iL}^{k} - \mathbf{U}_{iR}^{k+1}) = 0 \tag{12}$$

is recovered, because $\mathbf{\Pi}^+\mathbf{\Pi}^+ = \mathbf{\Pi}^+$, $\mathbf{\Pi}^+\mathbf{\Pi}^- = 0$.

Now, in order to *"regularize"* the problem we can add to (11) a small term

$$\mathbf{\Pi}^-(\mathbf{U}_{iL} - \mathbf{U}_{iR}) + \mathbf{\Pi}^+(\mathbf{U}_{iL} - \mathbf{U}_{iR}) - \epsilon(\mathbf{R}^{k+1} - \mathbf{R}^{k}) = 0 \tag{13}$$

so that $\mathbf{R}^{k+1}$ can be eliminated and a penalized version is obtained.

$$\mathbf{R}^{k+1} = \mathbf{R}^{k} + \frac{1}{\epsilon}\left[\mathbf{\Pi}^-(\mathbf{U}_{iL}^{k+1} - \mathbf{U}_{iR}^{k}) + \mathbf{\Pi}^+(\mathbf{U}_{iL}^{k} - \mathbf{U}_{iR}^{k+1})\right] = 0 \tag{14}$$

replacing in the iL-th and iR-th equations

$$\begin{aligned}
&\mathbf{F}_{iL}(\mathbf{U}_{i-1}^{k+1}, \mathbf{U}_{iL}^{k+1}) + \mathbf{\Pi}^+\mathbf{R}^{k} + \mathbf{\Pi}^-\mathbf{R}^{k+1} = 0 \\
&\mathbf{F}_{iL}(\mathbf{U}_{i-1}^{k+1}, \mathbf{U}_{iL}^{k+1}) + \mathbf{\Pi}^+\mathbf{R}^{k} + \mathbf{\Pi}^-\mathbf{R}^{k} + \frac{1}{\epsilon}\mathbf{\Pi}^-(\mathbf{U}_{iL}^{k+1} - \mathbf{U}_{iR}^{k}) = 0 \\
&\mathbf{F}_{iL}(\mathbf{U}_{i-1}^{k+1}, \mathbf{U}_{iL}^{k+1}) + \mathbf{R}^{k} + \frac{1}{\epsilon}\mathbf{\Pi}^-(\mathbf{U}_{iL}^{k+1} - \mathbf{U}_{iR}^{k}) = 0
\end{aligned} \tag{15}$$

and similarly, for the iR-th equation

$$\mathbf{F}_{iR}(\mathbf{U}_{iR}^{k+1}, \mathbf{U}_{i+1}^{k+1}) - \mathbf{R}^{k} - \frac{1}{\epsilon}\mathbf{\Pi}^+(\mathbf{U}_{iL} - \mathbf{U}_{iR}) = 0 \tag{16}$$

8 Rate of convergence

Consider the case of a scalar equation ($n_{\text{dof}} = 1$). In order to find a rate of convergence the continuum case is considered, in place of the discrete equations. So that the coupled problem is

$$P\begin{cases} u(0) = 0; \\ a u_{,x} = \kappa u_{,xx}; \;\; 0 \le x \le L; \\ u(1) = 1; \end{cases} \tag{17}$$

whose solution is

$$u(x) = \frac{\exp(ax/\kappa) - 1}{\exp(aL/\kappa) - 1}. \tag{18}$$

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The split problem is

$$
\begin{aligned}
P_L &\begin{cases} u^{k+1}(0) = 0; \\ a u_{,x}^{k+1} = \kappa u_{,xx}^{k+1}; \;\; 0 \le x \le x_{iL}; \\ (\kappa u_{,x}^{k+1})(x_{iL}) = r^k; \end{cases} \\
P_R &\begin{cases} u^{k+1}(x_{iR}) = s^k; \\ a u_{,x}^{k+1} = \kappa u_{,xx}^{k+1}; \;\; 0 \le x \le x_{iL}; \\ u^{k+1}(1) = 1; \end{cases} \\
\text{update: } &\begin{cases} r^{k+1} = \kappa u_{,x}^{k+1}(x_{iR}) \\ s^{k+1} = u^k(x_{iL}) \end{cases}
\end{aligned}
\tag{19}
$$

Note that, as the characteristic corresponding to (17) is right-going, a ND approach has been chosen. Solving the equations above leads to a map

$$(r, u_{iL})^k \to (r, u_{iL})^{k+1}. \tag{20}$$

Convergence of the loop can be determined from the convergence of this $\mathbb{R} \to \mathbb{R}$ map.

Solution to the P_L problem can be found by proposing a solution of the form

$$u^{k+1} = d + f \mathrm{e}^{ax/\kappa}, \tag{21}$$

where constants d, f can be determined from the boundary conditions

$$
\begin{aligned}
0 &= d + f, \\
r^k &= k \left(\frac{a}{\kappa}\right) f \, \mathrm{e}^{a x_{iL}/\kappa},
\end{aligned}
\tag{22}
$$

from which d, f can be determined

$$
\begin{aligned}
f &= \frac{1}{a} \mathrm{e}^{(-a x_{iL}/\kappa)} \, r^k, \\
d &= -f,
\end{aligned}
\tag{23}
$$

so that the solution to the left domain problem is

$$u^{k+1} = -\frac{1}{a} \mathrm{e}^{(-a x_{iL}/\kappa)} \, r^k \, (1 - \mathrm{e}^{(ax/\kappa)}), \tag{24}$$

and

$$s^{k+1} = u^{k+1}(x_{iL}) = -\frac{1}{a} \mathrm{e}^{(-a x_{iL}/\kappa)} \, r^k \, (1 - \mathrm{e}^{(a x_{iL}/\kappa)}). \tag{25}$$

On the other hand, the right domain solution can be again proposed in the form (21) but with other coefficients,

$$u^{k+1} = d' + f' \mathrm{e}^{ax/\kappa}. \tag{26}$$

Replacing in the b.c.'s in order to determine the coefficients d', f' the following eqs. are obtained

$$
\begin{aligned}
d' + f' \mathrm{e}^{a x_{iR}/\kappa} &= s^k, \\
d' + f' \mathrm{e}^{aL/\kappa} &= 1.
\end{aligned}
\tag{27}
$$

from which f' can be obtained as

$$f' = \frac{s^k - 1}{\mathrm{e}^{ax_{iR}/\kappa} - \mathrm{e}^{aL/\kappa}} \tag{28}$$

and then d' can be obtained as

$$\begin{aligned} d &= 1 - \frac{(s^k - 1)\,\mathrm{e}^{-aL/\kappa}}{\mathrm{e}^{ax_{iR}/\kappa} - \mathrm{e}^{aL/\kappa}}, \\ &= \frac{\mathrm{e}^{-ax_{iR}/\kappa} - s^k \mathrm{e}^{aL/\kappa}}{\mathrm{e}^{ax_{iR}/\kappa} - \mathrm{e}^{aL/\kappa}}. \end{aligned} \tag{29}$$

So that, the right domain solution is

$$u^{k+1} = \frac{\mathrm{e}^{-ax_{iR}/\kappa} - s^k \mathrm{e}^{aL/\kappa} + (s^k - 1)\mathrm{e}^{ax/\kappa}}{\mathrm{e}^{ax_{iR}/\kappa} - \mathrm{e}^{aL/\kappa}}. \tag{30}$$

and then,

$$r^{k+1} = \kappa u_{,x}^{k+1}(x_{iR}) = \kappa \frac{(s^k - 1)}{\mathrm{e}^{ax_{iR}/\kappa} - \mathrm{e}^{aL/\kappa}} \frac{a}{k}\, \mathrm{e}^{ax_{iR}/\kappa}. \tag{31}$$

So, the map is of the form

$$\begin{aligned} r^{k+1} &= c_1 s^k + d, \\ s^{k+1} &= c_2 r^k, \end{aligned} \tag{32}$$

with

$$\begin{aligned} c_1 &= \frac{a\,\mathrm{e}^{ax_{iR}/\kappa}}{\mathrm{e}^{ax_{iR}/\kappa} - \mathrm{e}^{aL/\kappa}} \\ c_2 &= -\frac{1}{a}\mathrm{e}^{(-ax_{iL}/\kappa)}\,(1 - \mathrm{e}^{(ax_{iL}/\kappa)}). \end{aligned} \tag{33}$$

The rate of convergence is given then by the spectral radius of the amplification matrix

$$\mathbf{G} = \begin{bmatrix} 0 & c_1 \\ c_2 & 0 \end{bmatrix} \tag{34}$$

The eigenvalues of G are

$$\lambda_{12} = \pm\sqrt{c_1 c_2} = \left[\mathrm{e}^{-a(L-x_i)/\kappa}\, \frac{1 - \mathrm{e}^{-ax_i/\kappa}}{1 - \mathrm{e}^{-a(L-x_i)/\kappa}}\right]^{1/2} \tag{35}$$

For advection dominated flows, we have

$$\begin{aligned} ax_i/\kappa &\gg 1, \\ a(L - x_i)/\kappa &\gg 1, \\ aL/\kappa &\gg 1, \end{aligned} \tag{36}$$

so that,

$$\begin{aligned} 1 - \mathrm{e}^{-ax_i/\kappa} &\approx 1 \\ 1 - \mathrm{e}^{-a(L-x_i)/\kappa} &\approx 1 \\ \mathrm{e}^{-a(L-x_i)/\kappa} &\ll 1 \end{aligned} \tag{37}$$

and the amplification factor is

$$\max|\lambda_{1,2}| \approx \mathrm{e}^{-a(L-x_i)/2\kappa} \ll 1 \tag{38}$$

and a very good convergence is expected. Note that if $a < 0$ then the amplification factor tends to

$$\max|\lambda_{1,2}| \approx \mathrm{e}^{-ax_i/2\kappa} \gg 1 \tag{39}$$