ON THE UNIFORM DOUBLING OF HUTCHINSON ORBITS OF CONTRACTIVE MAPPINGS

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Abstract. We are interested in the preservation of doubling properties along the Hutchinson orbit generated by successive applications of contraction mappings on a metric measure space. We construct some elementary examples, built on Muckenhoupt weights, showing that even when the initial and the limit points of the orbit are doubling, no iteration of the IFS remains doubling. We also obtain positive results under some quantitative assumptions on the separation of the images through the IFS. We also explore the completeness, in the Hausdorff-Kantorovich metric, of a version of the doubling property which is suitable for the application of Hutchinson type contractions.

Introduction

The basic scheme of the Banach fixed point theorem applied by Hutchinson in [7] to contractions acting on the Hausdorff metric space (the set $\mathcal{K}$ of all compact sets equipped with the Hausdorff distance), relies deeply on the completeness of this space. In [1] the authors consider the complete space $\mathcal{K} \times \mathcal{P}$, where $\mathcal{K}$ is the Hausdorff space of compact subsets of the fixed underlying compact metric space (X, d), and $\mathcal{P}$ is the set of probability measures on X with the Kantorovich distance for the weak star convergence. From the real analytical point of view (see [3]) we are specially interested in the subset of those pairs in $\mathcal{K} \times \mathcal{P}$ which are spaces of homogeneous type. It easy to see that this subset it is not complete. We provide a simple example in Section 2. So that the Banach fixed point theorem can not be directly applied to this family.

The results in [9] show that typically the limit fractal provided by the Hutchinson iteration scheme and equipped with the invariant measure, is a (normal) space of homogeneous type. In this note we construct examples showing that no point in the orbit has the doubling property even when the starting point, and the limit point, have both the doubling property. These examples show that the loose of the doubling property by iteration is related to the distance of the images of the starting set under the given family of contractions. We also prove the completeness of a version of the doubling property which, under the assumption of finite metric (Assouad) dimension, coincides with the classical doubling property. Moreover, we obtain sufficient conditions on an iterated function system (IFS) in such a way that this complete doubling classes are preserved by Hutchinson's algorithm. As a corollary we obtain uniform doubling orbits associated to such IFS.

In the first section we introduce the basic notation and definitions. Section 2 is devoted to prove, under the assumption of finiteness of metric dimension, that the

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doubling property is equivalent to a control by neighboring balls. We also prove in Section 2 that this version of the doubling property is closed as a subset of $\mathcal{K}\times\mathcal{P}$ and hence complete. Section 3 contains, for two classical IFS in dimensions one and two, examples showing that it may happen, for some starting doubling measures, that no point of the orbit is a space of homogeneous type. In Section 4 we prove our positive result and we illustrate it with Cantor and Sierpinski type sets.

1. Basic definitions and notation

Throughout this paper (X,d) shall be a compact metric space. We shall use $B_d(x,r)$ to denote the ball $\{y\in X: d(x,y)<r\}$, $r>0$. With $\overline{B}_d(x,r)$ we shall denote the "closed" ball $\{y\in X: d(x,y)\leq r\}$.

Let $\mathcal{K}=\{K\subseteq X: K\neq\emptyset,\ K \text{ compact}\}$. With $[A]_\varepsilon$ we shall denote the ε-enlargement of the set $A\subset X$; i.e. $[A]_\varepsilon=\bigcup_{x\in A}B_d(x,\varepsilon)=\{y\in X: d(y,A)<\varepsilon\}$. Here $d(x,A)=\inf\{d(x,y): y\in A\}$. Given A and B two sets in $\mathcal{K}$ the Hausdorff distance from A to B is given by

$$d_H(A,B)=\inf\{\varepsilon>0: A\subseteq[B]_\varepsilon \text{ and } \mathrm{B}\subseteq[\mathrm{A}]_\varepsilon\}.$$

Let us now introduce the Kantorovich-Hutchinson distance on the set of all Borel regular probability measures on the quasi-metric space (X,d). Let

$$\mathcal{P}(X)=\{\mu:\mu \text{ is a positive Borel measure on } X \text{ and } \mu(X)=1\},$$

and let $\mathcal{C}(X)$ be the space of continuous real valued functions on X. Let Lip_1 be the space of all Lipschitz continuous functions defined on X with Lipschitz constant less than or equal to one, i.e. $f\in\mathrm{Lip}_1$ if and only if $|f(x)-f(y)|\leq d(x,y)$ for every x and $y\in X$.

Since (X,d) is compact, $d_K(\mu,\nu)=\sup\left\{\left|\int f\,d\mu-\int f\,d\nu\right|: f\in\mathrm{Lip}_1\right\}$ gives a distance on $\mathcal{P}(X)$ such that the d_K-convergence of a sequence is equivalent to its weak star convergence to the same limit (see [5]).

We are now in position to describe the basic metric space introduced in [1] whose structure is of our interest. Let $\mathscr{X}=\mathcal{K}\times\mathcal{P}$. Given two elements (Y_i,μ_i) of $\mathscr{X}$, $i=1,2$, define

$$\delta\left((Y_1,\mu_1),(Y_2,\mu_2)\right)=d_H(Y_1,Y_2)+d_K(\mu_1,\mu_2),$$

so that $(\mathscr{X},\delta)$ becomes a complete metric space. Let

$$\mathcal{E}=\{(Y,\mu)\in\mathscr{X}:\mathrm{supp}(\mu)\subseteq Y\}$$

The basic result regarding the completeness of $(\mathcal{E},\delta)$ is given in the following statement.

Theorem 1.1. *The set $\mathcal{E}$ is closed in $(\mathscr{X},\delta)$. Hence $(\mathcal{E},\delta)$ is a complete quasi-metric subspace of $(\mathscr{X},\delta)$.*

The basic tool in Hutchinson method for the construction of self-similar fractals is the Banach fixed point theorem.

Corollary 1.2. *If $T:\mathcal{E}\to\mathcal{E}$ is a contraction mapping and $(Y_0,\mu_0)\in\mathcal{E}$, then $(Y_\infty,\mu_\infty)=\lim_{n\to\infty}T^n((Y_0,\mu_0))$ belongs to $\mathcal{E}$, and it is the unique point in $\mathcal{E}$ which satisfies $T((Y_\infty,\mu_\infty))=(Y_\infty,\mu_\infty)$.*

Like in the now classical theory of Hutchinson, the basic examples of application of Corollary 1.2 are induced by iterated function systems (IFS) $\Phi = \{\phi_1, \ldots, \phi_M\}$ of contractions on (X, d). This means that there exist $a_1, a_2, \ldots, a_M > 1$ such that

$$d(\phi_i(x), \phi_i(y)) \leq \frac{1}{a_i} d(x, y)$$

for every $x, y \in X$. Given an IFS Φ, let $T : \mathscr{X} \to \mathscr{X}$ the application defined by $T(Y, \mu) = (T_1 Y, T_2 \mu) = (Y', \mu')$, where

$$Y' = \bigcup_{i=1}^{M} \phi_i(Y),$$

and

$$\mu'(B) = \sum_{i=1}^{M} p_i \mu \left(\phi_i^{-1}(B \cap \phi_i(Y)) \right),$$

for some probabilistic sequence $0 < p_i < 1$, $\sum_{i=1}^{M} p_i = 1$ and every Borel subset B of Y'. This transformation T is called the transformation induced by Φ associated with the probabilities $\{p_i\}$. As usual, it is easy to see that T is contractive on $(\mathscr{X}, \delta)$ with contractivity factor $\lambda = 1/a$, where $a = \min_i a_i$, and that $\mathcal{E}$ is invariant under T.

With our notation we have that T_1 is a contraction on $\mathcal{K}$ and hence we obtain, as in Theorem 2.6 in [5], a fixed point Y_∞ for T_1 which is the only compact set in X satisfying

$$Y_\infty = \bigcup_{i=1}^{M} \phi_i(Y_\infty).$$

On the other hand T_2 is a contraction on $\mathcal{P}$ and we obtain , as in Theorem 2.8 in [5], a fixed point μ_∞ for T_2 which is the only probabilistic Borel measure such that

$$\mu_\infty(B) = \sum_{i=1}^{M} p_i \mu_\infty \left(\phi_i^{-1}(B) \right),$$

for every Borel set B.

2. Subspaces of $\mathcal{E}$: the doubling property

Given $(Y, \mu) \in \mathcal{E}$, we say that (Y, μ) is a *space of homogeneous type* (s.h.t.), or that μ is a *doubling measure* on Y if the inequalities

$$0 < \mu(B_d(y, \alpha r)) \leq A \mu(B_d(y, r))$$

hold for every $y \in Y$ and $r > 0$ and some constants $A \geq 1$ and $\alpha > 1$. We shall write $(Y, \mu) \in \mathcal{D}(\alpha, A)$ to keep record of the quantitative parameters of this doubling property.

Notice also that if $(Y, \mu) \in \mathcal{E}$ is a space of homogeneous type, then $\mathrm{supp}(\mu) = Y$. In fact, from the very definition of $\mathcal{E}$, we have that $\mathrm{supp}(\mu) \subseteq Y$. On the other hand, if $y \notin \mathrm{supp}(\mu)$ then there exists an open set G containing y with $\mu(G) = 0$. So that for some ball B in Y we should have $\mu(B) = 0$, which is impossible.

We shall introduce the doubling property of a measure on a metric space in another way. In Euclidean spaces the doubling property can be detected through the behavior of the measures of neighboring balls of similar size. This property is no longer true in general metric spaces. The basic property of $\mathbb{R}^n$ which makes this possible is the finiteness of its (metric) dimension.

Given constants $A \geq 1$ and $\alpha > 1$, we shall say that $(Y,\mu) \in \widetilde{\mathcal{D}}(\alpha, A)$ if $(Y,\mu) \in \mathcal{E}$ and for every choice of $y_1, y_2 \in Y$ and $r > 0$ with $d(y_1, y_2) < \alpha r$, we have that

$$\mu(B_d(y_1, r)) \leq A\mu(\overline{B}_d(y_2, r)).$$

A metric space (X, d) satisfies the weak homogeneity property, or it has finite metric (or Assouad) dimension, if there exists a constant $N \in \mathbb{N}$ such that no ball of radius $r > 0$ contains more than N points of any $r/2$-disperse (or separated) subset of X (see [4] and [2]). Let us point out that the expression "finite metric dimension" corresponds to the finiteness of the following concept of dimension. The metric dimension of X is the infimum of all those positive numbers s such that the inequality

$$\operatorname{card}(B(x, \lambda r) \cap A) \leq C\lambda^s$$

holds for some constant C, every $\lambda \geq 1$, every $x \in X$, every r-disperse subset A of X and every $r > 0$.

The next result shows that $\widetilde{\mathcal{D}}$ can be regarded as a different way of describing a space of homogeneous type when (X, d) has finite metric dimension.

Theorem 2.1. *Let $\alpha > 1$ and $A \geq 1$ be given constants. Then*

(1) *$\mathcal{D}(\alpha, A) \subseteq \widetilde{\mathcal{D}}(\alpha, \widetilde{A})$ for some constant $\widetilde{A}$;*
(2) *if (X, d) has finite metric dimension, then $\widetilde{\mathcal{D}}(\alpha, A) \subseteq \mathcal{D}(\alpha, A')$, for some constant A' depending only on A and N.*

Proof. Suppose that (Y,μ) belongs to $\mathcal{D}(\alpha, A)$ for some constants $\alpha > 1$ and $A \geq 1$, and take $y_1, y_2 \in Y$ and $r > 0$ satisfies $d(y_1, y_2) < \alpha r$. Then for every positive integer $n \geq \log_\alpha(1 + \alpha)$ we have

$$\mu\big(B_d(y_1, r)\big) \leq \mu\big(B_d(y_2, (\alpha + 1)r)\big) \leq A^n \mu\big(B_d(y_2, r)\big).$$

Hence $(Y,\mu) \in \widetilde{\mathcal{D}}(\alpha, A^n)$.

For the converse, suppose that $(Y,\mu) \in \widetilde{\mathcal{D}}(\alpha, A)$. For fixed $y \in Y$ and $r > 0$, we define

$$C(y, r) = B_d(y, \alpha r) - B_d(y, r)\,.$$

Let E be a fixed maximal r-disperse subset of $C(y, r)$. By the finiteness of the Assouad dimension of X, there exists a natural number N_0 which does not depend on y and r such that $N = \operatorname{card}(E) \leq N_0$, let us say $E = \{y_1, y_2, \ldots, y_N\}$. We claim that

$$C(y, r) \subseteq \bigcup_{\ell=1}^{N} B(y_\ell, r)\,.$$

In fact, is not, for some $z \in C(y, r)$ we would have that $d(z, y_l) \geq r$ for all $\ell = 1, 2 \ldots, N$. But it can not be possible since E is a maximal r-disperse set. Since

$d(y, y_\ell) < \alpha r$ for each $\ell = 1, 2, \ldots, N$, we have $\mu(B_d(y_\ell, r)) \leq A\mu(\overline{B}_d(y, r))$. Then

$$\begin{aligned}\mu(B_d(y, \alpha r)) &\leq \mu\left(B_d(y, r)\right) + \sum_{l=1}^{N} \mu(B_d(y_l, r)) \\ &\leq (1 + AN_0)\mu(\overline{B}_d(y, r)) \\ &\leq (1 + AN_0)\mu(B_d(y, \theta r)),\end{aligned}$$

for every $1 < \theta < \alpha$. In other words, (Y, μ) belongs to $\mathcal{D}(\alpha/\theta, 1 + AN_0)$, and the results holds taking $A' = (1 + AN_0)^p$, where p is a positive integer such that $\alpha \leq (\theta/\alpha)^p$. $\square$

We are interested in the subspace of $\mathcal{E}$ of all those $(Y, \mu) \in \mathcal{E}$ which are spaces of homogeneous type. This subspace of $\mathcal{E}$ is not closed in $\mathcal{E}$ in general, and this fact does not allow the application of the Banach fixed point theorem. In fact the set $\mathcal{D}(\alpha, \infty) = \bigcup_{A\geq 1} \mathcal{D}(\alpha, A)$ is not closed in $\mathcal{E}$. For example, consider $X = [0, 1]$ with d the usual distance. Take $Y_n = [0, 1]$ for each n and μ_n the measure with density $f_n(t) = n - 1 + 1/n$ on $[0, 1/n]$ and $f_n(t) = 1/n$ on $(1/n, 1]$. It is easy to see that $\mu_n \xrightarrow{\delta_K} \delta_0$, and that each $(Y_n, \mu_n) \in \mathcal{D}(2, A_n)$, with $A_n = 2n(n-1+1/n)$ as a possible doubling constant. Actually it is also easy to show that A_n can not be bounded above, since by taking the balls $B(x, r) = B(2/n, 1/n)$ we see that $A_n \geq \frac{n^2-n+4}{2}$. Since in each space of homogeneous type atoms are isolated (see [8]), the space $([0, 1], |\cdot|, \delta_0)$ can not be a space of homogeneous type.

Nevertheless if α and A are fixed, then the class $\widetilde{\mathcal{D}}(\alpha, A)$ is complete with the distance δ.

Theorem 2.2. *For every constants $A \geq 1$ and $\alpha > 1$ given, the space $\widetilde{\mathcal{D}}(\alpha, A)$ is a closed subset of $(\mathscr{X}, \delta)$.*

Proof. Notice that after Theorem 1.1, to check that $\widetilde{\mathcal{D}}(\alpha, A)$ is closed we only have to prove that given a sequence $\{(Y_n, \mu_n) : n \in \mathbb{N}\}$ in $\widetilde{\mathcal{D}}(\alpha, A)$ such that $(Y_n, \mu_n) \xrightarrow{\delta} (Y, \mu)$, we have that

$$\mu(B_d(y, r)) \leq A\mu(B_d(z, r + \varepsilon))$$

for every $y, z \in Y$ and $r > 0$ with $d(y, z) < \alpha r$ and $\varepsilon > 0$.

Now, for every natural number m let φ^m be the continuous function defined on $\mathbb{R}^+ \cup \{0\}$ such that $\varphi^m \equiv 1$ on $[0, 1]$, $\varphi^m \equiv 0$ out of $[0, 1 + 1/m)$ and φ^m is linear on $[1, 1 + 1/m]$. Then for each $s > 0$ we have that $\varphi^m\left(d(y, x)/s\right) = 1$ for every $x \in B_d(y, s)$ and that the set where $\varphi^m\left(d(y, \cdot)/s\right)$ is not zero is contained in $B_d(y, s + s/m)$.

Fix $y, z \in Y$ and $r > 0$ with $d(y, z) < \alpha r$, $\varepsilon > 0$ and $\eta > 0$. Let $\varepsilon_0 = \min\{\frac{\alpha r - d(y,z)}{2}, \eta\} > 0$. Since $Y_n \xrightarrow{d_H} Y$, there exists $N = N(\varepsilon_0)$ such that $Y \subseteq [Y_n]_{\varepsilon_0}$ if $n \geq N$, and from compacity for every $n \geq N$ we can choose $y_n, z_n \in Y_n$ such that $d(y_n, y) < \varepsilon_0$ and $d(z_n, z) < \varepsilon_0$. Then $d(y_n, z_n) < \alpha r$ for every $n \geq N$. Also we have that $d(y_n, y) < \eta$ and $d(z_n, z) < \eta$ for every $s > 0$ and $n \geq N$, then $B_d(y_n, s) \subseteq B_d(y, s + \eta)$, $B_d(y, s) \subseteq B_d(y_n, s + \eta)$, $B_d(z_n, s) \subseteq B_d(z, s + \eta)$ and $B_d(z, s) \subseteq B_d(z_n, s + \eta)$. Hence for $m \geq 1$ we have

$$\mu\left(B_d(y, r)\right) \leq \int \varphi^m\left(\frac{d(y, x)}{r}\right) d\mu(x)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \int \varphi^m\left(\frac{d(y,x)}{r}\right) d\mu_n(x)\\
&\leq \liminf_{n\to\infty} \mu_n\Big(B_d\big(y, r(1+1/m)\big)\Big)\\
&\leq \liminf_{n\to\infty} \mu_n\Big(B_d\big(y_n, r(1+1/m)+\eta\big)\Big)\\
&\leq A\liminf_{n\to\infty} \mu_n\Big(B_d\big(z_n, r(1+1/m)+2\eta\big)\Big)\\
&\leq A\liminf_{n\to\infty} \mu_n\Big(B_d\big(z, r(1+1/m)+3\eta\big)\Big)\\
&\leq A\liminf_{n\to\infty} \int \varphi^m\left(\frac{d(z,x)}{r(1+1/m)+3\eta}\right) d\mu_n(x)\\
&= A\int \varphi^m\left(\frac{d(z,x)}{r(1+1/m)+3\eta}\right) d\mu(x)\\
&\leq A\mu\Big(B_d\big(z, (1+1/m)[r(1+1/m)+3\eta]\big)\Big).
\end{aligned}
$$

The desired inequality is obtained by taking $m \geq \frac{3(r+3)}{\varepsilon}$ and $\eta = \frac{1}{m}$. □

3. Non-doubling orbits starting at doubling spaces

As we have already mentioned, the results in [9] show that the usual self-similar fractals, constructed from the iteration and fixed point technique introduced by Hutchinson in [7], are typically spaces of homogeneous type.

Let us show now that it may happen that the only point in the orbit satisfying the doubling property is (Y_0, μ_0) and of course the limit space (Y_∞, μ_∞) but no other $T^n(Y_0, \mu_0)$, $n \in \mathbb{N}$, is a space of homogeneous type. In these constructions we shall use Muckenhoupt type weights. Let us review the basic properties of the A_p classes (see [6]). Let (X, d, μ) be a space of homogeneous type and let $1 < p < \infty$. We say that a non-negative locally integrable function w defined on X belongs to $A_p = A_p(X, d, \mu)$ if there exists a constant C for which the inequality

$$
\left(\int_B w\, d\mu\right)\left(\int_B w^{-\frac{1}{p-1}}\, d\mu\right)^{p-1} \leq C\,(\mu(B))^p
$$

holds for every d-ball B of X. We have that if $w \in A_p$, then $(X, d, wd\mu)$ is also a space of homogeneous type. In fact, denote $d\nu = wd\mu$ en let $x \in X$ and $r > 0$ be given. From Hölder inequality and Muckenhoupt condition we have

$$
\begin{aligned}
\mu(B_d(x,r)) &= \int_{B_d(x,r)} w^{\frac{1}{p}} w^{-\frac{1}{p}}\, d\mu\\
&\leq \left(\int_{B_d(x,r)} w\, d\mu\right)^{\frac{1}{p}} \left(\int_{B_d(x,2r)} w^{-\frac{1}{p-1}}\, d\mu\right)^{\frac{p-1}{p}}\\
&\leq \nu(B_d(x,r))^{\frac{1}{p}} \left(\frac{C}{\int_{B_d(x,2r)} w\, d\mu}\right)^{\frac{1}{p}} \mu(B_d(x,2r))\\
&= \left(C\frac{\nu(B_d(x,r))}{\nu(B_d(x,2r))}\right)^{\frac{1}{p}} \mu(B_d(x,2r)).
\end{aligned}
$$

Then

$$\frac{\nu(B_d(x,r))}{\nu(B_d(x,2r))} \geq \frac{1}{C}\left(\frac{\mu(B_d(x,r))}{\mu(B_d(x,2r))}\right)^p \geq \frac{1}{CA^p},$$

where A is the doubling constant for μ. Then ν is doubling with constant CA^p. A particular case is obtained by taking $(X,d,\mu) = (\mathbb{R}^n, |\cdot|, dx)$. If w is any weight in A_p, then $(\mathbb{R}^n, |\cdot|, wdx)$ is a space of homogeneous type.

We shall consider the Cantor type IFS

$$\phi_1(x) = \frac{1}{k}x, \qquad \phi_2(x) = \frac{1}{k}x + \frac{k-1}{k}\,. \tag{3.1}$$

defined on $X = [0,1]$ with the usual distance $d(x,y) = |x-y|$, where k a fixed positive number. Let T_c be the application induced on $\mathscr{X}$ by this IFS with $p_1 = p_2 = 1/2$.

For the case $k=2$ we have $\Phi = \{\phi_1, \phi_2\}$ with $\phi_1(x) = x/2$ and $\phi_2(x) = x/2+1/2$, and the application T_c is given by $T_c(Y,\mu) = (Y',\mu')$, where

$$Y' = \phi_1(Y) \cup \phi_2(Y),$$

and

$$\mu'(B) = \frac{1}{2}\mu\left(\phi_1^{-1}(B\cap\phi_1(Y))\right) + \frac{1}{2}\mu\left(\phi_2^{-1}(B\cap\phi_2(Y))\right).$$

Let μ_0 be the absolutely continuous measure given by $d\mu_0 = \frac{1}{2}w(x)dx$, with $w(x) = x^{-1/2}$. It is not hard to check that it is a doubling measure on $[0,1]$ with respect to the standard Euclidean distance. Actually w is a Muckenhoupt A_2 weight. In fact, let $0 \leq a < b \leq 1$. If $b-a \leq \frac{a}{2}$, then

$$\begin{aligned}\left(\int_a^b w(x)\,dx\right)\left(\int_a^b w^{-1}(x)\,dx\right) &= \left(\int_a^b x^{-1/2}\,dx\right)\left(\int_a^b x^{1/2}\,dx\right)\\ &\leq \left(\frac{b}{a}\right)^{1/2}(b-a)^2\\ &\leq \left(\frac{3}{2}\right)^{1/2}(b-a)^2.\end{aligned}$$

On the other hand, if $b-a \geq \frac{a}{2}$ we have

$$\begin{aligned}\left(\int_a^b w(x)\,dx\right)\left(\int_a^b w^{-1}(x)\,dx\right) &\leq \left(\int_0^b x^{-1/2}\,dx\right)\left(\int_0^b x^{1/2}\,dx\right)\\ &= \frac{4}{3}b^2\\ &\leq 12(b-a)^2.\end{aligned}$$

Notice now that even when $(Y_0,\mu_0) = ([0,1], \frac{1}{2}w(x)dx)$ is a space of homogeneous type, $T_c((Y_0,\mu_0))$ is not. In order to show the above statement, take $0 < \varepsilon < 1/4$, $E_\varepsilon = [\frac{1}{2}-\varepsilon, \frac{1}{2}]$ and $F_\varepsilon = [\frac{1}{2}, \frac{1}{2}+\varepsilon]$. Notice that $Y_0' = Y_0 = [0,1]$ and that $d\mu_0' = \frac{\sqrt{2}}{4}v(x)dx$ with

$$v(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1/2,\\ \left(x-\frac{1}{2}\right)^{-1/2} & \text{if } 1/2 < x < 1.\end{cases}$$

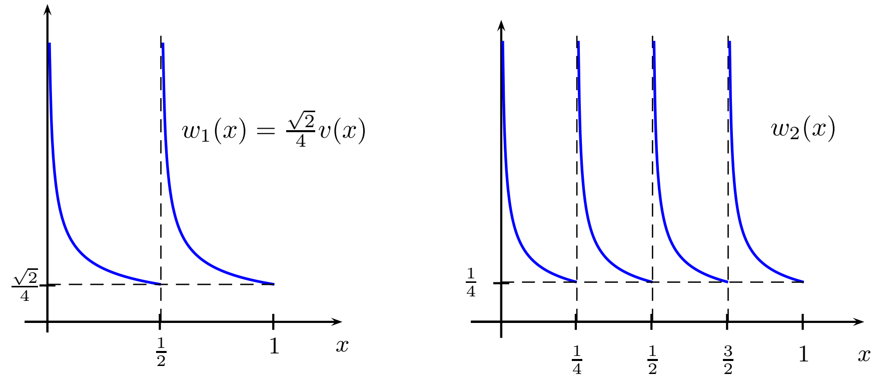


Figure 1: $T_c^n([0,1], \frac{1}{2}wdx) = ([0,1], w_n dx)$

(see Figure 1). Hence $\mu_0'(E_\varepsilon) = \frac{1}{2}\left(1 - \sqrt{1-2\varepsilon}\right)$ and $\mu_0'(F_\varepsilon) = \sqrt{2\varepsilon}/2$. Since E_ε and F_ε are balls with the same radii and non empty intersection, by taking $\varepsilon \to 0$ we realize the impossibility of the doubling for μ_0'. For the further iterations of T_c acting on $([0,1], \frac{1}{2}wdx)$, let us say $T_c^n([0,1], \frac{1}{2}wdx)$, the same situation appears at each point of the form $j/2^n$, $j = 1, 2, \ldots, 2^n - 1$. Hence no $T_c^n([0,1], \frac{1}{2}wdx)$ is a space of homogeneous type for $n \in \mathbb{N}$. But from uniqueness in the Banach fixed point theorem

$$\lim_{n\to\infty} T_c^n\left([0,1], \frac{1}{2}wdx\right) = ([0,1], dx),$$

which is, perhaps, the most elementary example of space of homogeneous type.

Notice that in the above example we have that $d\,(\phi_1([0,1], \phi_2([0,1]) = 0$, and also the attractor of the IFS is the initial set $[0,1]$. The second example is given by a similar construction associated to the classical Sierpinski contraction T_s, and shows that the difficulty for the doubling is in the contact and not in the invariance of the original set. In this case let X be the triangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(1,0)$ and $(0,1)$, and take $d((x_1,y_1),(x_2,y_2)) = \max\{|x_1-x_2|, |y_1-y_2|\}$ as the distance on X. Let

$$\phi_1(x,y) = (x/2, y/2)\,,$$
$$\phi_2(x,y) = \phi_1(x,y) + (0, 1/2)\,,$$
$$\phi_3(x,y) = \phi_1(x,y) + (1/2, 0)\,.$$

Define T_s on $\mathscr{X}$ by $T_s(Y,\mu) = (Y', \mu')$ with

$$Y' = \phi_1(Y) \cup \phi_2(Y) \cup \phi_3(Y) := Y_1' \cup Y_2' \cup Y_3'\,,$$

and

$$\mu'(B) = \frac{1}{3}\left(\mu\left(\phi_1^{-1}(B\cap Y_1')\right) + \mu\left(\phi_2^{-1}(B\cap Y_2')\right) + \mu\left(\phi_3^{-1}(B\cap Y_3')\right)\right),$$

for every Borel subset B of Y'.

Let us define a weight function $\widetilde{w}(x,y)$ on the basic triangle X, given by $\widetilde{w}(x,y) = \frac{1}{2}w(y)$ where w is the weight function defined on $[0,1]$ by $w(y) = y^{-1/2}$. Notice that for $(x,y)\in X$ and $r>0$, $B_d((x,y),\frac{r}{2})=(I_1\times I_2)\cap X$, where $I_1=(x-\frac{r}{2},x+\frac{r}{2})$ and $I_2=(y-\frac{r}{2},y+\frac{r}{2})$. Hence

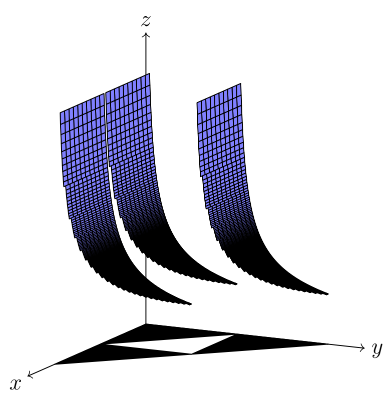
$$\int_{B_d\left((x,y),\frac{r}{2}\right)}\widetilde{w}(x,y)\,dxdy\leq r\int_{I_2}w(y)\,dy,$$

$$\int_{B_d\left((x,y),\frac{r}{2}\right)}\widetilde{w}^{-1}(x,y)\,dxdy\leq r\int_{I_2}w^{-1}(y)\,dy.$$

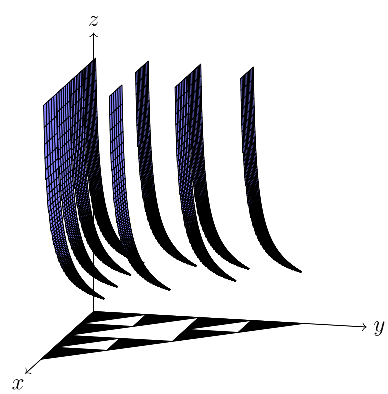
After multiplying term by term the above inequalities and using the fact that $w\in A_2([0,1],dy)$, we obtain the $A_2(X,d,dxdy)$ condition for $\widetilde{w}$:

$$\begin{aligned}&\left(\int_{B_d\left((x,y),\frac{r}{2}\right)}\widetilde{w}(x,y)\,dxdy\right)\left(\int_{B_d\left((x,y),\frac{r}{2}\right)}\widetilde{w}^{-1}(x,y)\,dxdy\right)\\&\leq r^2\int_{I_2}w(y)\,dy\int_{I_2}w^{-1}(y)\,dy\\&\leq\hat{C}r^4\\&\leq\widetilde{C}\mu^2\left(B_d\left((x,y),\frac{r}{2}\right)\right).\end{aligned}$$

So that, in particular, $(X,d,\widetilde{w}\,dxdy)$ is a space of homogeneous type. Notice that $d(\phi_i(X),\phi_j(X))=0$ for every $i,j=1,2,3$, and taking $Y_0=X$ again $T_s(Y_0,\widetilde{w}\,dxdy)$ is not a space of homogeneous type since precisely at each contact point of $\phi_i(X)$ and $\phi_j(X)$ for $i\neq j$, we have a singularity of $\widetilde{w}$ in one of these sets and boundedness on the other (see Figure 2). In fact, for $0<\varepsilon\leq\frac{1}{4}$, $E_\varepsilon=\left([0,\varepsilon]\times[\frac{1}{2}-\varepsilon,\frac{1}{2}]\right)\cap\phi_1(X)$ and



(a) $\widetilde{w}_1(x,y)$ defined on Y_1



(b) $\widetilde{w}_2(x,y)$ defined on Y_2

Figure 2: $T_s^n(Y_0,\widetilde{w}\,dxdy)=:(Y_n,\widetilde{w}_n dxdy)$

$F_\varepsilon=[0,\varepsilon]\times[\frac{1}{2},\frac{1}{2}+\varepsilon]$, with $d\mu_0=\widetilde{w}\,dxdy$, we have $\mu_0'\left(E_\varepsilon\right)=\frac{1}{6}\varepsilon\left(1-(1-2\varepsilon)^{3/2}\right)$ and $\mu_0'\left(F_\varepsilon\right)=\sqrt{2}\varepsilon^{3/2}/2$. These formulas show that $T_s(X,\widetilde{w}\,dxdy)$ is not a space of homogeneous type, since E_ε and F_ε are two neighboring balls with the same radii and $\mu_0'(F_\varepsilon)/\mu_0'(E_\varepsilon)$ tends to infinity when ε tend to zero. Again no $T_s^n(X,\widetilde{w}\,dxdy)$ is a space of homogeneous type and the limit space (Y_∞,μ_∞) is the Sierpinski triangle with the restriction of the Hausdorff measure of dimension $\log 3/\log 2$, which is doubling.

Our third example shows that even when some separation of the sets $\{\phi_i(X) : i = 1,\dots,M\}$ holds, no uniform doubling property for the whole orbit can be expected. In fact, let us consider now the application T_c induced by the IFS of Cantor type taking $k = 5/2$ in (3.1), and with $p_1 = p_2 = 1/2$. In other words, $\phi_1(x) = 2x/5$ and $\phi_2(x) = 2x/5 + 3/5$. Take again $w(x) = x^{-1/2}$ and $(Y_0, \mu_0) = ([0,1], \frac{1}{2}w(x)dx)$ as the starting space. Notice that for any fixed $\alpha > 1$, the space $(Y_0,\mu_0) \in \widetilde{\mathcal{D}}(\alpha, A)$. We claim that if $T_c^n(Y_0,\mu_0)$ belongs to $\widetilde{\mathcal{D}}(\alpha, A_n)$ for some constant $A_n \geq 1$, then $A_n \geq 2^{n/2}$, for each natural number n. If fact, let us write (Y_n, μ_n) to denote $T_c^n(Y_0,\mu_0)$. For a fixed n, take $y_1 = \left(\frac{2}{5}\right)^n$, $y_2 = \frac{3}{2}\left(\frac{2}{5}\right)^n$, and $r = d(y_1,y_2) = \frac{1}{2}\left(\frac{2}{5}\right)^n$. Notice that y_1 and y_2 belong to Y_n and they are the extreme points of the first "gap" of Y_n. It is easy to see that $\mu_n(B_d(y_1,r)) = C5^{-n/2}$ and $\mu_n(B_d(y_2,r)) = C\left(\frac{2}{5}\right)^{n/2}$, where C is a constant which does not depend on n. Then

$$\frac{\mu_n(B_d(y_2,r))}{\mu_n(B_d(y_1,r))} = 2^{n/2}.$$

Hence $\widetilde{\mathcal{D}}(\alpha, A)$ can not be invariant under T_c.

4. A positive result

As a corollary of Theorem 2.2 and the Banach fixed point theorem, we have immediately the following result.

Corollary 4.1. *Let $T : \mathcal{E} \to \mathcal{E}$ be a contraction mapping. Then if $T : \widetilde{\mathcal{D}}(\alpha, A) \to \widetilde{\mathcal{D}}(\alpha, A)$ for some $\alpha > 1$ and $A \geq 1$, there exists a unique fixed point (Y,μ) for T in $\widetilde{\mathcal{D}}(\alpha, A)$.*

Let us next present an application of the above result to the special case of a finite family of contractive similitudes on a metric space which strictly separate images.

Proposition 4.2. *Let (X,d) be a compact metric space and let $\{\phi_1,\dots,\phi_M\}$ be a finite sequence of contractive similitudes on X:*

$$d\left(\phi_i(x), \phi_i(y)\right) = \frac{1}{a_i} d(x,y)$$

for every $x, y \in X$, where $a_i > 1$ for $i = 1,\dots,M$. Let T be the mapping on $\mathscr{X}$ defined by $T(Y,\mu) = (Y', \mu')$, where

$$Y' = \bigcup_{i=1}^{M} \phi_i(Y),$$

$$\mu'(B) = \sum_{i=1}^{M} p_i \mu\left(\phi_i^{-1}(B \cap \phi_i(Y))\right),$$

for some probabilistic sequence $0 < p_i < 1$, $\sum_{i=1}^M p_i = 1$ and every Borel subset B of Y'. If $\operatorname{diam}(X) \leq 1$ and

$$\max_{i=1,\dots,M} a_i^{-1} \leq \min_{\substack{1 \leq i,j \leq M \\ i \neq j}} \{d(\phi_i(X), \phi_j(X))\} =: D, \tag{4.1}$$

then $T : \widetilde{\mathcal{D}}(2, A) \to \widetilde{\mathcal{D}}(2, A)$ for every $A \geq P := \max\{p_i^{-1} : i = 1,\dots,M\}$.

Proof. Set $Y_i' = \phi_i(Y)$ and $\mu_i'(B) = p_i\mu(\phi_i^{-1}(B \cap Y_i'))$ for $i = 1, 2, \ldots, M$. Let us first prove that, for $A \geq P$ and every $i = 1, 2, \ldots, M$, the space $(Y_i', \mu_i') \in \widetilde{\mathcal{D}}(2, A)$ provided that $(Y, \mu) \in \widetilde{\mathcal{D}}(2, A)$. In fact, for $(Y, d, \mu) \in \widetilde{\mathcal{D}}(2, A)$, take $y, z \in Y_i'$ and $r > 0$ such that $d(y, z) < 2r$. Since each ϕ_i is one to one we have that $d(\phi_i^{-1}(y), \phi_i^{-1}(z)) = a_i d(y, z) < 2a_i r$, and that $\phi_i^{-1}(B_d(x, r) \cap Y_i') = B_d(\phi_i^{-1}(x), a_i r) \cap Y$ for every $x \in X$. Then, for every $\varepsilon > 0$,

$$\begin{aligned} p_i^{-1}\mu_i'\left(B_d(y, r)\right) &= \mu\left(\phi_i^{-1}\left(B_d(y, r) \cap Y_i'\right)\right) \\ &= \mu\left(B_d(\phi_i^{-1}(y), a_i r)\right) \\ &\leq A\mu\left(B_d(\phi_i^{-1}(z), a_i r + a_i\varepsilon)\right) \\ &= A\mu\left(\phi_i^{-1}\left(B_d(z, r + \varepsilon) \cap Y_i'\right)\right) \\ &= p_i^{-1}A\mu_i'\left(B_d(z, r + \varepsilon)\right). \end{aligned}$$

To prove that (Y', μ') belongs to $\widetilde{\mathcal{D}}(2, A)$, take now $y, z \in Y'$ and $r > 0$ such that $d(x', y') < 2r$, and $\varepsilon > 0$. Let us consider two cases:

1) y and z belong to the same Y_i';
2) $y \in Y_i'$ and $z \in Y_j'$ with $i \neq j$.

Case 1: If $B_d(y, r)$ and $B_d(z, r + \varepsilon)$ do not intersect any other Y_j' for $j \neq i$, we can use the fact that $(Y_i', \mu_i') \in \widetilde{\mathcal{D}}(2, A)$ to obtain the desired estimate. On the other hand, if $B_d(y, r)$ or $B_d(z, r + \varepsilon)$ intersect Y_j' for some $j \neq i$, necessarily $r + \varepsilon \geq D$. Since we are assuming that $\operatorname{diam}(X) \leq 1$ we have that $\operatorname{diam}(Y_i') \leq a_i^{-1} \leq D$, so that $Y_i' \subseteq B_d(z, r + \varepsilon)$ and $\mu'(B_d(z, r + \varepsilon)) \geq p_i$. Hence

$$\mu'(B_d(y, r)) \leq 1 \leq p_i^{-1}\mu'(B_d(z, r + \varepsilon)) \leq P\mu'(B_d(z, r + \varepsilon)) \leq A\mu'(B_d(z, r + \varepsilon)),$$

which is the desired inequality.

Case 2: We may assume that $a_i \leq a_j$. Since $D \leq d(y, z) < 2r$, we have $r \geq D/2$. If $r + \varepsilon \geq D$ we easily have that

$$\mu'(B_d(y, r)) \leq 1 = p_j^{-1}\mu'(Y_j') \leq A\mu'(B_d(z, r + \varepsilon)).$$

If, instead, $r + \varepsilon \leq D$ we have

$$\mu'(B_d(y, r)) = \mu_i'(B_d(y, r)) = \mu(B_d(\phi_i^{-1}(y), a_i r))/M,$$

$$\mu'(B_d(z, r + \varepsilon)) = \mu_j'(B_d(z, r + \varepsilon)) = \mu(B_d(\phi_j^{-1}(z), a_j(r + \varepsilon)))/M.$$

So that, since $d(y, z) \leq 1 \leq 2r/D \leq 2a_i r$,

$$\begin{aligned} \mu'\left(B_d(y, r)\right) &= p_i\mu(B_d(\phi_i^{-1}(y), a_i r)) \\ &\leq Ap_i\mu(B_d(\phi_j^{-1}(z), a_i(r + \varepsilon))) \\ &\leq A\frac{p_j}{P}\mu(B_d(\phi_j^{-1}(z), a_j(r + \varepsilon))) \\ &\leq A\mu'(B_d(z, r + \varepsilon)). \end{aligned}$$

□

Notice that the hypothesis $\operatorname{diam}(X) \leq 1$ is not restrictive in the sense that the results holds if now $D \geq \operatorname{diam}(X)/a_i$ for every $i = 1, \ldots, M$.

Hence, under the hypotheses of the above theorem, any starting point (Y_0, μ_0) generates an orbit

$$\mathcal{O}_T(Y_0, \mu_0) = \{T^n(Y_0, \mu_0) : n \in \mathbb{N}_0\}$$

which is completely contained in $\widetilde{\mathcal{D}}(2, P)$. So that, in these cases, we obtain again the result of Mosco ([9], see also [10]). And actually $T^n(Y_0, \mu_0)$ are good approximations of the limit space (Y_∞, μ_∞), in the sense that each approximating space is a space of homogeneous type, with bounded doubling constant.

As a family of examples to which Proposition 4.2 applies, consider the Cantor type IFS defined in (3.1), with $k \geq 3$. The classical Cantor set C is obtained when $k = 3$ and $p_1 = p_2 = 1/2$, and actually the invariant measure μ_∞ is the restriction to C of $\mathcal{H}^s$ with $s = \frac{\log 2}{\log 3}$. Then for every $A \geq 2$ and every $(Y_0, \mu_0) \in \widetilde{\mathcal{D}}(2, A)$, the whole orbit $\{T_s^n(Y_0, \mu_0) : n \in \mathbb{N}\}$ is a uniform sequence of spaces of homogeneous type.

To give a higher dimensional example of application of Proposition 4.2, let us consider a Sierpinski type family of contractions. For each $k \geq 1$, let

$$\phi_1(x, y) = (x/k, y/k)\,,$$
$$\phi_2(x, y) = \phi_1(x, y) + (0, 1 - 1/k)\,,$$
$$\phi_3(x, y) = \phi_1(x, y) + (1 - 1/k, 0)\,.$$

Let X be the triangle in $\mathbb{R}^2$ with vertices at $(0, 0)$, $(1, 0)$ and $(0, 1)$. Taking $d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$ as the distance on X, we define T_s on $\mathscr{X}$ by $T_s(Y, \mu) = (Y', \mu')$ with

$$Y' = \phi_1(Y) \cup \phi_2(Y) \cup \phi_3(Y) := Y_1' \cup Y_2' \cup Y_3'\,,$$

and

$$\mu'(B) = \frac{1}{3}\left(\mu\left(\phi_1^{-1}(B \cap Y_1')\right) + \mu\left(\phi_2^{-1}(B \cap Y_2')\right) + \mu\left(\phi_3^{-1}(B \cap Y_3')\right)\right),$$

for every Borel subset B of Y'. Hence from Proposition 4.2, for $k \geq 3$ and $A \geq 3$ we have that $T_s : \widetilde{\mathcal{D}}(2, A) \to \widetilde{\mathcal{D}}(2, A)$. So that the whole orbit $\{T_s^n(Y_0, \mu_0) : n \in \mathbb{N}\}$ is a uniform sequence of spaces of homogeneous type for every $(Y_0, \mu_0) \in \widetilde{\mathcal{D}}(2, A)$. We can take for example $Y_0 = X$ and μ_0 to be twice the area Lebesgue measure on X. And of course also the limit (Y_∞, μ_∞) is a space of homogeneous type.

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