WHAT IS A SOBOLEV SPACE FOR THE LAGUERRE FUNCTION SYSTEMS?

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Abstract. We discuss the concept of Sobolev space associated to the Laguerre operator $L_\alpha = -y\,\frac{d^2}{dy^2} - \frac{d}{dy} + \frac{y}{4} + \frac{\alpha^2}{4y}, \quad y \in (0,\infty)$. We show that the natural definition does not fit with the concept of potential space, defined via the potentials $(L_\alpha)^{-s}$. An appropriate Laguerre-Sobolev spaces are defined in order to have the mentioned coincidence. An application is given to the almost everywhere convergence of solutions of the Schrödinger equation. Other Laguerre operators are also considered.

1. Introduction

We start with a naive description of our aim in writing this paper. Let $\mathbf{L}$ be a linear second order differential operator, selfadjoint with respect to a certain measure μ. Different techniques, see for example (**??**), allow us to define the "Riesz potentials" $\mathbf{L}^{-s}$, $s > 0$. In these circumstances we consider the "potential space" $\mathbf{L}^p_s$, $1 < p < \infty$, as the space $\mathbf{L}^{-s/2}(L^p(\mu))$, in other words the collection of functions f such that there exists $g \in L^p(\mu)$ with $f = \mathbf{L}^{-s/2}(g)$.

In general, the second order operator $\mathbf{L}$ admits a certain factorization $\mathbf{L} = \sum_i \mathcal{D}_i^* \mathcal{D}_i$, where $\mathcal{D}_i$ are first order differential operators with adjoints (respect to μ) $\mathcal{D}_i^*$. Then it is also usual to define the "Riesz transforms" $R_i = \mathcal{D}_i \circ \mathbf{L}^{-1/2}$ and analyze their boundedness properties on $L^p(\mu)$, see [**?**], [**?**]. Several motivations can be given for the study of these Riesz transforms. For example the boundedness in L^p of operators like $\mathcal{D}_i^2 \circ \mathbf{L}^{-1}$ (usually called Riesz transforms of second order) drive rather easily to "a priori" estimates in L^p for the equation $\mathbf{L}u = f$, just observe that the boundedness $\|\mathcal{D}_i^2 \circ \mathbf{L}^{-1} g\|_p \le C\|g\|_p$ can be written in this case as $\|\mathcal{D}_i^2 u\|_p \le C\|f\|_p$. A second motivation (in fact **the motivation** of this note) is the following. Given a natural number k, let us define the (Sobolev) space W^k_p as the collection of functions on L^p such that the $k-$derivatives $\mathcal{D}_i^k f$ belong to L^p. Suppose that the Riesz transforms of order k, $\mathcal{D}_i^k \circ \mathbf{L}^{-k/2}$, satisfy $\|\mathcal{D}_i^k \circ \mathbf{L}^{-k/2} f\|_p \sim \|f\|_p$. This last equivalence could be written (at least formally) as $\|\mathcal{D}_i^k f\|_p \sim \|\mathbf{L}^{k/2} f\|_p$. In other words the spaces W^k_p and $\mathbf{L}^p_k$ would coincide. As the spaces $\mathbf{L}^p_s$ have a meaning for all $s > 0$, (even no integer) one could say that the potential space $\mathbf{L}^p_s$ is the space of functions on L^p whose "$s-$derivative" is in L^p. We observe that if

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φ_n is an eigenfunction of $\mathbf{L}$ with eigenvalue $\lambda_n \neq 0$ then $\mathbf{L}^{-s}\varphi_n = \lambda_n^{-s}\varphi_n$. Hence the last interpretation of $s-$derivative is particularly simple to understand.

If the above ideas are directly applied to the Laguerre differential operator (for Laguerre functions) one finds something that can be consider a surprise. The expected definition of Sobolev space of order k, that is, the set of functions in L^p such that the derivatives (according to the natural factorization of the differential operator) of order k belong to L^p, does not fit with the definition of potential spaces, see Definitions **??**, **??** and **??** and Theorems **??** and **??**. The main purpose of the paper is to clarify and make precise which could be the most appropriated definition of Sobolev spaces for the Laguerre operator. Our work was inspired in [**?**] and [**?**].

It is a common fact that if a concept is developed for Laguerre functions then the analogous concept can be developed in an easier way for Hermite functions. That happens in this work and then we devote Section **??** to Hermite functions. A comment about the dimension is convenient here. The motivation of this paper is essentially one-dimensional, but in the case of Hermite functions, the theory has no added difficulty in several variables, so we present in that context our results for the Hermite operator.

The knowledge of a sharp enough power weighted theory for a Laguerre function system can be transferred to another Laguerre function system, see [**?**]. That is why we develop a weighted theory of Sobolev and potential spaces for a particular system of Laguerre functions and then we transfer in an easy way to another systems, see Section **??**.

Finally in the last section we present a simple application to the pointwise converge of solutions of Schrödinger equation.

We discuss quickly the case of the Hermite operator

$$H = -\Delta + |x|^2, \quad x \in \mathbb{R}^d. \tag{1}$$

H is self-adjoint on the set of infinitely differentiable functions with compact support $C_c^\infty(\mathbb{R})$. The Lebesgue measure will be the ambient measure.

For each $s > 0$, the Hermite potential, H^{-s}, is defined for $f \in L^2(\mathbb{R}, dx)$, by the formula

$$H^{-s} f(x) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-tH} f(x)\, t^s \,\frac{dt}{t}, \quad x \in \mathbb{R}^d, \tag{2}$$

where $\{e^{-tH}\}_{t\geq 0}$ is the heat semi-group associated to H. The corresponding potential spaces, $\mathcal{L}_s^p(w) = H^{-s/2}(L^p(w))$, are defined in (**??**) with respect to an absolute continuous measure $w(x)dx$, being w a weight in A_p. For the reader's convenience we remind that a positive function w is said to belong to the Muckenhoupt class $A_p, 1 < p < \infty$, if the Hardy-Littlewood maximal operator is bounded from $L^p(w(x)dx)$ into $L^p(w(x)dx)$, and w is said to belong to the class A_1, if the Hardy-Littlewood maximal operator is bounded from $L^1(w(x)dx)$ into weak-$L^1(w(x)dx)$.

The operator H can be factorized as $H = \frac{1}{2}\sum_{j=1}^d A_j A_{-j} + A_{-j} A_j$, see (**??**). Where A_j and A_{-j} are first order differential operators.

Definition 1. *Given $k \in \mathbb{N}$, the Hermite-Sobolev space of order k, denoted by $W^{k,p}(w)$, will be the set of functions $f \in L^p(w)$ such that*

$$\overbrace{A_j \cdots A_j}^{m\ times} f = A_j^m f \ \in \ L^p(w),\ 1 \le m \le k, 1 \le j \le d,$$

with the norm

$$\|f\|_{W^{k,p}(w)} = \sum_{j=1}^{d} \sum_{1 \le m \le k} \|A_j^m f\|_{L^p(w)} + \|f\|_{L^p(w)}.$$

The following theorem will be proved in Section **??**.

Theorem 1. *Let $k \in \mathbb{N}$, $1 < p < \infty$, and $w \in A_p$. Then,*

$$W^{k,p}(w) = \mathfrak{L}_k^p(w)$$

and the norms $\|\cdot\|_{W^{k,p}(w)}$ and $\|\cdot\|_{\mathfrak{L}_k^p(w)}$ are equivalent.

Of course in order to prove this theorem, we shall need previously to prove some boundedness result of higher order Riesz Transforms, see Theorem **??**.

Regarding the Laguerre operator

$$L_\alpha = -y\,\frac{d^2}{dy^2} - \frac{d}{dy} + \frac{y}{4} + \frac{\alpha^2}{4y}, \quad y \in (0,\infty), \tag{3}$$

selfadjoint in the set $\mathcal{C}_c(0,\infty)$, there is a natural domain of power weights y^δ for the boundedness on $L^p(\mathbb{R}^+, y^\delta dy)$ of classical operators associated to L_α, (see [**?**]), namely if $\alpha > -1$, $1 < p < \infty$ and $\delta \in \mathbb{R}$

$$(\mathrm{C}_\alpha) \qquad -\frac{\alpha}{2}p - 1 < \delta < p - 1 + \frac{\alpha}{2}p. \tag{4}$$

In a parallel way to the Hermite case, we can define appropriate potential spaces for Laguerre functions.

Definition 2. *Given $\alpha > -1$, $1 < p < \infty$, $s > 0$ and $\delta \in \mathbb{R}$ we define*

$$\mathfrak{W}^p_{\alpha,s}(y^\delta) = (L_\alpha)^{-s/2}[L^p(\mathbb{R}^+, y^\delta dy)]$$

with the norm $\|f\|_{\mathfrak{W}^p_{\alpha,s}(y^\delta)} = \|g\|_{p,\delta}$, where $(L_\alpha)^{-s/2} g = f$.

On the other hand the Laguerre operator can be factorized as $L_\alpha = (\delta^\alpha)^*\delta^\alpha + \dfrac{(\alpha+1)}{2}$, see (**??**). Following the thoughts that we developed for the Hermite case, one can give the following.

Definition 3. *We shall denote by $\mathcal{W}_\alpha^{k,p}(y^\delta)$ the set of functions f in $L^p(\mathbb{R}^+, y^\delta dy)$ such that $(\delta^\alpha)^m f \in L^p(\mathbb{R}^+, y^\delta dy)$, $0 \le m \le k$, with the norm*

$$\|f\|_{\mathcal{W}_\alpha^{k,p}(y^\delta)} = \sum_{m=0}^{k} \|(\delta^\alpha)^m f\|_{L^p(\mathbb{R}^+, y^\delta dy)}.$$

However, even we shall prove (see Theorem **??**) that the higher order Riesz transforms $(\delta^\alpha)^k (L_\alpha)^{-k/2}$ are bounded in $L^p(y^\delta dy)$ for δ satisfying (C_α), the "Sobolev" spaces $\mathcal{W}_\alpha^{k,p}(y^\delta)$ are different from the potential spaces $\mathfrak{W}^p_{\alpha,k}(y^\delta)$. In fact we have the following

Theorem 2. *Let p be in the range $1 < p < \infty$.*

(i) *Let* $\alpha > -1$, *and* δ *satisfying* (C_α). *Then* $\mathfrak{W}^{k,p}_\alpha(y^\delta) \subset \mathcal{W}^{k,p}_\alpha(y^\delta)$.
(ii) *Let* $-1 < \alpha \leq 0$. *Then* $\mathfrak{W}^{2,2}_\alpha \neq \mathcal{W}^{2,2}_\alpha$.
(iii) *Let* $\alpha > 0$, *and* δ *satisfying* $(C_{\alpha-1})$. *Then* $\mathfrak{W}^{2,p}_\alpha(y^\delta) = \mathcal{W}^{2,p}_\alpha(y^\delta)$.

This result suggests that the iteration of operators δ^α are no good substitutes for the notion of fractional derivative in this case. Looking at the actual action of these operators over the set of eigenfunctions of the operator L_α, see (**??**) and (**??**), it seems natural to consider the higher order Riesz transforms defined as

$$R^k_\alpha = \left(\delta^{\alpha+k-1} \circ \cdots \circ \delta^{\alpha+1} \circ \delta^\alpha\right)(L_\alpha)^{-k/2}.$$

It is proved in Theorem **??** that these Riesz transforms, R^k_α, are bounded on $L^p(\mathbb{R}^+, y^\delta dy)$ for δ satisfying (C_α). This would suggest the following alternative concept of the "Sobolev" spaces given in Definition **??**.

Definition 4. *The Laguerre-Sobolev spaces, that we denote by* $W^{k,p}_\alpha(y^\delta)$, *are the sets of functions* f *in* $L^p(\mathbb{R}^+, y^\delta dy)$ *such that*

$$\delta^{\alpha+m} \circ \ldots \circ \delta^{\alpha+1} \circ \delta^\alpha f \in L^p(\mathbb{R}^+, y^\delta dy), \ 0 \leq m \leq k-1$$

with the norm

$$\|f\|_{W^{k,p}_\alpha(y^\delta)} = \|f\|_{p,\delta} + \sum_{m=0}^{k-1} \left\|\delta^{\alpha+m} \circ \ldots \circ \delta^{\alpha+1} \circ \delta^\alpha f\right\|_{p,\delta}.$$

These spaces are the right spaces for the problem we are considering and the following theorem will be proved in Section **??**.

Theorem 3. *Let* $\alpha > -1$, $1 < p < \infty$, $k \in \mathbb{N}$ *and* δ *satisfies* (C_α). *Then,*

$$W^{k,p}_\alpha(y^\delta) = \mathfrak{W}^{k,p}_\alpha(y^\delta),$$

and the norms are equivalent.

Unweighted Sobolev spaces in the case of Hermite operator were considered previously by Thangavelu see [**?**] and the authors [**?**].

For the case of Laguerre functions, Laguerre potential spaces were introduced by Peetre and Sparr in 1975, they were also studied by Thangavelu in [**?**] and by Radha and Thangavelu in [**?**] and [**?**]. For some previous works contain the definition and power weighted L^p-boundedness of the first order Riesz transforms, see [**?**] and [**?**] for the system $\mathcal{L}^\alpha_k$, and [**?**] for the system φ^α_k, see Section **??**. Recently, power weighted L^p-boundedness of the higher order Riesz transforms of the form $(\mathbf{D}^\alpha)^k \mathbf{L}^{-k/2}$ for the system φ^α_k (see Section **??**) has been proved in [**?**]. From that result one can deduce, by using back the methods in Section **??**, our Theorem **??** about operators of the form $(\delta^\alpha)^k (L_\alpha)^{-k/2}$. However, we present our different proof since we think that it contains some explanation of the behaviour of the commuting properties of several operators. Finally for the case of Laguerre polynomials some results can be found in [**?**].

2. Hermite Sobolev spaces with weights

Let $H_n, n = 0, 1, \ldots$ be the family of Hermite polynomials. The Hermite function of order n is defined as $h_n(t) = \dfrac{H_n(t)\, e^{-t^2/2}}{(2^n n! \pi^{1/2})^{1/2}}$, $t \in \mathbb{R}$. Given a multi-index $\alpha =$

$(\alpha_j)_{j=1}^d \in \mathbb{N}^d$, the Hermite function of order α is defined as

$$h_\alpha(x) = \Pi_{j=1}^d h_{\alpha_j}(x_j), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

These functions are eigenvectors of the Hermite operator, see (**??**). In fact

$$H h_\alpha = (2|\alpha| + d)\, h_\alpha,$$

where $|\alpha| = \sum_{j=1}^d \alpha_j$, see [**?**].

We shall need the following lemmas. Their proofs can be found respectively in [**?**], [**?**] and [**?**].

Lemma 1. *Let $M \in \mathbb{N}$ and $f \in C_c^\infty$, then there exists a constant $C_{M,f} > 0$ such that*

$$\left| \int_{\mathbb{R}^d} f\, h_\alpha \right| \leq C_{M,f}\, (|\alpha| + 1)^{-M}, \quad \alpha \in \mathbb{N}^d.$$

Lemma 2. *Let $1 \leq p < \infty$ and $w \in A_p$, there exist constants $\epsilon_p > 0$ and C_w such that*

$$\|h_\alpha\|_{L^p(w)} \leq C_w\, (|\alpha| + 1)^{\epsilon_p}.$$

Lemma 3. *Let f be a linear combination of Hermite functions, the fractional integral H^{-s}, $s > 0$, see (**??**), has an integral representation*

$$H^{-s} f(x) = \int_{\mathbb{R}^d} K_s(x, y) f(y) dy,\ x \in \mathbb{R}^d,$$

where $K_s(x, y)$ is positive and symmetric. Moreover,

$$K_s(x, y) \leq C\, \phi_s(|x - y|), \quad x, y \in \mathbb{R}^d, \tag{5}$$

where $\phi_s(r)$, for $r \geq 0$, is defined by

$$\phi_s(r) = \begin{cases} \frac{\chi_{\{r<1\}}(r)}{r^{d-2s}} + e^{\frac{-r^2}{4}}\, \chi_{\{r\geq 1\}}(r), & \text{if } s < \frac{d}{2}, \\ \log\left(\frac{e}{r}\right)\, \chi_{\{r<1\}}(r) + e^{\frac{-r^2}{4}}\, \chi_{\{r\geq 1\}}(r), & \text{if } s = \frac{d}{2}, \\ \chi_{\{r<1\}}(r) + e^{\frac{-r^2}{4}}\, \chi_{\{r\geq 1\}}(r), & \text{if } s > \frac{d}{2}. \end{cases}$$

Theorem 4. *Let $1 \leq p < \infty$ and $s > 0$. If $w \in A_p$, then the operator H^{-s}, is bounded on $L^p(w)$.*

Proof. If $p > 1$, we just observe that the function $x \to \phi_s(|x|)$ is radial and decreasing for $|x| \to \infty$, therefore, $|H^{-s} f(x)| \leq M(|f|)(x)$ where M is the Hardy-Littlewood maximal operator and the the result follows.

In the case $p = 1$, we shall prove that $\displaystyle\int_{\mathbb{R}^d} K_s(x, y) w(x)\, dx \leq\ C\, w(y)$, whenever y is a Lebesgue point of w. Therefore,

$$\int_{\mathbb{R}^d} |H^{-s} f(x)|\, w(x)\, dx \leq \int_{\mathbb{R}^d} |f(y)| \int_{\mathbb{R}^d} K_s(x, y)\, w(x)\, dx\, dy \leq \int_{\mathbb{R}^d} |f(y)| w(y) dy.$$

If y is a Lebesgue point of $w \in A_1$, then

$$\frac{1}{|B(y, r)|} \int_{B(y,r)} w(x) dx \leq C\, w(y).$$

Hence, by estimate (??) and splitting into annuli, we have

$$\begin{aligned}\int_{\mathbb{R}^d} K_s(x,y)w(x)\,dx &\le C\sum_{k=-\infty}^{\infty}\int_{B(y,2^k)\setminus B(y,2^{k-1})}\phi_s(|y-x|)w(x)\,dx\\ &\le C\sum_{k=-\infty}^{\infty}\phi_s(2^k)\frac{2^{dk}}{|B(y,2^k)|}\int_{B(y,2^k)\setminus B(y,2^{k-1})}w(x)\,dx\\ &\le C\,w(y)\left(\sum_{k=-\infty}^{\infty}\phi_s(2^k)2^{dk}\right)\le Cw(y).\end{aligned}$$

□

Given $1\le p<\infty$, $s>0$ and $w\in A_p$, we define the potential spaces

$$\mathfrak{L}_s^p(w)=H^{-s/2}(L^p(w)), \tag{6}$$

with the norm $\|f\|_{\mathfrak{L}_s^p(w)}=\|g\|_{L^p(w)}$, where g is such that $H^{-s/2}g=f$.
The space $\mathfrak{L}_s^p(w)$ is well defined, since $H^{-s/2}$ is bounded and one to one in $L^p(w)$. If fact, suppose $g\in L^p(w)$ and $H^{-s/2}g=0$. Observe that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K_{s/2}(x,y)|g(x)||h_\alpha(y)|dy\,dx\le\|H^{-s/2}|g|\|_{L^p(w)}\|h_\alpha\|_{L^{p'}(w^{-p'/p})},$$

and this expression in finite by Theorem ?? and Lemma ?? since $w^{-p'/p}$ belongs to $A_{p'}$. Hence, by Fubini and the symmetry of $K_{s/2}$,

$$\int_{\mathbb{R}^d}g\,h_\alpha=(2n+1)^{s/2}\int_{\mathbb{R}^d}g\,H^{-s/2}h_\alpha=\ (2n+1)^{s/2}\int_{\mathbb{R}^d}H^{-s/2}g\,h_\alpha=0,$$

and this assures that $g=0$ (see Corollary 2.4 in [?]).

Remark 1. The space $\mathfrak{F}$ of finite linear combinations of Hermite functions is a dense subspace of $\mathfrak{L}_s^p(w)$, since $\mathfrak{F}=H^{-s/2}(\mathfrak{F})$ is dense in $L^p(w)$.

The operator H can be factorized as

$$H=\frac{1}{2}\sum_{j=1}^{d}A_jA_{-j}+A_{-j}A_j, \tag{7}$$

where

$$A_j=\frac{\partial}{\partial x_j}+x_j\quad\text{and}\quad A_{-j}=-\frac{\partial}{\partial x_j}+x_j.$$

It is easy to check that

$$A_jh_\alpha=\sqrt{2\alpha_j}\,h_{\alpha-e_j},\qquad A_{-j}h_\alpha=\sqrt{2(\alpha_j+1)}\,h_{\alpha+e_j}, \tag{8}$$

where e_j is the jth-coordinate vector in $\mathbb{N}^d$. From these formulas the operators A_j and A_j^* are called *annihilation* and *creation* operators respectively.

Definition 5. *The Hermite-Riesz transforms of order* m, $m\in\mathbb{N}$, *associated to* H *are defined by*

$$R_J^m=A_{j_1}\dots A_{j_m}H^{-m/2},\ \textit{where}\ J=(j_1,\dots,j_m), 1\le|j_i|\le d,\ 1\le i\le m.$$

In the case $j_1 = \cdots = j_m = j$, these operators will be denoted by R_j^m. The case $m = 1$ were introduced by S. Thangavelu, see [**?**]. He proved that they are bounded operators in $L^p(\mathbb{R}^d)$. Also in [**?**] and [**?**], it was shown that the operators R_J^m are Calderon-Zygmund operators and as a consequence they are bounded in $L^p(w)$ for $w \in A_p$, $1 < p < \infty$.

We shall now present a structural theorem for the spaces $\mathfrak{L}_s^p(w)$. The unweighted versions of this result can be found in [**?**] (Theorems 2, 6 and 7.)

Theorem 5. *Let $w \in A_p$, $1 < p < \infty$, and $s > 0$.*

i) If $t > s$, then $\mathfrak{L}_t^p(w) \subset \mathfrak{L}_s^p(w) \subset L^p(w)$ with continuous inclusions. Moreover, $\mathfrak{L}_s^p(w)$ and $\mathfrak{L}_t^p(w)$ are isometrically isomorphic.
ii) If $t > 0$, then $H^{-t/2}$ maps $\mathfrak{L}_s^p(w)$ into $\mathfrak{L}_{s+t}^p(w)$.
iii) If $s > 1$ and $1 \le |j| \le d$, then A_j is bounded from $\mathfrak{L}_s^p(w)$ into $\mathfrak{L}_{s-1}^p(w)$.
iv) The operators R_J^m, are bounded on $\mathfrak{L}_s^p(w)$.

Proof. Observe that $H^{-t/2} = H^{-s/2} \circ H^{-r}$, with $r = (t-s)/2$. Then (i) follows from Theorem **??**. (ii) also follows from Theorem **??** and the definition of the spaces $\mathfrak{L}_s^p(w)$.

In order to prove (iii) we shall need the two following results. They can be found respectively in [**?**] and [**?**]. For further reference, we stated them as Proposition **??** and Lemma **??**.

Proposition 1. *Let $1 < p < \infty$ and $m \in \ell^\infty(\mathbb{N}^d)$ such that*

$$\|\Delta^\ell m(\alpha)| \le C\,(1+|\alpha|)^{-|\ell|}, \quad \alpha \in \mathbb{N}^d, \quad \forall\, |\ell| \le d+1.$$

Consider the operator $\mathcal{T}_m f = \sum_\alpha m(\alpha)\langle f, h_\alpha\rangle h_\alpha$, defined at least for $f \in L^2(\mathbb{R})$. Then, $\mathcal{T}_m$ admits a bounded extension to $L^p(w)$ whenever the weight w belongs to the Muckenhoupt class A_p.

Remark 2. Observe that as $Hh_\alpha = (2|\alpha|+d)h_\alpha$, any operator of the type $F(H)f = \sum_\alpha F(2\alpha+d)\langle f, h_\alpha\rangle h_\alpha$ can be written as $\mathcal{T}_m f = \sum_\alpha m(\alpha)\langle f, h_\alpha\rangle h_\alpha$ with $m(\alpha) = F(2\alpha+d) = F(2(\alpha_1, \ldots, \alpha_d)+d)$.

Lemma 4. *Let $b \in \mathbb{R}^d$, then for all f in $\mathfrak{F}$, we have*

$$A_jH^bf = (H+2)^bA_jf, \quad 1 \le j \le d,$$
$$A_jH^bf = (H-2)^bA_jf, \quad -d \le j \le -1$$
$$H^bA_jf = A_j(H-2)^bf, \quad 1 \le j \le d \text{ and}$$
$$H^bA_jf = A_j(H+2)^bf, \quad -d \le j \le -1.$$

Where $H^bh_\alpha = (2|\alpha|+d)^bh_n$ and $(H+2)^bh_\alpha = (2|\alpha|+d+2)^bh_\alpha$, for all $\alpha \in \mathbb{N}_0^d$, and $(H-2)^bh_\alpha = (2|\alpha|+d-2)^bh_\alpha$, for all α with $|\alpha| \ge 1$.

We continue the proof of Theorem **??**. Let $1 \le j \le d$ (the case $-d \le j \le -1$ is similar). Let $f \in \mathfrak{F}$, by Lemma **??** we have

$$A_jf = H^{-(s-1)/2}\left(\frac{H}{H+2}\right)^{(s-1)/2} R_j\,H^{s/2}f.$$

As the function $\mathbf{m}(\alpha) = \left(\dfrac{2|\alpha|+d}{2|\alpha|+d+2}\right)^{(s-1)/2}$ satisfies the hypotheses of Proposition **??**, see also Remark **??**, the operator $\left(\frac{H}{H+2}\right)^{(s-1)/2}$ is bounded on $L^p(w)$.

Hence by using the boundedness in $L^p(w)$ of the Riesz transforms, we have

$$\|A_j f\|_{\mathfrak{L}^p_{s-1}(w)} = \left\| \left(\frac{H}{H+2}\right)^{(s-1)/2} R_j\, H^{s/2} f \right\|_{L^p(w)} \le C\|H^{s/2}f\|_{L^p(w)} = \|f\|_{\mathfrak{L}^p_s(w)}.$$

Finally (iv) follows from (ii) and (iii). □

The following technical result will be needed later.

Proposition 2. *Let $1 < p < \infty$ and $w \in A_p$. For $k \in \mathbb{N}$ the set $W^{k,p}(w)$ (see Definition* **??***) is a Banach space. Moreover, the sets $\mathfrak{F}$ and C_c^∞ are dense in $W^{k,p}(w)$.*

Proof. Observe that if $\{f_n\}_{n\ge1}$ is a Cauchy sequence in $W^{k,p}(w)$, the completeness of $L^p(w)$ implies that f_n converges to some f and $A_j^m f_n$ converges to some $g_{m,j}$ in $L^p(w)$, $1 \le m \le k$, $1 \le j \le d$. If ψ belongs to C_c^∞, also $(A_j^m)^*\psi$ belongs to C_c^∞, and if B is a ball containing the support of $(A_j^m)^*\psi$, then

$$\begin{aligned} \left| \int_{\mathbb{R}^d} f\,(A_j^m)^*\psi - \int_{\mathbb{R}^d} f_n\,(A_j^m)^*\psi \right| &\le C \int_B |f - f_n| \\ &\le C \left(\int_{\mathbb{R}^d} |f-f_n|^p w \right)^{1/p} \left(\int_B w^{-p'/p} \right)^{1/p'}, \end{aligned}$$

where the last integral is finite due to $w \in A_p$. Hence

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n\,(A_j^m)^*\psi = \int_{\mathbb{R}^d} f\,(A_j^m)^*\psi.$$

In the same way, $\lim_{n\to\infty} \int_{\mathbb{R}^d} A_j^m f_n\,\psi = \int_{\mathbb{R}^d} g_{m,j}\,\psi$. Therefore we have $\int_{\mathbb{R}^d} A_j^m f \psi = \int_{\mathbb{R}^d} g_{m,j}\,\psi$, for all ψ in C_c^∞, and thus $A_j^m f = g_{m,j}$ almost everywhere. These completes the proof that $W^{k,p}(w)$ is complete.

Now we will see that C_c^∞ is a dense set in $W^{k,p}(w)$ (we shall follow the ideas in [**?**], p. 123.) Let ψ be a function in C_c^∞ such that $\int_{\mathbb{R}^d}\psi = 1$. For every $\epsilon > 0$, consider $\psi_\epsilon(x) = \frac{1}{\epsilon^d}\psi\left(\frac{x}{\epsilon}\right)$. Given f in $W^{1,p}(w)$, the function $f * \psi_\epsilon$ belongs to C^∞ and approximates f in the $W^{1,p}(w)$-norm. In fact, it is easy to see that for all $m \ge 1$ and $1 \le j \le d$,

$$A_j(f*\psi_\epsilon) = A_j f * \psi_\epsilon + \epsilon f * (x_j\psi)_\epsilon\,, \quad A_j^m(f*\psi_\epsilon) = \sum_{n=0}^m \epsilon^n A_j^{m-n} f * (x_j^n\psi)_\epsilon.$$

Since $A_j^{m-n}f$ belongs to $L^p(w)$ and $x_j^n\psi$ belongs to C_c^∞, $0 \le n \le m$, we have

$$A_j^m(f*\psi_\epsilon) \to A_j^m f$$

in $L^p(w)$ as ϵ goes to 0. The functions $f*\psi_\epsilon$ do not necessary have compact support, but they can be modified as in the classical case (see [**?**], p. 123).

It remains to prove that any function in C_c^∞ can be approximated in the $W^{k,p}(w)$-norm by a function in $\mathfrak{F}$. We will show that any $f \in C_c^\infty$ is the limit, in the $W^{k,p}(w)$-norm, of a subsequence of the partial sums

$$S_N f = \sum_{|\alpha|\le N} \langle f, h_\alpha\rangle\, h_\alpha, \qquad N \ge 1,$$

where $\langle f, h_\alpha\rangle = \displaystyle\int f h_\alpha$. In [?], Lemma 2.3, it is proved that there exists a subsequence of the previous sequence converging to f in the $L^p(w)$-norm. Hence, it is enough to show that there exists a subsequence of

$$\{A_j^m(S_N(f))\}_{N\geq 1} = \{S_N(A_j^m f)\}_{N\geq 1}$$

converging to $A_j^m f$ in the $L^p(w)$-norm, where $1 \leq j \leq d$ and $1 \leq m \leq k$.

Let us fix j and m such that $1 \leq j \leq d$ and $1 \leq m \leq k$. Following the argument of [?], the sequence $\{S_N(A_j^m f)\}_{N\geq 1}$ converges to $A_j^m f$ in the L^2-norm. Hence we can take a subsequence $\{S_{N_k}(A_j^m f)\}_{k\geq 1}$ converging to $A_j^m f$ almost everywhere. By using (**??**), we have

$$\begin{aligned} S_N(A_j^m f) &= \sum_{|\alpha|\leq N} \langle A_j^m f, h_\alpha\rangle h_\alpha = \sum_{|\alpha|\leq N} \langle f, (A_j^*)^m h_\alpha\rangle h_\alpha \\ &= \sum_{|\alpha|\leq N} \prod_{l=1}^{m} \sqrt{2(\alpha_i + l)}\langle f, h_{\alpha+me_i}\rangle h_\alpha \,. \end{aligned}$$

Hence, by Lemma **??** (with $M \geq m$) and Hölder's inequality, we have

$$\begin{aligned} |S_N(A_j^m f)|^p &\leq C \left(\sum_{|\alpha|\leq N} \prod_{l=1}^{m} \sqrt{2(\alpha_j + l)}\, (|\alpha| + m + 1)^{-M} |h_\alpha| \right)^p \\ &\leq C \left(\sum_{|\alpha|\leq N} |\alpha + m + 1|^{-M/2} |h_\alpha| \right)^p \\ &\leq C \left(\sum_{\alpha} |\alpha + 1|^{-M/2} \right)^{p/p'} \sum_{\alpha} |\alpha + 1|^{-M/2} |h_\alpha|^p \\ &\leq C \sum_{\alpha} |\alpha + 1|^{-M/2} |h_\alpha|^p. \end{aligned}$$

From Lemma **??**, for a big enough M, the function

$$\sum_{\alpha} |\alpha + 1|^{-M/2} |h_\alpha|^p$$

belongs to $L^1(w)$. The dominated convergence theorem implies that

$$\{S_{N_k}(A_j^m f)\}_{k\geq 1} \to A_j^m f$$

in the $L^p(w)$-norm. Now we can repeat the lines above for every j and m, taking a subsequence of the previous subsequence in each step. □

Proof of Theorem **??**. Since $\mathfrak{F}$ is dense in both spaces, it is enough to show the equivalence of the norm for functions in $\mathfrak{F}$.

Let $f \in \mathfrak{F}$, and $f = H^{-k/2} g$. For $1 \leq j \leq d$ and $1 \leq m \leq k$, from the boundedness of R_j^m and H^{-k+m} (see the comments after Definition **??** and Theorem **??**) we have

$$\|(A_j)^m f\|_{L^p(w)} \leq \|R_j^m H^{\frac{-k+m}{2}} g\|_{L^p(w)} \leq \|g\|_{L^p(w)},$$

then

$$\|f\|_{W^{k,p}(w)} \leq C\|g\|_{L^p(w)} = C\|f\|_{\mathfrak{L}_k^p(w)}.$$

Now we shall prove the converse inequality. By using Lemma **??**, the following identities can be proved for each integer $k \geq 1$.

$$\begin{aligned} T_k &= \sum_{j=1}^d R_{-j}^k \, R_j^k = \sum_{j=1}^d (A_j^*)^k H^{-k/2} \, (A_j)^k H^{-k/2} \\ &= (H-2k)^{-k/2} \Big(\sum_{j=1}^d (A_j^*)^k \, (A_j)^k \Big) H^{-k/2} \\ &= (H-2k)^{-k/2} \Big\{ \sum_{j=1}^d \Big((H_j-1)(H_j-1-2)\ldots(H_j-1-2(k-1)) \Big) \Big\} H^{-k/2}, \end{aligned}$$

where $H_j = -\frac{\partial^2}{\partial x_j^2} + x_j^2$. Observe that $H = \sum_j H_j$. Consider the function

$$\mathbf{m}_k(\alpha) = \frac{(2|\alpha|+d-2k)^{k/2}(2|\alpha|+d)^{k/2}}{\sum_{j=1}^d (2\alpha_j)(2\alpha_j-2)\ldots(2\alpha_j-2(k-1))} \chi_{[dk,\infty)}(|\alpha|)$$

An appropriate smooth extension of $\mathbf{m}_k$ can be considered in order to apply Proposition **??**. Hence the operator $\mathcal{S}_{\mathbf{m}_k}$ defined as $\mathcal{S}_{\mathbf{m}_k} f = \sum_\alpha \mathbf{m}_k(\alpha)\langle f, h_\alpha\rangle h_\alpha$ is bounded in $L^p(w)$.

Denote by $\mathfrak{F}_k$ the finite dimensional space of linear combinations of Hermite functions h_α with $|\alpha| < k$. Given a function g in $\mathfrak{F}\backslash\mathfrak{F}_k$. We observe that $\mathcal{S}_{\mathbf{m}_k} \circ T_k \, g = g$ and therefore we have

$$\begin{aligned} \|g\|_{L^p(w)} &= \|\mathcal{S}_{\mathbf{m}_k} T_k g\|_{L^p(w)} \leq \; C_k \|T_k g\|_{L^p(w)} = C_k \sum_{j=1}^d \big\| R_{-j}^k \, R_j^k \, g \big\|_{L^p(w)} \\ &\leq C_k \sum_{j=1}^d \big\| R_j^k \, g \big\|_{L^p(w)} = C_k \sum_{j=1}^d \Big\| (A_j)^k H^{-k/2} g \Big\|_{L^p(w)} \end{aligned}$$

for some constant C_k independent of g. Therefore for $f \in \mathfrak{F} \setminus \mathfrak{F}_k$ with $f = H^{-k/2} g$, we have

$$\|f\|_{\mathfrak{L}_k^p(w)} = \|g\|_{L^p(w)} \leq C_k \|f\|_{W^{k,p}(w)}.$$

For the general case $g \in \mathfrak{F}$, we write $g = g_1 + g_2$ with $g_1 \in \mathfrak{F}_k$ and $g_2 \in \mathfrak{F} \setminus \mathfrak{F}_k$. We observe (one can use Lemma **??**) that $H^{k/2}$ is a bounded linear operator on the finite dimensional space $\mathfrak{F}_k$ (with the $L^p(w)$-norm). The same lemma also ensures that the projection $g \to g_1$ is bounded in $L^p(w)$, hence

$$\begin{aligned} \|g\|_{L^p(w)} &\leq \|H^{k/2} H^{-k/2} g_1\|_{L^p(w)} + \|g_2\|_{L^p(w)} \\ &\leq C_k \|H^{-k/2} g_1\|_{L^p(w)} + C_k \sum_{j=1}^d \Big\| (A_j)^k H^{-k/2} g_2 \Big\|_{L^p(w)} \\ &\leq C \|H^{-k/2} g\|_{L^p(w)} + C_k \sum_{j=1}^d \Big\| (A_j)^k H^{-k/2} g \Big\|_{L^p(w)}, \end{aligned}$$

where in the last inequality we have used $(A_j)^k H^{k/2} g_1 = 0$ and also the fact that the projection of the function $H^{-k/2} g$ from $\mathfrak{F}$ into $\mathfrak{F}_k$ is $H^{-k/2} g_1$. □

3. Laguerre setting

Let $L_n^\alpha, n = 0, 1, \ldots$ be the Laguerre polynomials of type $\alpha, \alpha > -1$. Consider the family of Laguerre functions $\mathcal{L}_n^\alpha$ defined as

$$\mathcal{L}_n^\alpha(y) = \left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{1/2} e^{-y/2} y^{\alpha/2} L_n^\alpha(y), \quad y \in \mathbb{R}^+, \ n \in \mathbb{N}_0. \tag{9}$$

For each $\alpha > -1$, $\{\mathcal{L}_n^\alpha\}_{n=0}^\infty$ is an orthonormal system in $L^2((0,\infty))$ and satisfy

$$L_\alpha \mathcal{L}_n^\alpha = \left(n + \frac{\alpha+1}{2}\right) \mathcal{L}_n^\alpha, \ n \in \mathbb{N}_0,$$

where L_α is defined in (**??**). It is known (probably it can be said that belongs to the folklore, see for example [**?**] Theorem 5.7.1) that if $\alpha > -1$, $1 < p < \infty$ and $\delta \in \mathbb{R}$ satisfy (C_α), see (**??**), then, the set S_α of finite linear combinations of Laguerre functions is dense in $L^p((0,\infty), y^\delta dy)$. This condition (C_α) will be crucial along this note.

Remark 3. Observe that if a pair (δ, p) satisfies condition (C_α) then it satisfies condition (C_β) for every $\beta > \alpha$.

Given $\alpha > -1$ and $s > 0$, we can define the operator $(L_\alpha)^{-s}$ analogously as in (**??**) just by making the substitution of $\{e^{-tH}\}_{t>0}$ by $\{e^{-tL_\alpha}\}_{t>0}$. We need the following two results that can be found in [**?**] and that we state as a unified theorem for further reference.

Theorem 6. *Let $\alpha > -1$, $1 < p < \infty$ and $\delta \in \mathbb{R}$ satisfying condition (C_α). Consider the function $\mu \in C^\infty([0,\infty))$ such that*

$$\left|\mu^{(k)}(t)\right| \leq C_k (1+t)^{-k}, \ k = 0, 1, 2, \ldots \tag{10}$$

for all $t > 0$ and $k \in \mathbb{N}_0$. Then, the operator

$$T_\mu f = \sum_{n=0}^\infty \mu(n) \langle f, \mathcal{L}_n^\alpha \rangle \mathcal{L}_n^\alpha,$$

defined at least for $f \in L^2(\mathbb{R})$, admits a bounded extension to $L^p(\mathbb{R}^+, y^\delta dy)$.

A consequence of this result is the following theorem.

Theorem 7. *Let $\alpha > -1$, $1 < p < \infty$ and $\delta \in \mathbb{R}$ satisfying relation (C_α). The operator $(L_\alpha)^{-s}$, $s > 0$, is bounded from $L^p((0,\infty), y^\delta)$ into itself.*

Proof. The multiplier $\mu(n) = \left(n + \dfrac{\alpha+1}{2}\right)^{-s}$ satisfies (**??**). □

Now we can see that the spaces in Definition **??** are well defined, we proceed as in the Hermite context. It is not difficult to prove that $(L_\alpha)^{-s/2}$ is one to one in $L^p(\mathbb{R}^+, y^\delta dy)$, using the fact that S_α is contained and dense in $L^{p'}(\mathbb{R}^+, y^{-\frac{p'}{p}\delta} dy)$, whenever δ satisfies (C_α). Moreover, since $S_\alpha = (L_\alpha)^{-s/2}(S_\alpha)$ and S_α are dense in $L^p(\mathbb{R}^+, y^\delta dy)$, then S_α is dense in $\mathfrak{W}^p_{\alpha,s}(y^\delta)$.

The operator L_α can be written as

$$L_\alpha = (\delta^\alpha)^* \delta^\alpha + \frac{(\alpha+1)}{2},$$

where

$$\delta^\alpha = \sqrt{x}\frac{d}{dx} + \frac{1}{2}\left(\sqrt{x} - \frac{\alpha}{\sqrt{x}}\right) \quad \text{and} \quad (\delta^\alpha)^* = -\sqrt{x}\frac{d}{dx} + \frac{1}{2}\left(\sqrt{x} - \frac{\alpha+1}{\sqrt{x}}\right). \tag{11}$$

The action of these operators on Laguerre functions is given by

$$\delta^\alpha(\mathcal{L}_0^\alpha) = 0, \qquad \delta^\alpha(\mathcal{L}_n^\alpha) = -\sqrt{n}\,\mathcal{L}_{n-1}^{\alpha+1}, \qquad \text{for } n \geq 1, \text{ and} \tag{12}$$

$$(\delta^\alpha)^*(\mathcal{L}_n^{\alpha+1}) = -\sqrt{n+1}\,\mathcal{L}_{n+1}^\alpha \qquad \text{for } n \geq 0. \tag{13}$$

The Riesz transforms were defined in [**?**], for $\alpha > -1$ by

$$R_\alpha = \delta^\alpha (L_\alpha)^{-1/2} \qquad \text{and} \qquad \tilde{R}_\alpha = (\delta^\alpha)^*(L_{\alpha+1})^{-1/2}.$$

In [**?**] it was proved that those operators are bounded on $L^p(\mathbb{R}^+, y^\delta dy)$ for δ satisfying (C_α). Given a positive integer k and $\alpha > -1$ we define the higher order Riesz transform of order k as

$$R_\alpha^k = \left(\delta^{\alpha+k-1} \circ \cdots \circ \delta^{\alpha+1} \circ \delta^\alpha\right)(L_\alpha)^{-k/2}$$

and

$$\tilde{R}_\alpha^k = \left((\delta^\alpha)^* \circ (\delta^{\alpha+1})^* \circ \cdots \circ (\delta^{\alpha+k-1})^*\right)(L_{\alpha+k})^{-k/2}.$$

Observe that $R_\alpha^1 = R_\alpha$ and $\tilde{R}_\alpha^1 = \tilde{R}_\alpha$.

Theorem 8. *Let $k \in \mathbb{N}$, $1 < p < \infty$, $\alpha > -1$ and δ satisfying (C_α). The operators R_α^k and $\tilde{R}_\alpha^k$ are bounded on $L^p(\mathbb{R}^+, y^\delta dy)$.*

In order to prove this theorem we shall need the following lemma, whose proof is left to the reader.

Lemma 5. *Let Φ be a continuous function and $\alpha > -1$. For every f in S_α, we have*

(1) $\delta^\alpha \Phi(L_\alpha) f = \Phi\Big(L_{\alpha+1} + \frac{1}{2} I_d\Big)\delta^\alpha f$.

(2) $(\delta^\alpha)^* \Phi(L_{\alpha+1}) f = \Phi\Big(L_\alpha - \frac{1}{2} I_d\Big)(\delta^\alpha)^* f$.

Now we can give the proof of Theorem **??** by using an induction argument on k.

Proof of Theorem **??**. As we mention above, the result is true for $k = 1$, see [**?**]. Let $k > 1$, for a function f in S_α, we have

$$\begin{aligned}
R_\alpha^k &= \left(\delta^{\alpha+k-1} \circ \cdots \circ \delta^{\alpha+1} \circ \delta^\alpha\right)(L_\alpha)^{-k/2} \\
&= \left(\delta^{\alpha+k-1} \circ \cdots \circ \delta^{\alpha+1}\right) \circ \delta^\alpha \circ (L_\alpha)^{-(k-1)/2}(L_\alpha)^{-1/2} \\
&= \left(\delta^{\alpha+k-1} \circ \cdots \circ \delta^{\alpha+1}\right) \circ \left(L_{\alpha+1} + \frac{1}{2} I_d\right)^{-(k-1)/2} \circ \delta^\alpha \circ (L_\alpha)^{-k/2} \\
&= \left(\delta^{\alpha+k-1} \circ \cdots \circ \delta^{\alpha+1}\right) \circ \left(L_{\alpha+1}\right)^{-(k-1)/2} \circ T_\mu \circ \delta^\alpha \circ (L_\alpha)^{-k/2} \\
&= R_{\alpha+1}^{k-1} \circ T_\mu \circ R_\alpha.
\end{aligned}$$

Where T_μ is the operator given by the multiplier (in the system $\{\mathcal{L}_k^{\alpha+1}\}_{k=0}^\infty$) $\mu(n) = \left[\dfrac{n + \frac{\alpha+1}{2}}{n + \frac{\alpha+2}{2} + \frac{1}{2}}\right]^{(k-1)/2}$. The function μ satisfies (**??**). Hence, by using Theorem **??**, we get that T_μ is bounded from $L^p((0,\infty), y^\delta dy)$ into $L^p((0,\infty), y^\delta dy)$ for δ

satisfying ($\mathrm{C}_{\alpha+1}$). On the other hand the induction hypothesis says that $R^{k-1}_{\alpha+1}$ is bounded from $L^p((0,\infty), y^\delta dy)$ into $L^p((0,\infty), y^\delta dy)$ for δ satisfying $(C_{\alpha+1})$. As we noticed, this range is bigger than the range (C_α), see Remark **??**.

In order to prove the boundedness of $\tilde{R}^k_\alpha$, we use again Lemma **??**. We write $\tilde{R}^k_\alpha = \tilde{R}^{k-1}_\alpha \circ T_\nu \circ \tilde{R}_{\alpha+k-1}$, where T_ν is the operator given by the multiplier $\nu(n) = \Big(n + \dfrac{\alpha+k}{2}\Big) \Big/ \Big(n + \dfrac{\alpha+k-1}{2}\Big)$. The proof continues along the same lines as for R^k_α, by using in this case the boundedness of the operators $\tilde{R}^k_\alpha$ and T_ν. □

Parallel to the Hermite setting we have the following structural theorem for the spaces $\mathfrak{W}^p_{\alpha,s}(y^\delta)$.

Theorem 9. *Let $\alpha > -1$, $1 < p < \infty$, $s > 0$, and δ satisfying (C_α).*

i) If $t > s$, then $\mathfrak{W}^p_{\alpha,s}(y^\delta) \subset \mathfrak{W}^p_{\alpha,t}(y^\delta) \subset L^p(y^\delta)$ with continuous inclusions. Moreover, $\mathfrak{W}^p_{\alpha,s}(y^\delta)$ and $\mathfrak{W}^p_{\alpha,t}(y^\delta)$ are isometrically isomorphic .

ii) If $t > 0$, then $(L_\alpha)^{-t/2}$ maps $\mathfrak{W}^p_{\alpha,s}(y^\delta)$ into $\mathfrak{W}^p_{\alpha,s+t}(y^\delta)$.

iii) If $s > 1$, then δ^α is bounded from $\mathfrak{W}^p_{\alpha,s}(y^\delta)$ into $\mathfrak{W}^p_{\alpha+1,s-1}(y^\delta)$.

iv) The operators R^k_α, are bounded from $\mathfrak{W}^p_{\alpha,s}(y^\delta)$ into $\mathfrak{W}^p_{\alpha+k,s}(y^\delta)$.

Proof. The statements *i)* and *ii)* follow from the boundedness of $(L_\alpha)^{-s/2}$ established in Theorem **??**. On the other hand, given a function $f \in \mathfrak{W}^p_{\alpha,s}(y^\delta)$, there exists a function $g \in L^p(y^\delta)$ such that $L_\alpha^{-s/2} g = f$. Consider $h = \left(\dfrac{L_{\alpha+1} + \frac{1}{2} I_d}{L_{\alpha+1}}\right)^{-\frac{s-1}{2}} R_\alpha g$, then

$$\begin{aligned} \delta^\alpha f &= \delta^\alpha L_\alpha^{-s/2} g = (L_{\alpha+1} + \frac{1}{2} I_d)^{-(s-1)/2} \delta^\alpha L_\alpha^{-1/2} g \\ &= (L_{\alpha+1} + \frac{1}{2} I_d)^{-(s-1)/2} R_\alpha g \\ &= (L_{\alpha+1})^{-\frac{s-1}{2}} h. \end{aligned}$$

By Theorem **??** and Theorem **??**, we have

$$\|\delta^\alpha f\|_{\mathfrak{W}^p_{\alpha+1,s-1}(y^\delta)} = \|h\|_{L^p(y^\delta)} \leq C \|g\|_{L^p(y^\delta)} = \|\delta^\alpha f\|_{\mathfrak{W}^p_{\alpha,s}(y^\delta)}.$$

In order to prove *iv*) we use *ii*) and *iii*). □

Given a function f, consider the Cesàro sums of g of order $r > 0$, that is

$$C^r_{N,\alpha}(g) = \frac{1}{\mathfrak{a}^r_N} \sum_{n=0}^N \mathfrak{a}^r_{N-n} \langle f, \mathcal{L}^\alpha_n \rangle \mathcal{L}^\alpha_n,$$

for $N \in \mathbb{N}$, with $\mathfrak{a}^r_n = \dfrac{\prod_{j=1}^n (j+r)}{n!}$, $0 \leq n \leq N$. The following proposition is an easy consequence of Theorem 1.13 in [**?**] and it shall be the key to prove a density result in $W^{k,p}_{\alpha,\delta}$, see Definition **??**.

Proposition 3. *Let $\alpha > -1$, $1 < p < \infty$ and δ satisfying (C_α), there exists $r \geq 1$ (possibly depending on α) such that the Cesàro sums of order r of a function f converge to f in the $L^p(y^\delta dy)$-norm as N goes to infinity.*

Proposition 4. *Let $\alpha > -1$, $1 < p < \infty$, $k \in \mathbb{N}$ and δ satisfying (C_α). Then S_α is a dense subspace of $W^{k,p}_{\alpha,\delta}$.*

Proof. By using (**??**) and (**??**), we have

$$
\begin{aligned}
\delta^\alpha C^r_{N,\alpha}(f) &= \frac{1}{\mathfrak{a}^r_N}\sum_{n=1}^{N}\mathfrak{a}^r_{N-n}\langle f,\mathcal{L}^\alpha_n\rangle\,(-\sqrt{n})\,\mathcal{L}^{\alpha+1}_{n-1}\\
&= \frac{1}{\mathfrak{a}^r_N}\sum_{n=1}^{N}\mathfrak{a}^r_{N-n}\langle f,(\delta^\alpha)^*\mathcal{L}^{\alpha+1}_{n-1}\rangle\,\mathcal{L}^{\alpha+1}_{n-1}\\
&= \frac{\mathfrak{a}^r_{N-1}}{\mathfrak{a}^r_N}\frac{1}{\mathfrak{a}^r_{N-1}}\sum_{n=0}^{N-1}\mathfrak{a}^r_{(N-1)-n}\langle \delta^\alpha f,\mathcal{L}^{\alpha+1}_n\rangle\,\mathcal{L}^{\alpha+1}_n\\
&= \frac{\mathfrak{a}^r_{N-1}}{\mathfrak{a}^r_N}C^r_{N-1,\alpha+1}(\delta^\alpha f),
\end{aligned}
$$

and inductively, if $m\in\mathbb{N}_0$ and $N>m$,

$$
\left(\delta^{\alpha+m-1}\circ\ldots\circ\delta^{\alpha+1}\circ\delta^\alpha\right)C^r_N(f)=\frac{\mathfrak{a}^r_{N-m}}{\mathfrak{a}^r_N}\,C^r_{N-m,\alpha+m}\left(\delta^{\alpha+m-1}\circ\ldots\circ\delta^{\alpha+1}\circ\delta^\alpha f\right). \tag{14}
$$

We choose r big enough as in Proposition **??**, therefore the sequence in S_α given by $f_N=C^r_{N,\alpha}f$, converges to f in $L^p(y^\delta dy)$-norm. Observe that the functions $\delta^{\alpha+m-1}\circ\ldots\circ\delta^{\alpha+1}\circ\delta^\alpha f$, where $1\le m\le k-1$, also belong to $L^p(\mathbb{R}^+,y^\delta dy)$, then equation (**??**), Proposition **??** and the fact that $\lim_{N\mapsto\infty}\frac{\mathfrak{a}^r_{N-m}}{\mathfrak{a}^r_N}=1$, imply that the sequence $\delta^{\alpha+m-1}\circ\ldots\circ\delta^{\alpha+1}\circ\delta^\alpha\circ C^r_N(f)$ converges to $\delta^{\alpha+m-1}\circ\ldots\circ\delta^{\alpha+1}\circ\delta^\alpha f$ in the $L^p(y^\delta dy)$-norm, for $1\le m\le k-1$. □

Now we give the proof of Theorem **??**.

Proof of Theorem **??**. As S_α is a dense subspace of $\mathfrak{W}^{k,p}_{\alpha,\delta}$ and $W^{k,p}_{\alpha,\delta}$, it is enough to show the equivalence of the norms for functions $f\in S_\alpha$. Let g such that $(L_\alpha)^{-k/2}g=f$. For $0\le m\le k-1$, we have

$$
\begin{aligned}
\|f\|_{W^{k,p}_{\alpha,\delta}} &= \|f\|_{p,\delta}+\sum_{m=1}^{k-1}\left\|\delta^{\alpha+m}\circ\ldots\circ\delta^{\alpha+1}\circ\delta^\alpha f\right\|_{p,\delta}\\
&= \|(L_\alpha)^{-k/2}g\|_{p,\delta}+\sum_{m=1}^{k-1}\left\|\delta^{\alpha+m}\circ\ldots\circ\delta^{\alpha+1}\circ\delta^\alpha(L_\alpha)^{-k/2}g\right\|_{p,\delta}\\
&= \|(L_\alpha)^{-k/2}g\|_{p,\delta}+\sum_{m=1}^{k-1}\left\|R^m_\alpha(L_\alpha)^{-(k-m)/2}g\right\|_{p,\delta}\\
&\le C\|g\|_{p,\delta}=C\|f\|_{\mathfrak{W}^{k,p}_{\alpha,\delta}}.
\end{aligned}
$$

Where in the last inequality we have used Theorem **??** and Theorem **??**.

For the converse inequality it is clearly enough to prove that for all functions $f\in S_\alpha$, there exists a constant C such that

$$
\|(L_\alpha)^{k/2}f\|_{p,\delta}\le C\left(\|f\|_{p,\delta}+\left\|\delta^{\alpha+k-1}\circ\ldots\circ\delta^{\alpha+1}\circ\delta^\alpha f\right\|_{p,\delta}\right). \tag{15}
$$

Let $\alpha>-1$ and $k\in\mathbb{N}$. We call Π^k_α the set of linear combinations of Laguerre functions of type α up to order k. If $f\in S_\alpha$, we split $f=f_1+f_2$, with $f_1\in\Pi^k_\alpha$

and $f_2 \in S_\alpha \setminus \Pi_\alpha^k$. Since $(L_\alpha)^k$ is a linear operator on a finite dimensional space Π_α^k, there exists a constant C that depends on k such that

$$\|(L_\alpha)^{k/2} f_1\|_{p,\delta} \le C\|f_1\|_{p,\delta}.$$

On the other hand, since $(L_\alpha)^{-k/2}$ is bounded on $L^p(\mathbb{R}^+, y^\delta dy)$ (Theorem **??**), we have

$$\|f_1\|_{p,\delta} = \|f - f_2\|_{p,\delta} \le \|f\|_{p,\delta} + \|(L_\alpha)^{-k/2}(L_\alpha)^{k/2} f_2\|_{p,\delta} \le \|f\|_{p,\delta} + C\|(L_\alpha)^{k/2} f_2\|_{p,\delta},$$

thus

$$\|(L_\alpha)^{k/2} f\|_{p,\delta} \le \|(L_\alpha)^{k/2} f_1\|_{p,\delta} + \|(L_\alpha)^{k/2} f_2\|_{p,\delta} \le C\left(\|f\|_{p,\delta} + \|(L_\alpha)^{k/2} f_2\|_{p,\delta}\right).$$

Therefore, it is enough to prove (**??**) for f_2. By using Lemma **??** we can easily show the following identity for each integer k

$$\begin{aligned} T_k &= \tilde{R}_\alpha^k \circ R_\alpha^k \\ &= (L_\alpha - \frac{k}{2})^{-k/2} \circ (L_\alpha - \frac{\alpha+1}{2} - k - 1) \circ (L_\alpha - \frac{\alpha+1}{2} - k - 2) \dots \\ &\qquad \dots \circ (L_\alpha - \frac{\alpha+1}{2} - 1) \circ (L_\alpha - \frac{\alpha+1}{2})(L_\alpha)^{-k/2} \end{aligned}$$

and consider the function

$$\mu_k(t) = \frac{\left(t + \frac{\alpha+1}{2}\right)^{k/2}\left(t + \frac{\alpha+1}{2} - \frac{k}{2}\right)^{k/2}}{\prod_{j=0}^{k-1}(t-j)} \chi_{[k,\infty)}(t).$$

which satisfies (**??**). Then, the proof follows the same lines as in the Hermite case in Theorem **??** using the multiplier Theorem **??** and Theorem **??** in order to control the operator $\tilde{R}_\alpha^k$. □

4. Alternative definitions of Riesz transforms. Consequences for Sobolev spaces

In this section we analyse the role of the "natural" Riesz transforms

$$(\delta^\alpha)^k (L_\alpha)^{-k/2},$$

relating to Sobolev spaces. Some commutation properties of the operators δ^α with the operator of multiplication by $x^{\ell/2}$ will be essential. We shall write $\delta^\alpha \frac{1}{x^{\ell/2}}$ and $x^{\ell/2}\delta^\alpha$ as a shorthand to denote the action $\delta^\alpha\left(\frac{1}{(\cdot)^{\ell/2}} f(\cdot)\right)(x)$ and $x^{\ell/2}\delta^\alpha(f)(x)$. We state the following lemma whose proof (by using (**??**)) is left to the reader.

Lemma 6. *Let $\beta, \alpha > -1$, and $\ell \in \mathbb{N}$.*

(i) $\delta^\beta = \delta^\alpha + \dfrac{\alpha - \beta}{2\sqrt{x}}$.

(ii) $\delta^\beta \dfrac{1}{x^{\ell/2}} = \dfrac{1}{x^{\ell/2}} \delta^{\beta+\ell}$.

(iii) *If $\beta > \ell - 1$, then* $\dfrac{1}{x^{\ell/2}}\delta^{\beta+\ell} = \dfrac{1}{x^{\ell/2}}\delta^{\beta-\ell} - \dfrac{\ell}{x^{(\ell+1)/2}}$.

(iv) *If $\beta > \ell - 1$, then* $\delta^\beta x^{\ell/2} = x^{\ell/2}\delta^{\beta-\ell}$.

(v) $(\delta^\beta)^* = -\delta^\alpha + \dfrac{1}{2}\Big(\sqrt{x} - \dfrac{\alpha}{\sqrt{x}}\Big) + \dfrac{1}{2}\Big(\sqrt{x} - \dfrac{\beta+1}{\sqrt{x}}\Big)$.

Lemma 7. *Let $\alpha > -1$, and $k \in \mathbb{N}$, then*

$$(\delta^\alpha)^k = \sum_{0\le p\le m+1, p+m=k-1} \frac{c_m}{x^{p/2}} \delta^{\alpha+m} \circ \cdots \circ \delta^\alpha.$$

Proof. Let $p < m+1$, by using Lemma **??** we have

$$\begin{aligned}\delta^\alpha\Big(\frac{1}{x^{p/2}}\delta^{\alpha+m}\circ\cdots\circ\delta^\alpha\Big) &= \frac{1}{x^{p/2}}\delta^{\alpha+p}\circ\delta^{\alpha+m}\circ\cdots\circ\delta^\alpha \\ &= \frac{1}{x^{p/2}}\Big(\delta^{\alpha+m+1} + \frac{m+1-p}{2\sqrt{x}}\Big)\circ\delta^{\alpha+m}\circ\cdots\circ\delta^\alpha \\ &= \frac{1}{x^{p/2}}\delta^{\alpha+m+1}\circ\delta^{\alpha+m}\circ\cdots\circ\delta^\alpha \\ &\quad + \frac{1}{x^{p/2}}\frac{m+1-p}{2\sqrt{x}}\delta^{\alpha+m}\circ\cdots\circ\delta^\alpha.\end{aligned}$$

If $p = m+1$ we have

$$\delta^\alpha\Big(\frac{1}{x^{p/2}}\delta^{\alpha+m}\circ\cdots\circ\delta^\alpha\Big) = \frac{1}{x^{p/2}}\delta^{\alpha+m+1}\circ\delta^{\alpha+m}\circ\cdots\circ\delta^\alpha.$$

Then

$$\begin{aligned}(\delta^\alpha)^{k+1} &= \delta^\alpha\Big(\sum_{0\le p\le m+1,\, p+m=k-1}\frac{c_m}{x^{p/2}}\delta^{\alpha+m}\circ\cdots\circ\delta^\alpha\Big) \\ &= \sum_{0\le p< m+1,\, p+m=k-1}\frac{c_m}{x^{p/2}}\delta^{\alpha+m+1}\circ\delta^{\alpha+m}\circ\cdots\circ\delta^\alpha \\ &\quad + \sum_{0\le p< m+1,\, p+m=k-1}\frac{c_m}{x^{p/2}}\frac{m+1-p}{2\sqrt{x}}\delta^{\alpha+m}\circ\cdots\circ\delta^\alpha \\ &\quad + \sum_{0\le p= m+1,\, p+m=k-1}\frac{1}{x^{p/2}}\delta^{\alpha+m+1}\circ\delta^{\alpha+m}\circ\cdots\circ\delta^\alpha \\ &= \sum_{0\le p< m,\, p+m=k}\frac{c_m}{x^{p/2}}\delta^{\alpha+m}\circ\cdots\circ\delta^\alpha \\ &\quad + \sum_{0\le q< m+1,\, q+m=k}\frac{c_m}{x^{q/2}}\delta^{\alpha+m}\circ\cdots\circ\delta^\alpha \\ &\quad + \sum_{0\le p= m,\, p+m=k}\frac{1}{x^{p/2}}\delta^{\alpha+m}\circ\cdots\circ\delta^\alpha \\ &= \sum_{0\le p\le m+1, p+m=k}\frac{c_m}{x^{p/2}}\delta^{\alpha+m}\circ\cdots\circ\delta^\alpha.\end{aligned}$$

The standard induction argument gives now the proof. □

Lemma 8. *Let $P_m(u,v)$ be a polynomial of degree m and variables u, v, i.e.*

$$P_m(u,v) = a_0u^m + a_1u^{m-1}v + \cdots + a_mv^m.$$

Assume that $\beta > m-1$, then

$$\delta^\beta\ P_m\Big(\sqrt{x}, \frac{1}{\sqrt{x}}\Big) = P_m^1\Big(\sqrt{x}, \frac{1}{\sqrt{x}}\Big)\ \delta^{\beta-m} + P_{m+1}^2\Big(\sqrt{x}, \frac{1}{\sqrt{x}}\Big)$$

where P_m^1 and P_{m+1}^2 are polynomials of degrees m and $m+1$.

Proof. Observe that

$$P_m(\sqrt{x}, \frac{1}{\sqrt{x}}) = a_0\, x^{m/2} + a_1\, x^{(m-2)/2} + \cdots + a_{m-1}\, x^{-(m-2)/2} + a_m\, x^{-m/2}.$$

Let $0 < \ell \leq m$, then by using Lemma **??** we have

$$\delta^{\beta}(x^{\ell/2}) \quad = \quad x^{\ell/2}\delta^{\beta-\ell} = x^{\ell/2}\Big(\delta^{\beta-m} + \frac{\ell-m}{2\sqrt{x}}\Big) = x^{\ell/2}\delta^{\beta-m} + \frac{\ell-m}{2}x^{(\ell-1)/2}.$$

Let $\ell = -q < 0$, again by Lemma **??** we have

$$\begin{aligned} \delta^{\beta}(x^{\ell/2}) &= \delta^{\beta}\frac{1}{x^{q/2}} = \frac{1}{x^{q/2}}\delta^{\beta-q} - \frac{q}{x^{(q+1)/2}} \\ &= \frac{1}{x^{q/2}}\delta^{\beta-m} + \frac{1}{x^{q/2}}\Big(\frac{q-m}{x^{1/2}}\Big) - \frac{q}{x^{(q+1)/2}} \\ &= \frac{1}{x^{q/2}}\delta^{\beta-m} - \frac{m}{x^{(q+1)/2}} \\ &= x^{\ell/2}\delta^{\beta-m} - m\, x^{(\ell-1)/2} \end{aligned}$$

□

Lemma 9. *Let ℓ, m natural numbers such that $0 < \ell \leq m$. Given $\alpha > -1$ and (δ, p) satisfying (C_α), then the operators $\frac{1}{x^{\ell/2}}\Big(L_{\alpha+m}\Big)^{-\ell/2}$ and $x^{\ell/2}(L_{\alpha+m})^{-\ell/2}$ are bounded on $L^p(\mathbb{R}^+, y^\delta dy)$.*

Proof. Case $1 = \ell = m$. We already mention that the operator

$$(\delta^{\alpha})^*(L_{\alpha+1})^{-1/2} = \Big\{ -\sqrt{x}\frac{d}{dx} + \frac{1}{2}\Big(\sqrt{x} - \frac{\alpha+1}{\sqrt{x}}\Big)\Big\}(L_{\alpha+1})^{-1/2},$$

is bounded in $L^p(\mathbb{R}^+, y^\delta dy)$, for p, δ satisfying (C_α), see (**??**). Also the operator

$$(\delta^{\alpha+1})(L_{\alpha+1})^{-1/2} = \Big\{\sqrt{x}\frac{d}{dx} + \frac{1}{2}\Big(\sqrt{x} - \frac{\alpha+1}{\sqrt{x}}\Big)\Big\}(L_{\alpha+1})^{-1/2}$$

is bounded in $L^p(\mathbb{R}^+, y^\delta dy)$, for p, δ satisfying ($\mathrm{C}_{\alpha+1}$). Hence both operators are bounded in $L^p(\mathbb{R}^+, y^\delta dy)$, for p, δ satisfying (C_α). Consequently the operator $\Big(\sqrt{x} - \frac{\alpha+1}{\sqrt{x}}\Big)(L_{\alpha+1})^{-1/2}$ is also bounded.

If $2(\alpha+1) < x$ then $0 < \sqrt{x} \leq 2\Big(\sqrt{x} - \frac{\alpha+1}{\sqrt{x}}\Big)$. We already know that $(L_\beta)^{-1/2}$ has positive kernel, hence for positive functions f we have

$$\begin{aligned} \sqrt{x}(L_{\alpha+1})^{-1/2}f(x) &\leq \sqrt{x}(L_{\alpha+1})^{-1/2}(f)(x)\chi_{[0,2(\alpha+1)]}(x) \\ &\quad + \Big(\sqrt{x} - \frac{\alpha+1}{\sqrt{x}}\Big)(L_{\alpha+1})^{-1/2}(f)(x)\chi_{[2(\alpha+1),\infty)}(x) \\ &\leq \sqrt{2(\alpha+1)}(L_{\alpha+1})^{-1/2}(f)(x)\chi_{[0,2(\alpha+1)]}(x) \\ &\quad + \Big(\sqrt{x} - \frac{\alpha+1}{\sqrt{x}}\Big)(L_{\alpha+1})^{-1/2}(f)(x)\chi_{[2(\alpha+1),\infty)}(x) \end{aligned}$$

The case $\ell = 1$ and $\ell < m$, can be proved as the previous one by using $(\delta^{\alpha+m-1})^*(L_{\alpha+m})^{-1/2}$ and $\delta^{\alpha+m}(L_{\alpha+m})^{-1/2}$. Then we would obtain boundedness in $L^p(\mathbb{R}^+, y^\delta dy)$, for δ, p satisfying ($\mathrm{C}_{\alpha+m-1}$). By Remark **??** we obtain boundedness for δ, p satisfying (C_α).

In order to prove the case $1 < \ell \le m$ we shall apply an induction argument. The operators

$$R^{\ell}_{\alpha+m} = \Big(\delta^{\alpha+m+\ell-1} \circ \cdots \circ \delta^{\alpha+m+1} \circ \delta^{\alpha+m}\Big)(L_{\alpha+m})^{\ell/2},$$

are bounded in $L^p(\mathbb{R}^+, y^\delta dy)$, for δ, p satisfying $(\mathrm{C}_{\alpha+m})$. On the other hand the operators

$$\tilde{R}^{\ell}_{\alpha+m-\ell} = \Big((\delta^{\alpha+m-\ell})^* \circ (\delta^{\alpha+m-\ell+1})^* \circ \cdots \circ (\delta^{\alpha+m-1})^*\Big)(L_{\alpha+m})^{-\ell/2}$$

are bounded in $L^p(\mathbb{R}^+, y^\delta dy)$, for δ, p satisfying $(\mathrm{C}_{\alpha+m-\ell})$, see Theorem **??**. In particular, both operators are bounded on $L^p(\mathbb{R}^+, y^\delta dy)$, for δ, p satisfying (C_α).

We observe that due to Lemma **??** we have for $j = 0, \ldots, \ell$

$$\begin{aligned}
(\delta^{\alpha+m-j})^* &= -\delta^{\alpha+m+(j-1)} \\
&\quad + \frac{1}{2}\Big(\sqrt{x} - \frac{\alpha+m+(j-1)}{\sqrt{x}}\Big) + \frac{1}{2}\Big(\sqrt{x} - \frac{(\alpha+m-j)+1}{\sqrt{x}}\Big) \\
&= -\delta^{\alpha+m+(j-1)} + \Big(\sqrt{x} - \frac{\alpha+m}{\sqrt{x}}\Big).
\end{aligned}$$

Therefore

$$\begin{aligned}
&(\delta^{\alpha+m-\ell})^* \circ (\delta^{\alpha+m-\ell+1})^* \circ \cdots \circ (\delta^{\alpha+m-1})^* \\
&= \Big(-\delta^{\alpha+m+\ell-1} + \Big(\sqrt{x} - \frac{\alpha+m}{\sqrt{x}}\Big)\Big) \circ \Big(-\delta^{\alpha+m+\ell-2} + \Big(\sqrt{x} - \frac{\alpha+m}{\sqrt{x}}\Big)\Big) \circ \ldots \\
&\qquad \cdots \circ \Big(-\delta^{\alpha+m} + \Big(\sqrt{x} - \frac{\alpha+m}{\sqrt{x}}\Big)\Big) \\
&= \Big(-\delta^{\alpha+m+\ell-1} + P_1(\sqrt{x}, \frac{1}{\sqrt{x}})\Big) \circ \Big(-\delta^{\alpha+m+\ell-2} + P_1(\sqrt{x}, \frac{1}{\sqrt{x}})\Big) \circ \ldots \\
&\qquad \cdots \circ \Big(-\delta^{\alpha+m} + P_1(\sqrt{x}, \frac{1}{\sqrt{x}})\Big),
\end{aligned}$$

where $P_1(\sqrt{x}, \frac{1}{\sqrt{x}})$ is the polynomial of first degree $\sqrt{x} - \frac{\alpha+m}{\sqrt{x}}$. Hence, by using Lemma **??** and an induction argument we get

$$\begin{aligned}
&(\delta^{\alpha+m-\ell})^* \circ (\delta^{\alpha+m-\ell+1})^* \circ \cdots \circ (\delta^{\alpha+m-1})^* \\
&= (-1)^\ell \delta^{\alpha+m+\ell-1} \circ \delta^{\alpha+m+\ell-2} \circ \cdots \circ \delta^{\alpha+m} + P_1(\sqrt{x}, \frac{1}{\sqrt{x}})\delta^{\alpha+m+\ell-2} \circ \cdots \circ \delta^{\alpha+m} \\
&\qquad + P_2(\sqrt{x}, \frac{1}{\sqrt{x}})\delta^{\alpha+m+\ell-3} \circ \cdots \circ \delta^{\alpha+m} + \cdots + P_\ell(\sqrt{x}, \frac{1}{\sqrt{x}}),
\end{aligned}$$

where as usual $P_j(\sqrt{x}, \frac{1}{\sqrt{x}})$ denotes a polynomial of degree j. Now by using Lemma **??** we get

$$\begin{aligned}
\tilde{R}^{\ell}_{\alpha+m-\ell} &= (-1)^\ell R^{\ell}_{\alpha+m} + P_1(\sqrt{x}, \frac{1}{\sqrt{x}}) \circ (L_{\alpha+m+\ell-1})^{-1/2} \circ T_{m,\alpha+m+\ell-1} \circ R^{(\ell-1)/2}_{\alpha+m} \\
&\quad + P_{\ell-1}(\sqrt{x}, \frac{1}{\sqrt{x}}) \circ (L_{\alpha+m+1})^{-(\ell-1)/2} \circ T_{m,\alpha+m+1} \circ R^{1}_{\alpha+m} \\
&\quad + \cdots + P_\ell(\sqrt{x}, \frac{1}{\sqrt{x}}) \circ (L_{\alpha+m})^{-\ell/2}.
\end{aligned}$$

Where $T_{m,\alpha+m+\ell-1}$ are multipliers analogous to the multipliers appearing in Theorem **??**. By using Theorem **??**, Theorem **??** and induction hypotheses on ℓ, the operators $\tilde{R}^{\ell}_{\alpha+m-\ell}$, $R^{\ell}_{\alpha+m}$, $P_1(\sqrt{x}, \frac{1}{\sqrt{x}}) \circ (L_{\alpha+m+\ell-1})^{-1/2} \circ T_{m,\alpha+m+\ell-1} \circ R^{(\ell-1)/2}_{\alpha+m}$ and $P_{\ell-1}(\sqrt{x}, \frac{1}{\sqrt{x}}) \circ (L_{\alpha+m+1})^{-(\ell-1)/2} \circ T_{m,\alpha+m+1} \circ R^{1}_{\alpha+m}$ are bounded for (δ, p) satisfying (C_α) hence the operator $P_\ell(\sqrt{x}, \frac{1}{\sqrt{x}}) \circ (L_{\alpha+m})^{-\ell/2}$ will be bounded in the same range. Therefore by using induction hypotheses on ℓ again, we get that an operator of the type $(ax^{\ell/2} + bx^{-\ell/2}) \circ (L_{\alpha+m})^{-\ell/2}$ will be bounded. By using an argument similar to the beginning of this proof we get the lemma.

□

Theorem 10. *Let $\alpha > -1$ then the "Riesz" transforms $(\delta^\alpha)^k (L_\alpha)^{-k/2}$ are bounded in $L^p(y^\delta dy)$ for (δ, p) satisfying (C_α).*

Proof. Let $0 \le p \le m+1$ and $p + m = k - 1$, by using Lemma **??** we have

$$(\frac{1}{x^{p/2}} \delta^{\alpha+m} \circ \cdots \circ \delta^\alpha)(L_\alpha)^{-k/2} = \frac{1}{x^{p/2}} (L_{\alpha+m+1})^{-p/2} T_\mu R^{m+1}_\alpha$$

Where T_μ is a multiplier defined on the system $\{\mathcal{L}^{\alpha+m+1}_n\}_{n\ge 0}$ which satisfies the hypothesis in Theorem **??**. Now Lemmas **??** and **??**, and Theorem **??** give the result. □

In order to analyze the possible coincidence for certain α, δ and p of the spaces $\mathcal{W}^{k,p}_\alpha(y^\delta)$, see Definition **??**, with the spaces considered in Section **??** we shall need the following lemma whose statement is just a reformulation of Lemma **??**.

Lemma 10. *Let ℓ, m natural numbers such that $0 < \ell \le m+1$. Given $\alpha > 0$ and (δ, p) satisfying $(C_{\alpha-1})$, then the operators $\frac{1}{x^{\ell/2}} \Big(L_{\alpha+m} \Big)^{-\ell/2}$ are bounded on $L^p(\mathbb{R}^+, y^\delta dy)$.*

Now we present the proof of Theorem **??**.

*Proof of Theorem **??**.* By using Theorem **??** and same arguments in the beginning of the proof of Theorem **??**, it is easy to prove (i). To see (ii), consider the function f with support in $[0,1]$ such that $f(y) = y^{(\alpha+1)/2}$ for $0 < y < 1/2$, $f(1) = 0$ and f smooth in $[1/3, 1]$. It is easy to see that f, $\delta_\alpha f$ and $\delta_\alpha \circ \delta_\alpha f$ belong to $L^2(\mathbb{R}^+, dy)$. However, for $y \sim 0$, $\delta_{\alpha+1} \circ \delta_0 f \sim y^{\frac{\alpha-1}{2}}$, that is to say $\delta_{\alpha+1} \circ \delta_\alpha f$ is not in $L^2(\mathbb{R}^+, dy)$.

Finally, let f be a function in $\mathcal{W}^{2,p}_\alpha(y^\delta)$ then we have $\delta^\alpha f$, $(\delta^\alpha)^2 f \in L^p(y^\delta)$, therefore (by Theorem **??**) there exist $h \in L^p$ such that $(\delta^\alpha) f = (L_\alpha)^{-1/2} h$. Hence, by using Lemma **??**, we have

$$\delta^{\alpha+1} \circ \delta^\alpha f \quad = \quad \delta^\alpha \circ \delta^\alpha f - \frac{1}{2\sqrt{x}} \delta^\alpha f = \delta^\alpha \circ \delta^\alpha f - \frac{1}{2\sqrt{x}} (L_\alpha)^{-1/2} h.$$

Lemma **??** gives (iii). □

5. Other Laguerre systems

The Laguerre functions $\{\varphi^\alpha_k\}_{k=0}^\infty$, $\alpha > -1$. We consider the orthonormal system in $L^2((0,\infty), dy)$ given by $\varphi^\alpha_k(y) = \mathcal{L}^\alpha_k(y^2)(2y)^{1/2}$, where $\mathcal{L}^\alpha_k$ are the functions defined in (**??**). The functions φ^α_k are eigenfunctions of the operator

$$\mathbf{L}_\alpha = \frac{1}{4} \left\{ -\frac{d^2}{dy^2} + y^2 + \frac{1}{y^2} \left(\alpha^2 - \frac{1}{4} \right) \right\}.$$

In fact,

$$\mathbf{L}_\alpha(\varphi_k^\alpha) = \left(k + \frac{\alpha+1}{2}\right)\ \varphi_k^\alpha.$$

The operator $\mathbf{L}_\alpha$ can be "factorized" as:

$$\mathbf{L}_\alpha - \left(\frac{\alpha+1}{2}\right) = (\mathbf{D}_\alpha)^*\mathbf{D}_\alpha,$$

being $\mathbf{D}^\alpha = \frac{1}{2}\left\{\frac{d}{dy} + y - \frac{1}{y}(\alpha + \frac{1}{2})\right\}$ and $(\mathbf{D}^\alpha)^* = \frac{1}{2}\left\{-\frac{d}{dy} + y - \frac{1}{y}(\alpha + \frac{1}{2})\right\}$, where $(\mathbf{D}^\alpha)^*$ is the formal adjoint of $\mathbf{D}^\alpha$ with respect to the Lebesgue measure. Then

$$\mathbf{D}^\alpha(\varphi_k^\alpha) = -\sqrt{k}\varphi_{k-1}^{\alpha+1} \quad \text{and} \quad (\mathbf{D}^{\beta-\mathbf{1}})^*(\varphi_k^\beta) = -\sqrt{k+1}\varphi_{k+1}^{\beta-1}.$$

As in Sections **??** and **??** the Riesz transforms can be defined as

$$\mathbf{R}_\alpha^k = \mathbf{D}^{\alpha+\mathbf{k-1}} \circ \cdots \circ \mathbf{D}^\alpha(\mathbf{L}_\alpha)^{-k/2}, \quad \text{alternatively} \quad (\mathbf{D}^\alpha)^k(\mathbf{L}_\alpha)^{-k/2},\ \alpha > -1.$$

Let V be the operator defined by $Vf(y) = (2y)^{1/2}f(y^2)$. Let $2\delta = \gamma + \frac{p}{2} - 1$, then $\|Vf\|_{L^p(y^\gamma\,dy)} = 2^{\frac{1}{2}-\frac{1}{p}}\|f\|_{L^p(y^\delta\,dy)}$.

Proposition 1. *Let $1 < p < \infty$, and δ, γ be real numbers. Let T be an operator defined over the set of finite linear combination of Laguerre functions $\{\mathcal{L}_k^\alpha\}_k$. The operator T has a bounded extension from $L^p((0,\infty), y^\delta dy)$ into $L^p((0,\infty), y^\delta dy)$ if and only if the operator $\mathbf{T} = V\,T\,V^{-1}$ has a bounded extension from $L^p((0,\infty), y^\gamma dy)$ into $L^p((0,\infty), y^\gamma dy)$, where $2\delta = \gamma + \frac{p}{2} - 1$.*

An easy consequence of the above proposition, and Theorems **??** and **??** is the following.

Theorem 11. *Let $\alpha > -1$ and let f be a finite linear combination of Laguerre functions $\{\mathcal{L}_k^\alpha\}_k$. Then*

(i) $e^{-tL_\alpha}f = V^{-1}e^{-t\mathbf{L}_\alpha}Vf$,
(ii) *Given* $s > 0$, $(L_\alpha)^{-s}f = V^{-1}(\mathbf{L}_\alpha)^{-s}Vf$ *and*
(iii) $\delta^\alpha f = V^{-1}\mathbf{D}^\alpha Vf$.
(iv) $R_\alpha^k f = V^{-1}\mathbf{R}_\alpha^k Vf$.

Proposition 2. *Let $\alpha > -1$, $1 < p < \infty$, and γ be real number. Let S be any one of the operators $\mathbf{L}^{-s}$, $s > 0$, $\mathbf{R}_\alpha^k$, $(\mathbf{D}^\alpha)^k\mathbf{L}^{-k/2}$, $s > 0$. Then the operator $\mathbf{S}$ has a bounded extension from $L^p((0,\infty), y^\gamma dy)$ into itself, for γ satisfying*

$$(\mathbf{C}_\alpha) \quad -1 - \alpha p - \frac{p}{2} < \gamma < \alpha p + \frac{3p}{2} - 1. \tag{16}$$

Now in a parallel way as we did in Section **??** and **??**, we can define potential spaces and Sobolev spaces for the class of Laguerre functions $\{\varphi_k^\alpha\}_k$, $\alpha > -1$. Thus, given $\alpha > -1$, $1 < p < \infty$, $s > 0$ and γ satisfying $(\mathbf{C}_\alpha)$, see (**??**), we define

$$\mathfrak{U}_{\alpha,s}^p(y^\gamma) = (\mathbf{L}_\alpha)^{-s/2}[L^p(\mathbb{R}^+, y^\gamma dy)]$$

with the norm $\|f\|_{\mathfrak{U}_{\alpha,s}^p(y^\gamma)} = \|g\|_{p,\gamma}$, where $(\mathbf{L}_\alpha)^{-s/2}g = f$.

We shall denote by $U_\alpha^{k,p}(y^\delta)$, the set of functions f in $L^p(\mathbb{R}^+, y^\gamma dy)$ such that

$$\mathbf{D}^{\alpha+m} \circ \ldots \circ \mathbf{D}^{\alpha+1} \circ \mathbf{D}^\alpha f \in L^p(\mathbb{R}^+, y^\gamma dy),\ 0 \le m \le k-1$$

with the norm

$$\|f\|_{U_\alpha^{k,p}(y^\gamma)} = \|f\|_{p,\gamma} + \sum_{m=0}^{k-1} \left\|\mathbf{D}^{\alpha+m} \circ \ldots \circ \mathbf{D}^{\alpha+1} \circ \mathbf{D}^{\alpha} f\right\|_{p,\gamma}.$$

Finally, $\mathcal{U}_\alpha^{k,p}(y^\delta)$, will denote the set of functions f in $L^p(\mathbb{R}^+, y^\gamma dy)$ such that $(\mathbf{D}^\alpha)^m f \in L^p(\mathbb{R}^+, y^\gamma dy)$, $0 \le m \le k$, with the norm

$$\|f\|_{\mathcal{U}_\alpha^{k,p}(y^\gamma)} = \sum_{m=0}^{k} \|(\mathbf{D}^\alpha)^m f\|_{p,\gamma}.$$

The following theorems are direct consequences of Theorems **??**, **??** and Propositions **??** and **??**.

Theorem 12. *Let $\alpha > -1$, $1 < p < \infty$, $k \in \mathbb{N}$ and γ satisfies $(\mathbf{C}_\alpha)$. Then,*

(i) *$U_{\alpha,\gamma}^{k,p} = \mathfrak{U}_{\alpha,\gamma}^{k,p}$, and the norms are equivalent.*
(ii) *Let γ satisfying $(\mathbf{C}_\alpha)$. Then $\mathfrak{U}_\alpha^{k,p}(y^\gamma) \subset \mathcal{U}_\alpha^{k,p}(y^\gamma)$.*
(ii) *Let $-1 < \alpha \le 0$. Then $\mathfrak{U}_\alpha^{2,2} \ne \mathcal{U}_\alpha^{2,2}$.*
(iii) *Let γ satisfying $(\mathbf{C}_{\alpha-\mathbf{1}})$. Then $\mathfrak{U}_\alpha^{2,p}(y^\gamma) = \mathcal{U}_\alpha^{2,p}(y^\gamma)$.*

Analogous results could be obtained for the systems of Laguerre fucntionss $\ell_k^\alpha(y) = \mathcal{L}_k^\alpha(y) y^{-\alpha/2}$ and $\psi_k^\alpha(y) = \sqrt{2} y^{-\alpha} \mathcal{L}_k^\alpha(y^2)$, $\alpha > -1$. These systems are eigenfunctions of the differential operators

$$\mathbb{L}_\alpha = -y\frac{d^2}{dy^2} - (\alpha+1)\frac{d}{dy} + \frac{y}{4}.$$

and

$$\mathfrak{L}_\alpha = -\frac{1}{4}\Big\{\frac{d^2}{dy^2} + \Big(\frac{2\alpha+1}{y}\Big)\frac{d}{dy} - y^2\Big\}.$$

We leave to the interested reader the easy work, but boring unless the statements were needed for some application, of establishing the corresponding Theorem **??** in these systems.

6. Schrödinger equation

Consider the equation

$$\begin{cases} i\dfrac{\partial u(y,t)}{\partial t} = L_\alpha u(y,t) \\ u(y,0) = f(y) \qquad\qquad y \in \mathbb{R}^+, t \in \mathbb{R}, \end{cases} \tag{17}$$

for some initial data f. Consider its solution

$$u(y,t) = e^{itL_\alpha} f(y),$$

for f in the space $L^2(\mathbb{R}^+, dy)$. From some general result in [**?**] one can get that if $f \in \mathfrak{W}_{\alpha,s}^2$, with $s > 1$, then $\lim_{t\to 0} e^{itL_\alpha} f(y) = f(y)$ a.e y. On the other hand it is known, see [**?**] and [**?**], that $\lim_{t\to 0} e^{it\Delta} f(y) = f(y)$ for f such that $\Delta^{1/8} f \in L^2$. We give the following intermediate result.

Theorem 13. *If $f \in \mathfrak{W}_{\alpha,s}^2$ with $s > \frac{1}{2}$ then,*

$$\lim_{t\to 0} e^{itL_\alpha} f(y) = f(y), \tag{18}$$

for almost everywhere $y \in \mathbb{R}^+$.

Proof. It is enough to prove that the maximal function

$$T^*f = \sup_{t\in\mathbb{R}} |e^{itL_\alpha} f|$$

satisfies the inequality

$$\int_I T^*f \le C\,\|f\|_{\mathfrak{w}^2_{\alpha,s}},$$

for all compact interval I of the real line not containing the origin, and C a constant that may depend on the interval I but not on f.

From [?] (Theorem 8.91.2, pp. 241) and (??), if I is an interval that does not contains the origin, then there exist constants C and n_0 such that

$$\mathcal{L}^\alpha_n(x) \le \frac{C}{n^{1/4}} \tag{19}$$

for all $x \in I$ and $n \ge n_0$.

Now, if f belongs to $\mathfrak{W}^2_{\alpha,s}$ we can write

$$f(y) = \sum_{n=0}^{\infty} a_n\, \mathcal{L}^\alpha_n(y),$$

and thus

$$\|f\|_{\mathfrak{W}^2_{\alpha,s}} = \left(\sum_{n=0}^{\infty} |a_n|^2\,(n+\frac{\alpha+1}{2})^s\right)^{1/2}.$$

By Tonelli's theorem, estimate (??) and Hölder's inequality, we get

$$\begin{aligned}\int_I |T^*f(y)|\,dy &\le \int_I \sup_{t>0}\left|\sum_{n=0}^{\infty} a_n\, e^{it(n+\frac{\alpha+1}{2})}\,\mathcal{L}_n(y)\right| dy \le \sum_{n=0}^{\infty}|a_n| \int_I |\mathcal{L}^\alpha_n(y)|\,dy \\ &\le C\left(C+\sum_{n=n_0}^{\infty}\frac{1}{n^{1/2}(n+\frac{\alpha+1}{2})^s}\right)^{1/2}\left(\sum_{n=0}^{\infty}|a_n|^2\,(n+\frac{\alpha+1}{2})^s\right)^{1/2} \\ &\le C\left(C+\sum_{n=n_0}^{\infty}\frac{1}{n^{1/2+s}}\right)^{1/2}\|f\|_{\mathfrak{W}^2_{\alpha,s}}.\end{aligned}$$

Since $s > 1/2$, we have $1/2 + s > 1$ and the last series is convergent.

□

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