MAXIMAL OPERATORS, RIESZ TRANSFORMS AND LITTLEWOOD-PALEY FUNCTIONS ASSOCIATED WITH BESSEL OPERATORS ON BMO

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Abstract. In this paper we study boundedness properties of certain harmonic analysis operators (maximal operators for heat and Poisson semigroups, Riesz transform and Littlewood-Paley g-functions) associated with Bessel operators, on the space $BMO_o(\mathbb{R})$ that consists of the odd functions with bounded mean oscillation on $\mathbb{R}$.

1. Introduction

By $BMO_o(\mathbb{R})$ we denote the space constituted by all those odd functions with bounded mean oscillation on $\mathbb{R}$. This space can be characterized as follows. An odd function $f \in L^1_{loc}(\mathbb{R})$ is in $BMO(\mathbb{R})$, that is, f has bounded mean oscillation on $\mathbb{R}$, if and only if, for all $1 \leq p < \infty$ (equivalently, for some $1 \leq p < \infty$) there exists $C_p > 0$ such that, for every interval $I = (a, b)$

$$\frac{1}{|I|}\int_I |f(x) - f_I|^p \, dx \leq C_p, \quad 0 < a < b < \infty, \tag{1}$$

and also

$$\frac{1}{|I|}\int_I |f(x)|^p \, dx \leq C_p, \quad 0 = a < b < \infty. \tag{2}$$

Here, as usual, $|I|$ denotes the length of I and $f_I = \frac{1}{|I|}\int_I f(x)\,dx$. Moreover, for every $1 \leq p < \infty$, $\inf\{C_p > 0 : (1) \text{ and } (2) \text{ hold}\}$ is equivalent to the usual $\|f\|_{BMO(\mathbb{R})}$ (see, for instance, [14, Chapter 1] definitions and properties concerning to $BMO(\mathbb{R})$). $BMO_o(\mathbb{R})$ coincides with the dual $H^1_o(\mathbb{R})'$ of the subspace $H^1_o(\mathbb{R})$ of $H^1(\mathbb{R})$ that consists of all the odd functions in the Hardy space $H^1(\mathbb{R})$. The space $H^1_o(\mathbb{R})$ was studied in [4] and [10], where several characterizations of $H^1_o(\mathbb{R})$ are obtained. In the sequel we denote by BMO_+ the space that consists of all those $f \in L^1_{loc}([0,\infty))$ such that the odd extension f_o of f to $\mathbb{R}$ is in $BMO(\mathbb{R})$. On BMO_+ we consider the natural norm. Our objective in this paper is to

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study the behavior on BMO_+ of maximal operator, Riesz transform and Littlewood-Paley g-functions associated with Bessel operators.

Muckenhoupt and Stein [12] began the development of harmonic analysis related to Bessel operators. They considered the Bessel operator B_λ, $\lambda > 0$, defined by $B_\lambda = -x^{-2\lambda} D x^{2\lambda} D$, with $D = \frac{d}{dx}$. In [12] Poisson integrals and conjugate of Poisson integrals associated with B_λ were introduced. Recently, L^p-boundedness properties for the higher order Riesz transform ([5]) and for the Littlewood-Paley g-functions ([6]) in the B_λ context have been established.

Here we consider the Bessel operator $\Delta_\lambda = -x^{-\lambda} D x^{2\lambda} D x^{-\lambda}$, with $\lambda > 0$. If J_ν denotes the Bessel function of the first kind and order ν, for every $y > 0$, the function $\varphi_y(x) = \sqrt{xy} J_{\lambda-\frac{1}{2}}(xy)$, $x \in (0,\infty)$, is an eigenfunction of Δ_λ and

$$\Delta_\lambda(\sqrt{xy} J_{\lambda-\frac{1}{2}}(xy)) = y^2 \sqrt{xy} J_{\lambda-\frac{1}{2}}(xy), \quad x, y \in (0,\infty).$$

The Poisson kernel associated with the operator Δ_λ is given by

$$P^\lambda(t,x,y) = \int_0^\infty e^{-tz} \varphi_x(z) \varphi_y(z)\, dz, \quad t, x, y \in (0,\infty).$$

According to [12, (16.4)] (see also [19]) we have that

$$P^\lambda(t,x,y) = \frac{2\lambda t (xy)^\lambda}{\pi} \int_0^\pi \frac{(\sin\theta)^{2\lambda-1}}{((x-y)^2 + t^2 + 2xy(1-\cos\theta))^{\lambda+1}} d\theta, \quad t, x, y \in (0,\infty).$$

The Poisson integral $P_t^\lambda(f)$ is defined by

$$P_t^\lambda(f)(x) = \int_0^\infty P^\lambda(t,x,y) f(y) dy, \quad t, x > 0.$$

The family $\{P_t^\lambda\}_{t>0}$ constitutes a semigroup of linear and bounded operators in $L^p(0,\infty)$, $1 \le p \le \infty$. L^p-boundedness properties of the maximal operator

$$P_*^\lambda(f) = \sup_{t>0} |P_t^\lambda(f)|$$

were established in [7] and [8].

The heat kernel associated with the operator Δ_λ is

$$W^\lambda(t,x,y) = \int_0^\infty e^{-tz^2} \varphi_x(z) \varphi_y(z)\, dz, \quad t, x, y \in (0,\infty).$$

According to [18, 13.31(1)], we can write

$$W^\lambda(t,x,y) = \frac{1}{\sqrt{2t}} \left(\frac{xy}{2t}\right)^{\frac{1}{2}} I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2t}\right) e^{-\frac{x^2+y^2}{4t}}, \quad t, x, y \in (0,\infty),$$

where I_ν denotes the modified Bessel function of the first kind and order ν. The heat integral $W_t^\lambda(f)$ of f is defined by

$$W_t^\lambda(f)(x) = \int_0^\infty W^\lambda(t,x,y)f(y)\,dy, \quad t,x>0.$$

Then, $\{W_t^\lambda\}_{t>0}$ is a semigroup of bounded and linear operators in $L^p(0,\infty)$, $1 \le p \le \infty$. The maximal operator associated with $\{W_t^\lambda\}_{t>0}$ is given by

$$W_*^\lambda(f) = \sup_{t>0} |W_t^\lambda(f)|$$

and it was investigated on L^p-spaces in [7].

Bennett, DeVore and Sharpley ([2, Th. 4.2 (b)]) proved that if $\mathcal{M}$ denotes the (uncentered) Hardy-Littlewood maximal operator on $\mathbb{R}^n$, then, for every $f \in BMO(\mathbb{R}^n)$, either $\mathcal{M}f \in BMO(\mathbb{R}^n)$ or $\mathcal{M}f \equiv \infty$. The function $f(x) = \log_+|x|$, $x \in \mathbb{R}^n$, is an example of the second situation. In [9] it was introduced a BMO type space on $\mathbb{R}^n$ associated with Schrödinger operators where the maximal operator $\mathcal{M}$ is bounded. This is the case for the maximal operators, W_*^λ, P_*^λ and the Hardy-Littlewood maximal operator $\mathcal{M}_0$ on $(0,\infty)$, on BMO_+ as we state in the following proposition.

Proposition 1. *Let $\lambda > 0$. We denote by $\mathcal{N}$ the operators $\mathcal{M}_0$, W_*^λ or P_*^λ. There exists $C > 0$ such that*

$$\|\mathcal{N}f\|_{BMO_+} \le C\|f\|_{BMO_+}, \quad f \in BMO_+.$$

Riesz transforms in the Δ_λ-setting were studied in [3]. The operator Δ_λ admits the factorization $\Delta_\lambda = D_\lambda^* D_\lambda$, where $D_\lambda = x^\lambda D x^{-\lambda}$ and D_λ^* represents the (formal) adjoint of D_λ in $L^2(0,\infty)$. Following the ideas developed by Stein in [13], the Riesz transform R_λ is defined by

$$R_\lambda f = D_\lambda \Delta_\lambda^{-\frac{1}{2}} f, \quad f \in C_c^\infty(0,\infty).$$

Here $C_c^\infty(0,\infty)$ denotes the space of smooth functions with compact support in $(0,\infty)$. The operator R_λ can be extended to $L^p(0,\infty)$ as a bounded operator on $L^p(0,\infty)$, for every $1 < p < \infty$, and to $L^1(0,\infty)$ as a bounded operator from $L^1(0,\infty)$ into $L^{1,\infty}(0,\infty)$. Moreover, for each $f \in L^p(0,\infty)$, $1 < p < \infty$,

$$R_\lambda f(x) = \lim_{\varepsilon\to 0} \int_{0,|x-y|>\varepsilon}^\infty R_\lambda(x,y)f(y)dy, \quad \text{a.e. } x \in (0,\infty), \tag{3}$$

being

$$R_\lambda(x,y)=\int_0^\infty D_{\lambda,x}P^\lambda(t,x,y)\,dt,\quad x,y\in(0,\infty),\ x\neq y.$$

According to [1, (1.6)] (also see [7]) we get

$$|R_\lambda(x,y)|\leq C(xy)^\lambda\begin{cases}\dfrac{x}{y^{2\lambda+2}}, & 2x\leq y,\\ \dfrac{1}{x^{2\lambda+1}}, & 0<y<\frac{x}{2},\end{cases}\tag{4}$$

and

$$\left|R_\lambda(x,y)-\frac{1}{\pi}\frac{1}{x-y}\right|\leq C\frac{1}{y}\left(1+\log_+\frac{\sqrt{xy}}{|x-y|}\right),\quad 0<\frac{x}{2}<y<2x.\tag{5}$$

Then, we can prove that the Riesz transform R_λ is well defined on $L^\infty(0,\infty)$. This fact establishes a difference between the behavior of R_λ and the Hilbert transform on bounded functions ([16, p. 294]).

The vertical Littlewood-Paley g-function associated with the heat semigroup $\{W_t^\lambda\}_{t>0}$ for the Bessel operator Δ_λ is defined by

$$g_{h,\lambda}(f)(x)=\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}W_t^\lambda(f)(x)\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}},$$

and the corresponding one for the Poisson semigroup $\{P_t^\lambda\}_{t>0}$ is given by

$$g_{P,\lambda}(f)(x)=\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}P_t^\lambda(f)(x)\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}.$$

The behavior of the Riesz transforms and g-functions on BMO_+ is established in the next proposition.

Proposition 2. *Let $\lambda>0$. We denote by $\mathcal{N}$ the operators R_λ, $g_{h,\lambda}$ and $g_{P,\lambda}$. There exists $C>0$ such that*

$$\|\mathcal{N}f\|_{BMO_+}\leq C\|f\|_{BMO_+},\quad f\in BMO_+.$$

As for maximal operators the property stated in Proposition 2 for $g_{h,\lambda}$ and $g_{P,\lambda}$ contrasts with the corresponding one for vertical classical Littlewood-Paley g-functions (see [17]).

This paper is organizated as follows. In Section 2 we prove Proposition 1 and the proof of Proposition 2 is showed in Section 3 and Section 4 where we establish the estimates for the Riesz transform and for the g-functions, respectively.

Throughout this paper we always denote by C a suitable positive constant that can change from a line to the other one.

2. Maximal operators in BMO_+.

In this section we present a proof of Proposition 1. We divide the proof in three parts. Each part is concerned with one of the maximal operators under considerations.

(i) By $\mathcal{M}_0$ we denote the Hardy-Littlewood maximal operator on $(0,\infty)$, that is, if $f \in L^1_{loc}([0,\infty))$,

$$\mathcal{M}_0(f)(x) = \sup_{x\in I} \frac{1}{|I|}\int_I |f(y)|\,dy, \quad x \in (0,\infty),$$

where the supremum is taken over all the bounded intervals I on $(0,\infty)$ such that $x \in I$.

Assume that $f \in BMO_+$, then $f_o \in BMO_o(\mathbb{R})$. Let $a > 0$, we write $f_o = f_1 + f_2$ where $f_1 = f_o\chi_{(-2a,2a)}$. Since $f_o \in L^1_{loc}(\mathbb{R})$, $\mathcal{M}f_1(x) < \infty$, a.e. $x \in \mathbb{R}$, where $\mathcal{M}$ denotes the Hardy-Littlewood maximal operator on $\mathbb{R}$. Moreover, if $x \in (-a,a)$ and I is a bounded interval such that $x \in I$ and $I \cap (-2a,2a)^c \neq \emptyset$, by denoting $J = (-b,b)$, where $b = \max\{|y|, y \in I\}$, we have

$$(6) \qquad \frac{1}{|I|}\int_I |f_2(y)|\,dy = \frac{1}{|I|}\int_{I\cap(-2a,2a)^c} |f_o(y)|\,dy \leq C\frac{1}{|J|}\int_J |f_o(y)|\,dy \leq C\|f\|_{BMO_+}.$$

Note that $|I| \leq |J| \leq 2(|I| + a) \leq 4|I|$. Hence $\mathcal{M}(f_2)(x) < \infty$, a.e. x, $|x| \leq a$. Then, we obtain that $\mathcal{M}f_o(x) < \infty$, a.e. x, $|x| \leq a$. Hence, we conclude that $\mathcal{M}f_o(x) < \infty$, a. e. $x \in \mathbb{R}$.

Since $f \in BMO(0,\infty)$, a wellknown result due to Bennett, DeVore and Sharpley ([2, Theorem 4.2]) implies that $\mathcal{M}_0 f \in BMO(0,\infty)$ and $\|\mathcal{M}_0 f\|_{BMO(0,\infty)} \leq C\|f\|_{BMO_+}$. Moreover, for every $a > 0$,

$$(7) \qquad \frac{1}{a}\int_0^a \mathcal{M}_0(f)(x)\,dx \leq C\|f\|_{BMO_+}.$$

Indeed, let $a > 0$. As above we write $f = f_1 + f_2$, where $f_1 = f\chi_{(0,2a)}$. Then, by proceeding as in (6) we get

$$(8) \qquad \mathcal{M}_0(f_2)(x) \leq 2\|f\|_{BMO_+}, \quad x \in (0,a).$$

Also, since $\mathcal{M}_0$ is bounded on $L^2(0,\infty)$, it has

$$(9) \quad \frac{1}{a}\int_0^a |M_0 f_1(x)|dx \leq \left(\frac{1}{a}\int_0^a |M_0 f_1(x)|^2\,dx\right)^{\frac{1}{2}} \leq C\left(\frac{1}{a}\int_0^{2a} |f(x)|^2 dx\right)^{\frac{1}{2}} \leq C\|f\|_{BMO_+}.$$

From (8) and (9) we deduce that (7) holds.

By combining the above arguments we conclude that $(\mathcal{M}_0 f)_o \in BMO_o(\mathbb{R})$ and $\|\mathcal{M}_0 f\|_{BMO_+} \leq C\|f\|_{BMO_+}$.

(ii) We now analyze the maximal operator W_*^λ associated with the heat semigroup $\{W_t^\lambda\}_{t>0}$. Assume that $f \in BMO_+$.

According to [11, (5.16.5)] we have that

$$0 \leq W^\lambda(t,x,y) \leq C\frac{1}{\sqrt{t}}e^{-\frac{(x-y)^2}{4t}}, \quad t,x,y \in (0,\infty), \tag{10}$$

and also,

$$W^\lambda(t,x,y) \leq C \begin{cases} \dfrac{1}{\sqrt{t}}\left(\dfrac{xy}{t}\right)^\lambda e^{-\frac{y^2+x^2}{4t}}, & \dfrac{xy}{2t} \leq 1, \\ \dfrac{1}{\sqrt{t}}\left(\dfrac{xy}{t}\right)^\lambda e^{-\frac{(x-y)^2}{4t}}, & \dfrac{xy}{2t} \geq 1. \end{cases} \tag{11}$$

It is wellknown that

$$\sup_{t>0}\int_0^\infty \frac{1}{\sqrt{t}}e^{-\frac{(x-y)^2}{4t}}|f(y)|\,dy \leq C\mathcal{M}_0(f)(x), \quad x \in (0,\infty). \tag{12}$$

Then

$$W_*^\lambda(f)(x) \leq C\mathcal{M}_0(f)(x), \quad x \in (0,\infty).$$

Hence, by (7), for every $a>0$, we have

$$\frac{1}{a}\int_0^a W_*^\lambda(f)(x)\,dx \leq C\|f\|_{BMO_+}. \tag{13}$$

On the other hand, we can write

$$\begin{aligned} &\left|\sup_{t>0}|W_t^\lambda(f)(x)| - \sup_{t>0}\left|\int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y)f(y)\,dy\right|\right| \\ &\qquad \leq \sup_{t>0}\int_0^{\frac{x}{2}} W^\lambda(t,x,y)|f(y)|\,dy + \sup_{t>0}\int_{2x}^\infty W^\lambda(t,x,y)|f(y)|\,dy, \quad x \in (0,\infty). \end{aligned}$$

From (10) it follows that

$$\begin{aligned} \int_0^{\frac{x}{2}} W^\lambda(t,x,y)|f(y)|\,dy &\leq C\int_0^{\frac{x}{2}} \frac{1}{\sqrt{t}}e^{-\frac{(x-y)^2}{4t}}|f(y)|\,dy \leq C\int_0^{\frac{x}{2}} \frac{1}{\sqrt{t}}e^{-\frac{x^2}{16t}}|f(y)|\,dy \\ &\leq \frac{C}{x}\int_0^{\frac{x}{2}} |f(y)|\,dy \leq C\|f\|_{BMO_+}, \quad t,x \in (0,\infty). \end{aligned}$$

By using (11) we get

$$\int_{2x}^\infty W^\lambda(t,x,y)|f(y)|\,dy \leq C\left(\int_{2x,\frac{xy}{2t}\leq 1}^\infty \left(\frac{xy}{t}\right)^\lambda \frac{e^{-\frac{y^2}{4t}}}{\sqrt{t}}|f(y)|dy + \int_{2x,\frac{xy}{2t}>1}^\infty \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{t}}|f(y)|dy\right)$$

$$\begin{aligned}
&\leq C\int_{2x}^{\infty}\left(\frac{xy}{t}\right)^{\lambda}\frac{e^{-\frac{y^2}{16t}}}{\sqrt{t}}|f(y)|\,dy \leq Cx^{\lambda}\int_{2x}^{\infty}\frac{|f(y)|}{y^{\lambda+1}}\,dy = Cx^{\lambda}\sum_{k=1}^{\infty}\int_{2k^{2/\lambda}x}^{2(k+1)^{2/\lambda}x}\frac{|f(y)|}{y^{\lambda+1}}\,dy\\
&\leq Cx^{\lambda}\sum_{k=1}^{\infty}\frac{1}{(2k^{2/\lambda}x)^{\lambda+1}}\int_0^{2(k+1)^{2/\lambda}x}|f(y)|\,dy \leq C\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{1}{2(k+1)^{2/\lambda}x}\int_0^{2(k+1)^{2/\lambda}x}|f(y)|\,dy\\
&\leq C\|f\|_{BMO_+},\quad t,x\in(0,\infty).
\end{aligned}$$

Hence, we have proved that

$$\sup_{t>0}|W_t^{\lambda}(f)(x)| - \sup_{t>0}\left|\int_{\frac{x}{2}}^{2x}W^{\lambda}(t,x,y)f(y)\,dy\right| \in L^{\infty}(0,\infty),$$

and

$$\left\|\sup_{t>0}|W_t^{\lambda}(f)(x)| - \sup_{t>0}\left|\int_{\frac{x}{2}}^{2x}W^{\lambda}(t,x,y)f(y)\,dy\right|\right\|_{\infty} \leq C\|f\|_{BMO_+}. \tag{14}$$

Moreover,

$$\sup_{t>0}\left|\int_{\frac{x}{2}}^{2x}W^{\lambda}(t,x,y)f(y)\,dy\right| - \sup_{t>0}\left|\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right| \in L^{\infty}(0,\infty),$$

and

$$\left\|\sup_{t>0}\left|\int_{\frac{x}{2}}^{2x}W^{\lambda}(t,x,y)f(y)\,dy\right| - \sup_{t>0}\left|\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|\right\|_{\infty} \leq C\|f\|_{BMO_+}. \tag{15}$$

Indeed, we can write

$$\begin{aligned}
&\left|\sup_{t>0}\left|\int_{\frac{x}{2}}^{2x}W^{\lambda}(t,x,y)f(y)\,dy\right| - \sup_{t>0}\left|\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|\right|\\
&\qquad\leq \sup_{t>0}\int_{\frac{x}{2}}^{2x}\left|W^{\lambda}(t,x,y) - \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\right||f(y)|\,dy,\quad x\in(0,\infty).
\end{aligned}$$

According to (10), it follows that

$$\begin{aligned}
&\int_{\frac{x}{2},\frac{xy}{2t}\leq 1}^{2x}|W^{\lambda}(t,x,y) - \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}||f(y)|\,dy\\
&\qquad\leq C\int_{\frac{x}{2},\frac{xy}{2t}\leq 1}^{2x}\frac{1}{\sqrt{t}}\left(\left(\frac{xy}{t}\right)^{\lambda}+1\right)e^{-\frac{x^2+y^2}{4t}}|f(y)|\,dy\\
&\qquad\leq C\int_{\frac{x}{2}}^{2x}\frac{|f(y)|}{\sqrt{x^2+y^2}}\,dy \leq \frac{C}{x}\int_0^{2x}|f(y)|\,dy \leq C\|f\|_{BMO_+},\quad t,x\in(0,\infty).
\end{aligned}$$

Also, by using [11, 5.16.5], we get

$$\int_{\frac{x}{2},\frac{xy}{2t}\geq 1}^{2x}\left|W^\lambda(t,x,y)-\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\right||f(y)|\,dy\leq C\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{t}}\left(\frac{t}{xy}\right)^{\frac{1}{2}}e^{-\frac{(x-y)^2}{4t}}|f(y)|\,dy$$

$$\leq\frac{C}{x}\int_0^{2x}|f(y)|\,dy\leq C\|f\|_{BMO_+},\quad t,x\in(0,\infty).$$

Hence (15) is established.

Now we denote by $\{W_t\}_{t>0}$ the classical heat semigroup, that is, we write

$$W_t(f_o)(x)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{(x-y)^2}{4t}}f_o(y)\,dy,\quad t\in(0,\infty)\text{ and }x\in\mathbb{R}.$$

Since f_o is an odd function we can write

$$W_t(f_o)(x)=\frac{1}{\sqrt{4\pi t}}\int_0^\infty\left(e^{-\frac{(x-y)^2}{4t}}-e^{-\frac{(x+y)^2}{4t}}\right)f(y)\,dy,\quad t\in(0,\infty)\text{ and }x\in\mathbb{R}. \tag{16}$$

Moreover $W_t(f_o)$ is odd, for every $t>0$. By splitting the integral it gets

$$\left|W_t(f_o)(x)-\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|$$

$$\leq\frac{1}{\sqrt{4\pi t}}\int_0^{\frac{x}{2}}\left|e^{-\frac{(x-y)^2}{4t}}-e^{-\frac{(x+y)^2}{4t}}\right||f(y)|\,dy$$

$$+\frac{1}{\sqrt{4\pi t}}\int_{2x}^{\infty}\left|e^{-\frac{(x-y)^2}{4t}}-e^{-\frac{(x+y)^2}{4t}}\right||f(y)|\,dy+\frac{1}{\sqrt{4\pi t}}\int_{\frac{x}{2}}^{2x}e^{-\frac{(x+y)^2}{4t}}|f(y)|\,dy$$

$$\leq\frac{C}{\sqrt{t}}\left(\int_0^{\frac{x}{2}}\left|\frac{(x-y)^2-(x+y)^2}{4t}\right|e^{-\frac{(x-y)^2}{4t}}|f(y)|dy\right.$$

$$\left.+\int_{2x}^{\infty}\left|\frac{(x-y)^2-(x+y)^2}{4t}\right|e^{-\frac{(x-y)^2}{4t}}|f(y)|dy+\int_{\frac{x}{2}}^{2x}e^{-\frac{(x+y)^2}{4t}}|f(y)|dy\right)$$

$$\leq C\left(\int_0^{\frac{x}{2}}\frac{xy}{t^{\frac{3}{2}}}e^{-\frac{x^2}{16t}}|f(y)|dy+\int_{2x}^{\infty}\frac{xy}{t^{\frac{3}{2}}}e^{-\frac{y^2}{16t}}|f(y)|dy+\int_{\frac{x}{2}}^{2x}\frac{|f(y)|}{x+y}dy\right)$$

$$\leq C\left(\frac{1}{x}\int_0^{\frac{x}{2}}|f(y)|dy+x\int_{2x}^{\infty}\frac{1}{y^2}|f(y)|dy+\frac{1}{x}\int_{\frac{x}{2}}^{2x}|f(y)|dy\right)$$

$$\leq C\left(\frac{1}{x}\int_0^{2x}|f(y)|dy+x\sum_{k=1}^{\infty}\int_{2xk^2}^{2x(k+1)^2}\frac{1}{y^2}|f(y)|dy\right)$$

$$\leq C\left(\frac{1}{x}\int_0^{2x}|f(y)|dy+\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{1}{2x(k+1)^2}\int_0^{2x(k+1)^2}|f(y)|dy\right) \leq C\|f\|_{BMO_+},$$

for every $t, x\in(0,\infty)$. Hence,

$$\sup_{t\in(0,\infty)}|W_t(f_o)(x)|-\sup_{t>0}\left|\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|\in L^\infty(0,\infty),$$

and

$$\left\|\sup_{t\in(0,\infty)}|W_t(f_o)(x)|-\sup_{t>0}\left|\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|\right\|_\infty\leq C\|f\|_{BMO_+}. \tag{17}$$

We deduce from (14), (15) and (17) that

$$\sup_{t>0}|W_t^\lambda(f)(x)|-\sup_{t>0}|W_t(f_o)(x)|\in L^\infty(0,\infty)$$

and

$$\left\|\sup_{t>0}|W_t^\lambda(f)|-\sup_{t>0}|W_t(f_o)|\right\|_\infty\leq C\|f\|_{BMO_+}. \tag{18}$$

According to (13) and (18), to see that

$$\sup_{t>0}|W_t^\lambda(f)|\in BMO_+ \text{ and } \left\|\sup_{t>0}|W_t(f)|)\right\|_{BMO_+}\leq C\|f\|_{BMO_+},$$

it is sufficient to see that $\sup_{t>0}|W_t(f_o)|)\in BMO(\mathbb{R})$ and that

$$\|\sup_{t>0}|W_t(f_o)|)\|_{BMO(\mathbb{R})}\leq C\|f_o\|_{BMO(\mathbb{R})}.$$

We have to show that $\sup_{t>0}|W_t(f_o)(x)|<\infty$, a.e. $x\in\mathbb{R}$ (see [15]). From (7) and (12) we get

$$\frac{1}{a}\int_0^a\sup_{t>0}|W_t(f_o)(x)|dx\leq C\|f\|_{BMO_+},\quad a>0.$$

Then, since $\sup_{t>0}|W_t(f_o)|$ is even, $\sup_{t>0}|W_t(f_o)(x)|<\infty$, a.e. $x\in\mathbb{R}$. Thus we prove that $W_*^\lambda(f)\in BMO_+$ and $\|W_*^\lambda(f)\|_{BMO_+}\leq C\|f\|_{BMO_+}$.

(iii) Let $f\in BMO_+$. By using subordination formula we can write

$$P^\lambda(t,x,y)=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-u}}{\sqrt{u}}W^\lambda\left(\frac{t^2}{4u},x,y\right)du,\quad t,x,y\in(0,\infty). \tag{19}$$

Then,

$$\sup_{t>0}|P_t^\lambda(f)|\leq C\sup_{t>0}|W_t^\lambda(f)|.$$

Hence, from (13) we deduce that

$$\frac{1}{a}\int_0^a P_*^\lambda(f)(x)dx \le C\|f\|_{BMO_+}, \quad a>0. \tag{20}$$

Moreover, by (18), it follows

$$\sup_{t>0}|P_t^\lambda(f)(x)| - \sup_{t>0}|P_t(f_o)(x)| \in L^\infty(0,\infty),$$

and

$$\left\|\sup_{t>0}|P_t^\lambda(f)(x)| - \sup_{t>0}|P_t(f_o)(x)|\right\|_\infty \le C\|f\|_{BMO_+}, \tag{21}$$

where

$$P_t(f_o)(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{t}{(x-y)^2+t^2}f_o(y)\,dy, \quad t>0,\ \ x\in\mathbb{R}. \tag{22}$$

From (12) it infers that $\sup_{t>0}|P_t(f_o)(x)| \le C\mathcal{M}_0(f)(x)$, $x\in(0,\infty)$. Then, by (7),

$$\frac{1}{a}\int_0^a \sup_{t>0}|P_t(f_o)(x)|dx \le C\|f\|_{BMO_+}, \quad a>0.$$

Hence, since $\sup_{t>0}|P_t(f_o)|$ is even, $\sup_{t>0}|P_t(f_o)(x)|<\infty$, a.e. $x\in\mathbb{R}$. It deduces that $\sup_{t>0}|P_t(f_o)|\in BMO(\mathbb{R})$ (see [15]). (20) and (21) allow us to conclude that $P_*^\lambda f\in BMO_+$, and $\|P_*^\lambda f\|_{BMO_+}\le C\|f\|_{BMO_+}$.

3. Riesz transform in BMO_+.

Our objective is to show Proposition 2 for the Riesz transforms R_λ. Firstly, note that, by (4) and (5), the Riesz transform R_λ is defined on $L^\infty(0,\infty)$. Indeed, let $f\in L^\infty(0,\infty)$. It is known (see, for instance, [16, p. 294]) that the limit

$$\lim_{\varepsilon\to 0}\int_{0,|x-y|>\varepsilon}^\infty f(y)\left(\frac{1}{x-y}+\chi_{(1,\infty)}(y)\frac{1}{y}\right)dy \tag{23}$$

exists for almost every $x\in(0,\infty)$. We now prove that

$$R_\lambda f(x) = \int_0^{\frac{x}{2}} R_\lambda(x,y)f(y)\,dy + \lim_{\varepsilon\to 0}\int_{\frac{x}{2},|x-y|>\varepsilon}^{2x} R_\lambda(x,y)f(y)dy + \int_{2x}^\infty R_\lambda(x,y)f(y)dy,$$

for almost all $x\in(0,\infty)$. According to (4), we get

$$\int_0^{\frac{x}{2}}|R_\lambda(x,y)||f(y)|dy \le C\frac{1}{x^{\lambda+1}}\int_0^{\frac{x}{2}} y^\lambda|f(y)|dy \le C\|f\|_\infty, \ \ x\in(0,\infty),$$

and

$$\int_{2x}^\infty |R_\lambda(x,y)||f(y)|dy \le Cx^{\lambda+1}\int_{2x}^\infty\frac{|f(y)|}{y^{\lambda+2}}dy \le C\|f\|_\infty, \ \ x\in(0,\infty).$$

On the other hand, it has

$$\int_{\frac{x}{2},|x-y|>\varepsilon}^{2x} R_\lambda(x,y)f(y)dy = \int_{\frac{x}{2},|x-y|>\varepsilon}^{2x}\left(R_\lambda(x,y)f(y) - \frac{1}{\pi}\frac{1}{x-y}\right)f(y)dy$$

$$+\frac{1}{\pi}\int_{\frac{x}{2},|x-y|>\varepsilon}^{2x}\frac{1}{x-y}f(y)dy, \quad \varepsilon, x\in(0,\infty). \tag{24}$$

From (5) it deduces that, for every $x \in (0,\infty)$,

$$\int_{\frac{x}{2}}^{2x}\left|R_\lambda(x,y) - \frac{1}{\pi}\frac{1}{x-y}\right||f(y)|dy \leq C\|f\|_\infty \int_{\frac{x}{2}}^{2x}\frac{1}{y}\left(1+\log_+\frac{\sqrt{xy}}{|x-y|}\right)dy \leq C\|f\|_\infty,$$

because $\int_{\frac{x}{2}}^{2x}\frac{1}{y}\left(1+\log_+\frac{\sqrt{xy}}{|x-y|}\right)dy = \int_{\frac{1}{2}}^{2}\left(1+\log_+\frac{\sqrt{u}}{|1-u|}\right)du$. Moreover, we write

$$\begin{aligned}\int_{\frac{x}{2},|x-y|>\varepsilon}^{2x}\frac{1}{x-y}f(y)dy &= \int_{0,|x-y|>\varepsilon}^{\infty}\left(\frac{1}{x-y}+\frac{\chi_{(1,\infty)}(y)}{y}\right)f(y)dy \\ &- \int_{\frac{x}{2},|x-y|>\varepsilon}^{2x} f(y)\frac{\chi_{(1,\infty)}(y)}{y}dy - \int_{2x,|x-y|>\varepsilon}^{\infty}\left(\frac{1}{x-y}+\frac{\chi_{(1,\infty)}(y)}{y}\right)f(y)dy \\ &- \int_{0,|x-y|>\varepsilon}^{\frac{x}{2}}\left(\frac{1}{x-y}+\frac{\chi_{(1,\infty)}(y)}{y}\right)f(y)dy, \quad \varepsilon, x\in(0,\infty).\end{aligned}$$

Note that, for each $x\in(0,\infty)$,

$$\int_{\frac{x}{2}}^{2x}\left|f(y)\chi_{(1,\infty)}(y)\right|\frac{dy}{y} \leq C\|f\|_\infty,$$

that

$$\begin{aligned}&\int_{2x}^{\infty}\left|\frac{1}{x-y}-\frac{\chi_{(1,\infty)}(y)}{y}\right||f(y)|dy \\ &\qquad \leq C\|f\|_\infty\left(\int_{2x}^{2x+1}\left(\frac{1}{|x-y|}+\frac{1}{y}\right)dy + \int_{2x+1}^{\infty}\frac{x}{|x-y|y}dy\right) \leq C\left(\frac{1}{x}+1\right)\|f\|_\infty,\end{aligned}$$

and

$$\int_0^{\frac{x}{2}}\left|\frac{1}{x-y}-\frac{\chi_{(1,\infty)}(y)}{y}\right||f(y)|dy \leq C\,(1+x)\,\|f\|_\infty.$$

Then, by (23) and (24) we conclude that the limit

$$R_\lambda(f)(x) = \lim_{\varepsilon\to 0}\int_{0,|x-y|>\varepsilon}^{\infty} R_\lambda(x,y)f(y)dy \tag{25}$$

exists for almost every $x \in (0,\infty)$. This property shows different behaviour of Hilbert transform (see (23)) and R_λ-transform on $L^\infty(0,\infty)$.

We now prove Proposition 2 for Riesz transform R_λ.

Assume that $f \in BMO_+$. If we consider f_e the even extension of f to $\mathbb{R}$, according to [16, p. 294],

$$H(f_e)(x) = \lim_{\varepsilon \to 0} \int_{-\infty, |x-y|>\varepsilon}^{+\infty} \left(\frac{1}{x-y} + \frac{\chi_{(-1,1)^c}(y)}{y} \right) f_e(y) dy \in BMO(\mathbb{R}),$$

because $f_e \in BMO(\mathbb{R})$. Since f_e is even we can write

$$\begin{aligned} H(f_e)(x) &= \lim_{\varepsilon \to 0} \int_{0, |x-y|>\varepsilon}^{\infty} \left(\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y} \right) f(y) dy \\ &\quad + \int_{-\infty, |x-y|>\varepsilon}^{0} \left(\frac{1}{x-y} + \frac{\chi_{(-\infty,-1)}(y)}{y} \right) f(-y) dy \\ &= \lim_{\varepsilon \to 0} \int_{0, |x-y|>\varepsilon}^{\infty} \left(\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y} \right) f(y) dy \\ &\quad + \int_{0, |x+y|>\varepsilon}^{\infty} \left(\frac{1}{x+y} - \frac{\chi_{(1,\infty)}(y)}{y} f(y) \right) dy \\ &= \lim_{\varepsilon \to 0} \int_{\frac{x}{2}, |x-y|>\varepsilon}^{2x} \left(\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y} \right) f(y) dy \\ &\quad + \int_{0}^{\frac{x}{2}} \frac{2x}{x^2 - y^2} f(y) dy + \int_{2x}^{\infty} \frac{2x}{x^2 - y^2} f(y) dy \\ &\quad + \int_{\frac{x}{2}}^{2x} \left(\frac{1}{x+y} - \frac{\chi_{(1,\infty)}(y)}{y} \right) f(y) dy, \quad x \in (0, \infty). \end{aligned}$$

Note that $H(f_e)$ is odd. We are going to see that

$$H(f_e)(x) - \lim_{\varepsilon \to 0} \int_{\frac{x}{2}, |x-y|>\varepsilon}^{2x} \left(\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y} \right) f(y) dy \in L^\infty(0, \infty) \tag{26}$$

and

$$\left\| H(f_e)(x) - \lim_{\varepsilon \to 0} \int_{\frac{x}{2}, |x-y|>\varepsilon}^{2x} \left(\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y} \right) f(y) dy \right\|_\infty \leq C \|f\|_{BMO_+}. \tag{27}$$

We have to analyze three terms. It gets, as in the proof of (17) in the previous section,

$$\left| \int_0^{\frac{x}{2}} \frac{2x}{x^2 - y^2} f(y) dy \right| \leq C \frac{1}{x} \int_0^x |f(y)| dy \leq C \|f\|_{BMO_+}, \quad x \in (0, \infty),$$

and

$$\left| \int_{2x}^{\infty} \frac{2x}{x^2 - y^2} f(y) dy \right| \leq C x \int_{2x}^{\infty} |f(y)| \frac{dy}{y^2} \leq C \|f\|_{BMO_+}, \quad x \in (0, \infty).$$

Also, we obtain

$$\left| \int_{\frac{x}{2}}^{2x} \left(\frac{1}{x+y} - \frac{\chi_{(1,\infty)}(y)}{y} \right) f(y) dy \right| \leq \frac{C}{x} \int_0^{2x} |f(y)| dy \leq C \|f\|_{BMO_+}, \quad x \in (0, \infty).$$

Thus (26) and (27) are established. By using (25), (4) and (5) we have that

$$\begin{aligned}
&\left|R_\lambda(f)(x)-\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\frac{x}{2},|x-y|>\varepsilon}^{2x}\left(\frac{1}{x-y}+\frac{\chi_{(1,\infty)}(y)}{y}\right)f(y)dy\right|\\
&\qquad\le C\left(\int_{\frac{x}{2}}^{2x}\frac{1}{y}\left(1+\log_+\frac{\sqrt{xy}}{|x-y|}\right)|f(y)|dy+\int_{\frac{x}{2}}^{2x}\frac{\chi_{(1,\infty)}(y)}{y}|f(y)|dy\right.\\
&\qquad\quad\left.+\frac{1}{x^{\lambda+1}}\int_0^{\frac{x}{2}}y^\lambda|f(y)|dy+x^{\lambda+1}\int_{2x}^\infty\frac{|f(y)|}{y^{\lambda+2}}dy\right)\\
&\qquad\le C\left(\left(\left(\int_{\frac{x}{2}}^{2x}\frac{1}{y}\left(1+\log_+\frac{\sqrt{xy}}{|x-y|}\right)^2dy\right)^{\frac{1}{2}}\left(\frac{1}{x}\int_{\frac{x}{2}}^{2x}|f(y)|^2dy\right)^{\frac{1}{2}}\right.\right.\\
&\qquad\quad\left.+\frac{1}{x}\int_0^x|f(y)|dy+x\int_{2x}^\infty\frac{|f(y)|}{y^2}dy\right)\le C\|f\|_{BMO_+},\qquad x\in(0,\infty),
\end{aligned}$$

because

$$\int_{\frac{x}{2}}^{2x}\frac{1}{y}\left(1+\log_+\frac{\sqrt{xy}}{|x-y|}\right)^2dy=\int_{\frac{1}{2}}^{2}\frac{1}{u}\left(1+\log_+\frac{\sqrt{u}}{|1-u|}\right)^2du<\infty,\ \ x\in(0,\infty).$$

Hence,

$$R_\lambda(f)(x)-\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\frac{x}{2},|x-y|>\varepsilon}^{2x}\left(\frac{1}{x-y}+\frac{\chi_{(1,\infty)}(y)}{y}\right)f(y)dy\in L^\infty(0,\infty),\tag{28}$$

and

$$\left\|R_\lambda(f)(x)-\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\frac{x}{2},|x-y|>\varepsilon}^{2x}\left(\frac{1}{x-y}+\frac{\chi_{(1,\infty)}(y)}{y}\right)f(y)dy\right\|_\infty\le C\|f\|_{BMO_+}.\tag{29}$$

By combining (26), (27),(28) and (29) we conclude that

$$R_\lambda(f)-H(f_e)\in L^\infty(0,\infty)\quad\text{and}\quad\|R_\lambda(f)-H(f_e)\|_\infty\le C\|f\|_{BMO_+}.\tag{30}$$

Moreover, since $H(f_e)\in BMO(\mathbb{R})$ and $H(f_e)$ is odd, for every $a\in(0,\infty)$,

$$\begin{aligned}
\frac{1}{a}\int_0^a|H(f_e)(x)|dx&=\frac{1}{2a}\int_{-a}^a|H(f_e)(x)|dx=\frac{1}{2a}\int_{-a}^a\left|H(f_e)(x)-\frac{1}{2a}\int_{-a}^aH(f_e)(u)du\right|dx\\
&\le C\|H(f_e)\|_{BMO(\mathbb{R})}\le C\|f_e\|_{BMO(\mathbb{R})}\le C\|f\|_{BMO_+}.
\end{aligned}$$

Then, from (30), for every $a\in(0,\infty)$,

$$\frac{1}{a}\int_0^a|R_\lambda(f)(x)|dx\le\frac{1}{a}\int_0^a|R_\lambda(f)(x)-H(f_e)(x)|dx+\frac{1}{a}\int_0^a|H(f_e)(x)|dx\le C\|f\|_{BMO_+}.$$

Hence $R_\lambda(f)\in BMO_+$ and $\|R_\lambda f\|_{BMO_+}\le C\|f\|_{BMO_+}$. Thus the proof of Proposition 2 for R_λ is finished.

4. Littlewood-Paley g-functions in BMO_+

In this section we prove Proposition 2 for the Littlewood-Paley g-functions $g_{h,\lambda}$ and $g_{P,\lambda}$ associated with the heat and the Poisson semigroups for Δ_λ, respectively.

Firstly we study $g_{h,\lambda}$. Let $f \in BMO_+$. Minkowski inequality implies that

$$\left| g_{h,\lambda}(f)(x) - \left\{ \int_0^\infty \left| t\frac{t}{\partial t} \int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y) f(y) dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \right|$$

$$\leq \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t} \left(\int_0^{\frac{x}{2}} + \int_{2x}^\infty \right) W^\lambda(t,x,y) f(y) dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}$$

$$\leq \left(\int_0^{\frac{x}{2}} + \int_{2x}^\infty \right) |f(y)| \left\{ \int_0^\infty t \left| \frac{\partial}{\partial t} W^\lambda(t,x,y) \right|^2 dt \right\}^{\frac{1}{2}} dy, \quad x \in (0,\infty).$$

According to [7, Lemma 8] we have that

$$\left\{ \int_0^\infty t \left| \frac{\partial}{\partial t} W^\lambda(t,x,y) \right|^2 dt \right\}^{\frac{1}{2}} \leq C \begin{cases} \dfrac{y^\lambda}{x^{\lambda+1}}, \; 0 < y < \dfrac{x}{2}, \\ \dfrac{x^\lambda}{y^{\lambda+1}}, \; 2x < y < \infty. \end{cases} \tag{31}$$

From (31) we deduce that, for every $x \in (0,\infty)$,

$$\int_0^{\frac{x}{2}} |f(y)| \left\{ \int_0^\infty t \left| \frac{\partial}{\partial t} W^\lambda(t,x,y) \right|^2 dt \right\}^{\frac{1}{2}} dy \leq C \int_0^{\frac{x}{2}} \frac{|f(y)| y^\lambda}{x^{\lambda+1}} dy \leq \frac{C}{x} \int_0^x |f(y)| dy \leq C \|f\|_{BMO_+},$$

and, as in the proof of (14) in Section 2,

$$\int_{2x}^\infty |f(y)| \left\{ \int_0^\infty t \left| \frac{\partial}{\partial t} W^\lambda(t,x,y) \right|^2 dt \right\}^{\frac{1}{2}} dy \leq C x^\lambda \int_{2x}^\infty \frac{|f(y)|}{y^{\lambda+1}} dy \leq C \|f\|_{BMO_+}.$$

Hence, we conclude that

$$g_{h,\lambda}(f)(x) - \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t} \int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y) f(y) dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \in L^\infty(0,\infty), \tag{32}$$

and

$$\left\| g_{h,\lambda}(f)(x) - \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t} \int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y) f(y) dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \right\|_\infty \leq C \|f\|_{BMO_+}. \tag{33}$$

By using again Minkowski inequality and [7, Lemma 8] we get

$$\left|\left\{\int_0^\infty \left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y)f(y)dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}} - \left\{\int_0^\infty \left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}\right|$$

$$\begin{aligned}
&\leq \left\{\int_0^\infty \left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}\left(W^\lambda(t,x,y)-\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\right)f(y)dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}\\
&\leq \int_{\frac{x}{2}}^{2x}|f(y)|\left\{\int_0^\infty t\left|\frac{\partial}{\partial t}\left(W^\lambda(t,x,y)-\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\right)\right|^2dt\right\}^{\frac{1}{2}}dy\\
&\leq C\frac{1}{x}\int_{\frac{x}{2}}^{2x}|f(y)|dy \leq C\|f\|_{BMO_+},\quad x\in(0,\infty).
\end{aligned}$$

Then

$$\left\{\int_0^\infty \left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y)f(y)dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}} - \left\{\int_0^\infty \left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}} \in L^\infty(0,\infty) \tag{34}$$

and

$$\left\|\left\{\int_0^\infty \left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y)f(y)dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}} - \left\{\int_0^\infty \left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}\right\|_\infty \leq C\|f\|_{BMO_+}. \tag{35}$$

We denote by

$$g_h(f_o)(x) = \left\{\int_0^\infty \left|t\frac{\partial}{\partial t}W_t(f_o)(x)\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}},\quad x\in\mathbb{R}.$$

We are going to see that

$$g_h(f_o)(x) - \left\{\int_0^\infty \left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}} \in L^\infty(0,\infty) \tag{36}$$

and

$$\left\| g_h(f_o)(x) - \left\{ \int_0^\infty \left| t \frac{\partial}{\partial t} \int_{\frac{x}{2}}^{2x} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} f(y)dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \right\|_\infty \leq C\|f\|_{BMO_+}. \tag{37}$$

Note firstly that according to (16) the Minkowski inequality leads to

$$\begin{aligned}
(38)\quad & \left| g_h(f_o)(x) - \left\{ \int_0^\infty \left| t \frac{\partial}{\partial t} \int_{\frac{x}{2}}^{2x} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} f(y)dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \right| \\
&\leq \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t} \int_0^\infty \frac{1}{\sqrt{4\pi t}} \left(e^{-\frac{(x-y)^2}{4t}} - e^{\frac{(x+y)^2}{4t}} \right) f(y)dy - t\frac{\partial}{\partial t} \int_{\frac{x}{2}}^{2x} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} f(y)dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \\
&\leq \int_{\frac{x}{2}}^{2x} |f(y)| \left\{ \int_0^\infty t \left| \frac{\partial}{\partial t} \left(\frac{1}{\sqrt{4\pi t}} e^{-\frac{(x+y)^2}{4t}} \right) \right|^2 dt \right\}^{\frac{1}{2}} dy \\
&\quad + \left(\int_0^{\frac{x}{2}} + \int_{2x}^\infty \right) |f(y)| \left\{ \int_0^\infty t \left| \frac{\partial}{\partial t} \left(\frac{1}{\sqrt{4\pi t}} \left[e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right] \right) \right|^2 dt \right\}^{\frac{1}{2}} dy \\
&= T_1(f)(x) + T_2(f)(x), \quad x \in (0,\infty).
\end{aligned}$$

It is not hard to see that

$$\left| \frac{\partial}{\partial t} \left(\frac{1}{\sqrt{4\pi t}} e^{-\frac{(x+y)^2}{4t}} \right) \right| \leq C \frac{1}{t^{\frac{3}{2}}} e^{-\frac{(x+y)^2}{8t}}, \quad t, x, y \in (0,\infty).$$

Then

$$\begin{aligned}
T_1(f)(x) &\leq C \int_{\frac{x}{2}}^{2x} |f(y)| \left\{ \int_0^\infty \frac{1}{t^2} e^{-\frac{(x+y)^2}{4t}} dt \right\}^{\frac{1}{2}} dy \leq C \int_{\frac{x}{2}}^{2x} \frac{|f(y)|}{x+y} dy \\
&\leq C \frac{1}{x} \int_0^{2x} |f(y)|dy \leq C\|f\|_{BMO_+}, \quad x \in (0,\infty).
\end{aligned}$$

Also, by using the mean value theorem, we get

$$\begin{aligned}
(39)\quad & \left| \frac{\partial}{\partial t} \left(\frac{1}{\sqrt{4\pi t}} \left[e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right] \right) \right| \\
&\leq C \frac{1}{t^{\frac{3}{2}}} \left(\left| e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right| + \left| \frac{(x-y)^2}{4t} e^{-\frac{(x-y)^2}{4t}} - \frac{(x+y)^2}{4t} e^{-\frac{(x+y)^2}{4t}} \right| \right) \\
&\leq C \frac{xy}{t^{\frac{5}{2}}} e^{-\frac{(x-y)^2}{8t}}, \qquad t \in (0,\infty) \text{ and } 0 < y < \frac{x}{2}, \text{ or } y > 2x.
\end{aligned}$$

Hence

$$\begin{aligned} T_2(f)(x) &\leq C\left(\int_0^{\frac{x}{2}}+\int_{2x}^{\infty}\right)|f(y)|xy\left\{\int_0^{\infty}\frac{e^{-\frac{(x-y)^2}{8t}}}{t^4}dt\right\}^{\frac{1}{2}}dy \leq C\left(\int_0^{\frac{x}{2}}+\int_{2x}^{\infty}\right)|f(y)|\frac{xy}{|x-y|^3}dy \\ &\leq C\int_0^{\frac{x}{2}}|f(y)|\frac{y}{x^2}dy + x\int_{2x}^{\infty}\frac{|f(y)|}{y^2}dy \leq C\frac{1}{x}\int_0^x |f(y)|dy + x\int_{2x}^{\infty}\frac{|f(y)|}{y^2}dy, \end{aligned}$$

and by proceeding as in the proof of (17) we obtain that $T_2(f)(x) \leq C\|f\|_{BMO_+}, x \in (0,\infty)$. From (38) we deduce (36) and (37). By using (33), (35) and (37) we conclude that $g_{h,\lambda}(f) \in BMO_+$ provided that $g_h(f_o) \in BMO(\mathbb{R})$, and that there exists $C > 0$ such that

$$\frac{1}{a}\int_0^a g_h(f_o)(x)dx \leq C, \quad a \in (0,\infty).$$

Since $f_o \in BMO(\mathbb{R})$, $g_h(f_o) \in BMO(\mathbb{R})$ when $g_h(f_o)(x) < \infty$, a.e. $x \in \mathbb{R}$ ([17]). Let $a > 0$. We write $f_o = f_1 + f_2 + f_3$, where

$$\begin{aligned} f_1(x) &= \frac{1}{2a}\int_0^{2a} f(y)dy := f_{(0,2a)}, \quad x \in (0,\infty), \\ f_2(x) &= (f(x) - f_{(0,2a)})\chi_{(0,2a)}(x), \quad x \in (0,\infty), \\ f_3(x) &= (f(x) - f_{(0,2a)})\chi_{(2a,\infty)}(x), \quad x \in (0,\infty), \end{aligned}$$

and $f_i(x) = -f_i(-x), \quad x \in (-\infty,0), \quad \text{and} \quad i = 1,2,3.$

Note that, for each $x \in (0,\infty)$,

$$g_h(f_1)(x) = \left|f_{(0,2a)}\right|\left\{\int_0^{\infty}\left|t\frac{\partial}{\partial t}\int_0^{\infty}\frac{1}{\sqrt{4\pi t}}\left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}\right)dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}} \tag{40}$$

$$= \frac{2}{\sqrt{\pi}}\left|f_{(0,2a)}\right|\left\{\int_0^{\infty}\left|t\frac{\partial}{\partial t}\int_0^{\frac{x}{2\sqrt{t}}} e^{-u^2}du\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}$$

$$\leq C\|f\|_{BMO_+}\left\{\int_0^{\infty}\left|t\frac{x}{4t^{\frac{3}{2}}}e^{-\frac{x^2}{4t}}\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}} = C\|f\|_{BMO_+}\left\{\int_0^{\infty}\frac{x^2}{t^2}e^{-\frac{x^2}{2t}}dt\right\}^{\frac{1}{2}} \leq C\|f\|_{BMO_+}.$$

Since $g_h(f_1)$ is even, $g_h(f_1)(x) \leq C\|f\|_{BMO_+}$, $x \in \mathbb{R}$. It is wellknown that g_h is a bounded operator from $L^2(\mathbb{R})$ into itself. Then

$$\begin{aligned} \int_0^a g_h(f_2)(x)dx &\leq \sqrt{a}\left\{\int_{-\infty}^{+\infty}|g_h(f_2)(x)|^2dx\right\}^{\frac{1}{2}} \leq C\sqrt{a}\left\{\int_0^{2a}\left|f(x) - f_{(0,2a)}\right|^2dx\right\}^{\frac{1}{2}} \\ &\leq Ca\|f\|_{BMO_+}. \end{aligned} \tag{41}$$

Hence, since $g_h(f_2)(x)$ is even, $g_h(f_2)(x) < \infty$, a.e. $x \in (-a,a)$.

Finally, by proceeding as in the proof of (17), we obtain

$$\begin{aligned}\int_0^a g_h(f_3)(x)dx &= \int_0^a g_h((f-f_{(0,2a)})\chi_{(2a,\infty)})(x)dx \leq C\int_0^a x\int_{2x}^\infty |f(y)-f_{(0,2a)}|\frac{dy}{y^2}dx\\ &\leq C\left(\int_0^a x\int_{2x}^\infty |f(y)|\frac{dy}{y^2}\,dx + |f_{(0,2a)}|\int_0^a x\int_{2x}^\infty \frac{dy}{y^2}dx\right) \leq Ca\|f\|_{BMO_+}. \qquad (42)\end{aligned}$$

Then, $g_h(f_3)(x) < \infty$, a.e. $x \in (-a,a)$. We conclude that $g_h(f_o)(x) < \infty$, a.e. $x \in (-a,a)$. Hence, since $a > 0$ is arbitrary $g_h(f_0)(x) < \infty$, a.e. $x \in \mathbb{R}$, and then $g_h(f_o) \in BMO(\mathbb{R})$. Moreover, from (40), (41) and (42) we obtain that,

$$\frac{1}{a}\int_0^a |g_h(f)(x)|dx \leq C\|f\|_{BMO_+}, \quad a > 0.$$

Thus, we deduce that $g_{h,\lambda}(f) \in BMO_+$ and $\|g_{h,\lambda}(f)\|_{BMO_+} \leq C\|f\|_{BMO_+}$.

To analyze the Littlewood-Paley g-function $g_{P,\lambda}$ associated with the Poisson semigroup $\{P_t^\lambda\}_{t>0}$ for the Bessel operator, we can proceed as for the $g_{h,\lambda}$ case. We compare $g_{P,\lambda}$ with the g-function for the classical Poisson semigroup on $\mathbb{R}$.

According to the results established in [6, (2.11)] we can write

$$\left\{\int_0^\infty \left|t\frac{\partial}{\partial t}\left[P^\lambda(t,x,y) - \frac{1}{\pi}\chi_{\{\frac{x}{2}<y<2x\}}(y)\frac{t}{(x-y)^2+t^2}\right]\right|^2 \frac{dt}{t}\right\}^{\frac{1}{2}}$$
$$\leq C(xy)^\lambda \begin{cases} x^{-2\lambda-1}, & 0 < y \leq \frac{x}{2},\\ y^{-2\lambda-1}\left(1+\log\left(1+\frac{xy}{|x-y|^2}\right)\right), & \frac{x}{2} < y < 2x,\\ y^{-2\lambda-1}, & y \geq 2x.\end{cases}$$

Then, by using Minkowski inequality we get, for every $x \in (0,\infty)$,

$$\left|g_{P,\lambda}(f)(x) - \frac{1}{\pi}\left\{\int_0^\infty \left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}\frac{t}{(x-y)^2+t^2}f(y)dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}\right| \qquad (43)$$
$$\leq \left\{\int_0^\infty \left|t\frac{\partial}{\partial t}\left(\int_0^\infty P^\lambda(t,x,y)f(y)dy - \frac{1}{\pi}\int_{\frac{x}{2}}^{2x}\frac{t}{(x-y)^2+t^2}f(y)dy\right)\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}$$
$$\leq \int_0^\infty |f(y)|\left\{\int_0^\infty \left|t\frac{\partial}{\partial t}\left(P^\lambda(t,x,y) - \frac{1}{\pi}\chi_{\{\frac{x}{2}<y<2x\}}(y)\frac{t}{(x-y)^2+t^2}\right)\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}} \leq C\|f\|_{BMO_+}.$$

On the other hand a straightforward manipulation allows us to write, for each $x \in \mathbb{R}$,

$$P_t(f_o)(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{t}{(x-y)^2+t^2}f_o(y)dy = \frac{1}{\pi}\int_0^\infty\left(\frac{t}{(x-y)^2+t^2} - \frac{t}{(x+y)^2+t^2}\right)f(y)dy.$$

We are going to see that

$$g_P(f_o)(x) - \left\{ \int_0^\infty \left| \frac{t}{\pi} \frac{\partial}{\partial t} \int_{\frac{x}{2}}^{2x} \frac{t}{(x-y)^2+t^2} f(y)dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \in L^\infty(0,\infty) \tag{44}$$

and

$$\left\| g_P(f_o)(x) - \left\{ \int_0^\infty \left| \frac{t}{\pi} \frac{\partial}{\partial t} \int_{\frac{x}{2}}^{2x} \frac{t}{(x-y)^2+t^2} f(y)dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \right\|_\infty \leq C\|f\|_{BMO_+}, \tag{45}$$

where

$$g_P(f_o)(x) = \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t} P_t(f_o)(x) \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}, \quad x \in \mathbb{R},$$

being $P_t f_o$ as in (22). Indeed, Minkowski inequality implies, for every $x \in (0,\infty)$,

$$\begin{aligned}
&\left| g_P(f_o)(x) - \left\{ \int_0^\infty \left| \frac{t}{\pi} \frac{\partial}{\partial t} \int_{\frac{x}{2}}^{2x} \frac{t}{(x-y)^2+t^2} f(y)dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \right| \\
&\quad \leq \frac{1}{\pi} \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t} \left[\int_0^\infty \left(\frac{t}{(x-y)^2+t^2} - \frac{t}{(x+y)^2+t^2} \right) f(y)dy \right.\right.\right. \\
&\quad \left.\left.\left. - \int_{\frac{x}{2}}^{2x} \frac{t}{(x-y)^2+t^2} f(y)dy \right] \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \\
&\quad \leq \frac{1}{\pi} \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t} \int_{\frac{x}{2}}^{2x} \frac{t}{(x+y)^2+t^2} f(y)dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \\
&\quad + \frac{1}{\pi} \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t} \left(\int_0^{\frac{x}{2}} + \int_{2x}^\infty \right) \left(\frac{t}{(x-y)^2+t^2} - \frac{t}{(x+y)^2+t^2} \right) f(y)dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \\
&\quad \leq \frac{1}{\pi} \int_{\frac{x}{2}}^{2x} |f(y)| \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t} \frac{t}{(x+y)^2+t^2} \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} dy \\
&\quad + \frac{1}{\pi} \left(\int_0^{\frac{x}{2}} + \int_{2x}^\infty \right) |f(y)| \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t} \left(\frac{t}{(x-y)^2+t^2} - \frac{t}{(x+y)^2+t^2} \right) \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} dy.
\end{aligned} \tag{46}$$

Moreover, we have that

$$\int_0^\infty \left| t\frac{\partial}{\partial t} \frac{t}{(x+y)^2+t^2} \right|^2 \frac{dt}{t} \leq C \int_0^\infty \frac{t}{((x+y)^2+t^2)^2} dt \leq \frac{C}{(x+y)^2} \leq \frac{C}{x^2}, \quad \frac{x}{2} < y < 2x,$$

and

$$\int_0^\infty \left| t\frac{\partial}{\partial t}\left(\frac{t}{(x-y)^2+t^2} - \frac{t}{(x+y)^2+t^2}\right)\right|^2 \frac{dt}{t} \leq C \begin{cases} \frac{x^2}{y^4}, \ 2x < y, \\ \frac{y^2}{x^4}, \ 0 < y \leq \frac{x}{2}. \end{cases}$$

Hence, (46) leads to

$$\left| g_p(f_o)(x) - \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} \frac{t}{(x-y)^2+t^2} f(y)dy\right|^2 \frac{dt}{t}\right\}^{\frac{1}{2}} \right|$$

$$\leq C\left(\frac{1}{x}\int_{\frac{x}{2}}^{2x} |f(y)|dy + \frac{1}{x}\int_0^{\frac{x}{2}} |f(y)|dy + x\int_{2x}^\infty \frac{|f(y)|}{y^2}dy\right) \leq C\|f\|_{BMO_+}, \quad x \in (0,\infty).$$

Thus (44) and (45) are established. From (43), (44) and (45) we deduce that

$$g_{P,\lambda}(f) - g_P(f_o) \in L^\infty(0,\infty) \text{ and } \|g_{P,\lambda}(f) - g_P(f_o)\|_\infty \leq C\|f\|_{BMO_+}.$$

To see that $g_{P,\lambda}(f) \in BMO_+$ and $\|g_{P,\lambda}(f)\|_{BMO_+} \leq C\|f\|_{BMO_+}$ we can proceed as in the proof of the corresponding property for $g_{h,\lambda}$.

References

[1] K. F. Andersen and R. A. Kerman, *Weighted norm inequalities for generalized Hankel conjugate transformations*, Studia Math. **71** (1981/82), 15–26.

[2] C. Bennett, R. A. DeVore, and R. Sharpley, *Weak-L^∞ and BMO*, Ann. of Math. (2) **113** (1981), 601–611.

[3] J. J. Betancor, D Buraczewski, J. C. Fariña, T. Martínez, and J. L. Torrea, *Riesz transforms related to Bessel operators*, Proc. Roy. Soc. Edinburgh Sect. A **137** (2007), 701–725.

[4] J. J. Betancor, J. Dziubański, and J. L. Torrea, *Remarks on Hardy spaces associated with Bessel operators*, J. Analyse Math. (to appear).

[5] J. J. Betancor, J. C. Fariña, T. Martínez, and L. Rodríguez-Mesa, *Higher order Riesz transforms associated with Bessel operators*, Ark. Mat. **46** (2008), 219–250.

[6] J. J. Betancor, J. C. Fariña, and A. Sanabria, *On Littlewood-Paley functions associated with Bessel operators*, Glasg. Math. J. **51** (2009), 55–70.

[7] J. J. Betancor, E. Harboure, A. Nowak, and B. Viviani, *Mapping properties of fundamental operators in harmonic analysis related to Bessel operators*, preprint, (arxiv:0802.3497) (2008).

[8] J. J. Betancor and K. Stempak, *On Hankel conjugate functions*, Stud. Sci. Math. Hung. **41** (2004), 59–91.

[9] J. Dziubański, G. Garrigós, T. Martínez, J. L. Torrea, and J. Zienkiewicz, *BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality*, Math. Z. **249** (2005), 329–356.

[10] S. Fridli, *Hardy spaces generated by an integrability condition*, J. Approx. Theory **113** (2001), 91–109.

[11] N. N. Lebedev, *Special functions and their applications*, Dover Publications Inc., New York, 1972.

[12] B. Muckenhoupt and E. M. Stein, *Classical expansions and their relation to conjugate harmonic functions*, Trans. Amer. Math. Soc. **118** (1965), 17–92.

[13] E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Annals of Mathematics Studies, No. 63, Princeton University Press, Princeton, N.J., 1970.

[14] ———, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, N.J., 1993.

[15] K. Stempak and J. L. Torrea, *BMO results for operators associated to Hermite expansions*, Illinois J. Math. **49** (2005), 1111–1131.

[16] A. Torchinsky, *Real-variable methods in harmonic analysis*, Pure and Applied Mathematics, vol. 123, Academic Press Inc., Orlando, FL, 1986.

[17] S. L. Wang, *Some properties of the Littlewood-Paley g-function*, Classical real analysis, Contemp. Math., vol. 42, Amer. Math. Soc., Providence, RI, 1985, pp. 191–202.

[18] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, Cambridge, 1995.

[19] A. Weinstein, *Discontinuous integrals and generalized potential theory*, Trans. Amer. Math. Soc. **63** (1948), 342–354.

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