ON HAAR BASES FOR GENERALIZED DYADIC HARDY SPACES

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Abstract. In this note we prove that Haar type systems are unconditional basis in the generalized dyadic Hardy space $H_1^{\mathcal{D}}$ in the setting of spaces of homogeneous type. As a consequence, we obtain an alternative proof of the unconditionality of such basis in Lebesgue spaces on spaces of homogeneous type.

1. Introduction

The theory of Hardy spaces on non-isotropic settings and non-Euclidean spaces is not new. Calderón and Torchinsky [6] initiated the study of Hardy spaces on $\mathbb{R}^n$ with anisotropic dilations and Macías and Segovia study in [12] the Hardy spaces in the general setting of space of homogeneous type. In 1980 L. Carlesson study the existence of unconditional basis on the Hardy spaces $H^1(\mathbb{R}^n)$, [7]. More precisely, he gives an explicit wavelet basis that is unconditional in $H^1(\mathbb{R}^n)$. In the proof given in [7] the regularity of the basic functions is crucial. In the books [10] and [14] conditions on the wavelets for unconditionality in $H^1(\mathbb{R}^n)$ can also found. This conditions are given in terms of regularity of the functions of the wavelet basis and therefore the same proof does not hold in the dyadic context for the Haar wavelet in $\mathbb{R}^n$. Notice that on an abstract metric space (X, d), for which no smoothness better than Lipschitz continuity makes sense, the first basic prototype of localized wavelet is the Haar wavelet. In this note we shall proof that Haar type basis $\mathcal{H}$ are unconditional bases for the atomic dyadic Hardy space $H_1^{\mathcal{D}}$, in the general context of space of homogeneous type. As a consequence, we shall obtain a new proof of the unconditionality of Haar systems in Lebesgue spaces on spaces of homogeneous type. We would like to point out that in [9] the authors prove that the usual Haar basis in the Euclidean context is an unconditional basis in the weighted dyadic Hardy space $H^p_{dy}(w)$ for every $0 < p \leq 1$ and every $w \in A_\infty^{dy}$ using the maximal approach to the definition of Hardy spaces. Our proof is addressed from an atomic approach to the definition of Hardy spaces and therefore, for the particular case of $\mathbb{R}^n$ and $w = 1$, we obtain a new proof of the result in [9].

The paper is organized as follows. In Section 2 we introduce the basic definitions of dyadic family in the class $\mathfrak{D}(\delta)$ and of Haar system $\mathcal{H}$ associated. Also, In Section 2 we define the dyadic Hardy space $H_1^{\mathcal{D}}$ and the dyadic bounded mean oscillation

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spaces $BMO^{\mathcal{D}}$ associated to a dyadic family $\mathcal{D}$. Section 3 is devoted to prove our main result which is stated at the end of Seccion 2.

2. Definitions, notation and statement of the result

Let us recall the basic properties of the general theory of spaces of homogeneous type. Assume that X is a set, a nonnegative symmetric function d on $X \times X$ is called a quasi-distance if there exists a constant K such that

$$d(x,y) \leq K[d(x,z) + d(z,y)], \tag{2.1}$$

for every $x, y, z \in X$, and $d(x,y) = 0$ if and only if $x = y$.

We shall say that (X, d, μ) is a space of homogeneous type if d is a quasi-distance on X, μ is a positive Borel measure defined on a σ-algebra of subsets of X which contains the balls, and there exists a constant A such that the inequalities

$$0 \ < \ \mu(B(x, 2r)) \ \leq \ A \ \mu(B(x,r)) \ < \ \infty$$

hold for every $x \in X$ and every $r > 0$.

The sets $\{(x,y) \in X \times X : d(x,y) < 1/n\}$ define a basis of a metrizable uniform structure on X and the balls $B(x,r) = \{y : d(x,y) < r\}$ form a basis of neighborhoods of x for the topology induced by the uniform structure. It is well known that the d-balls are generally not open sets. Moreover, sometimes some balls are not even Borel measurable subsets of X. Nevertheless in [11], R. Macias and C. Segovia prove that if d is a quasi-distance on X, then there exist a distance ρ and a number $\alpha \geq 1$ such that d is equivalent to ρ^α. Hence we shall assume along this paper that (X, d, μ) is a space of homogeneous type with d a distance on X, in other words that $K = 1$ in (2.1). In order to be able to apply Lebesgue Differentiation Theorem we shall also assume that continuous functions are dense in $L^1(X, \mu)$.

The construction of dyadic type families of subsets in metric or quasi-metric spaces with some inner and outer metric control of the sizes of the dyadic sets is given in [8]. These families satisfy all the relevant properties of the usual dyadic cubes in $\mathbb{R}^n$. Actually the only properties of Christ's cubes needed in our further analysis are contained in the next definition which we borrow from [3].

Definition 2.1. The class $\mathfrak{D}(\delta)$ of all dyadic families. We say that $\mathcal{D} = \bigcup_{j \in \mathbb{Z}} \mathcal{D}^j$ is a dyadic family on X with parameter $\delta \in (0,1)$, briefly that $\mathcal{D}$ belong $\mathfrak{D}(\delta)$, if each $\mathcal{D}^j$ is a family of open subsets Q of X, such that

- *(d.1) For every $j \in \mathbb{Z}$ the cubes in $\mathcal{D}^j$ are pairwise disjoints.*
- *(d.2) For every $j \in \mathbb{Z}$ the family $\mathcal{D}^j$ covers almost all X in the sense that $\mu(X - \bigcup_{Q \in \mathcal{D}^j} Q) = 0$.*
- *(d.3) If $Q \in \mathcal{D}^j$ and $i < j$, then there exists a unique $\tilde{Q} \in \mathcal{D}^i$ such that $Q \subseteq \tilde{Q}$.*
- *(d.4) If $Q \in \mathcal{D}^j$ and $\tilde{Q} \in \mathcal{D}^i$ with $i \leq j$, then either $Q \subseteq \tilde{Q}$ or $Q \cap \tilde{Q} = \emptyset$.*
- *(d.5) There exist two constants a_1 and a_2 such that for each $Q \in \mathcal{D}^j$ there exists a point $x \in Q$ for which $B(x, a_1\delta^j) \subseteq Q \subseteq B(x, 2a_2\delta^j)$.*

The following properties for a dyadic family $\mathcal{D}$ in the class $\mathfrak{D}(\delta)$ follow from the above definition (see [3]).

Proposition 2.2. *Let $\mathcal{D}$ be a dyadic family in the class $\mathfrak{D}(\delta)$. Then* (d.6) *There exists a positive integer N depending only on the doubling constant such that for every $j \in \mathbb{Z}$ and all $Q \in \mathcal{D}^j$ the inequalities $1 \leq \#(\mathcal{L}(Q)) \leq N$ hold, where $\mathcal{L}(Q) = \{Q^{'} \in \mathcal{D}^{j+1} : Q^{'} \subseteq Q\}$.*

(d.7) *For every dyadic cube Q in $\mathcal{D}$ we have that $\mu(\partial Q) = 0$, where ∂Q is the boundary of Q;*

(d.8) *There exists a finite and positive constant C such that for each $j \in \mathbb{Z}$ and every dyadic cube Q in $\mathcal{D}^j$ we have that $\mu(Q) \leq C\mu(Q^{'})$ for all dyadic cubes $Q^{'}$ in $\mathcal{D}^{j+1}$ with $Q^{'} \subseteq Q$;*

(d.9) *X is bounded if and only if there exists a dyadic cube Q in $\mathcal{D}$ such that $X = Q$;*

(d.10) *The families $\tilde{\mathcal{D}}^j = \{Q \in \mathcal{D}^j : \#(\{Q^{'} \in \mathcal{D}^{j+1} : Q^{'} \subseteq Q\}) > 1\}$, $j \in \mathbb{Z}$ are pairwise disjoints.*

Associated with the construction of Christ, in [2] the authors introduce the concept of quadrant on a space of homogeneous type. We extend such definition for our general context of dyadic families in $\mathfrak{D}(\delta)$

Definition 2.3. Let $\mathcal{D}$ be a dyadic family in the class $\mathfrak{D}(\delta)$. We define, for each dyadic cube Q in $\mathcal{D}$, the quadrant of X that contain the cube Q, $\mathcal{C}(Q)$, by

$$\mathcal{C}(Q) = \bigcup_{\{Q^{'} \in \mathcal{D} : Q \subseteq Q^{'}\}} Q^{'}.$$

Given a dyadic family $\mathcal{D}$ in the class $\mathfrak{D}(\delta)$ we can define Haar type systems associated to $\mathcal{D}$.

Definition 2.4. Haar system associated to $\mathcal{D} \in \mathfrak{D}(\delta)$. Let $\mathcal{D}$ be a dyadic family on X such that $\mathcal{D} \in \mathfrak{D}(\delta)$. A system $\mathcal{H}$ of Borel measurable simple real functions h on X is a Haar system associated to $\mathcal{D}$ if it satisfies

(h.1) *For each $h \in \mathcal{H}$ there exists a unique $j \in \mathbb{Z}$ and a cube $Q = Q(h) \in \tilde{\mathcal{D}}^j$ such that $\{x \in X : h(x) \neq 0\} \subseteq Q$, and this property does not hold for any cube in $\mathcal{D}^{j+1}$.*

(h.2) *For every $Q \in \tilde{\mathcal{D}} = \bigcup_{j \in \mathbb{Z}} \tilde{\mathcal{D}}^j$ there exist exactly $M_Q = \#(\mathcal{L}(Q)) - 1 \geq 1$ functions $h \in \mathcal{H}$ such that (h.1) holds. We shall write $\mathcal{H}_Q$ to denote the set of all these functions h.*

(h.3) *For each $h \in \mathcal{H}$ we have that $\int_X h d\mu = 0$.*

(h.4) *For each $Q \in \tilde{\mathcal{D}}$ let V_Q denote the vector space of all functions on Q which are constant on each $Q^{'} \in \mathcal{L}(Q)$. Then the system $\{\frac{\chi_Q}{(\mu(Q))^{1/2}}\} \bigcup \mathcal{H}_Q$ is an orthonormal basis for V_Q.*

In [1] and [2] the authors built wavelets of Haar type which are supported on Christ's dyadic cubes. Such construction is applicable to any dyadic family $\mathcal{D}$ in the class $\mathfrak{D}(\delta)$ and therefore there are always systems of functions that satisfy *(h.1)* to *(h.4)* for any dyadic family $\mathcal{D}$. The following result is an easy consequence of Definition 2.4. We shall denote with $\mathcal{L}^p(X,\mu), (p \geq 1)$ the space $L^p(X,\mu)$ when $\mu(X) = \infty$ and the space $L^p_0 = \{f \in L^p(X,\mu) : \int_X f d\mu = 0\}$ if $\mu(X) < \infty$.

Theorem 2.5. *Let $\mathcal{D}$ be a dyadic family on X such that $\mathcal{D}$ belong to class $\mathfrak{D}(\delta)$. Then every Haar type system $\mathcal{H}$ associated to $\mathcal{D}$ is an orthonormal basis in $\mathcal{L}^2(X,\mu)$.*

Now we shall introduce the dyadic Hardy space $H_1^{\mathcal{D}}$ on a space of homogeneous type (X,d,μ) following the lines in [14] for the Hardy space H_1 on $\mathbb{R}^n$. We start giving the definition of dyadic atom associated to a dyadic family $\mathcal{D}$ in the class $\mathfrak{D}(\delta)$.

Definition 2.6. Let $\mathcal{D}$ be a dyadic family in the class $\mathfrak{D}(\delta)$. For $1<q\leq\infty$ we shall say that a function a defined on X is a dyadic q-atom associated to $\mathcal{D}$, briefly that $a\in\mathcal{A}_{q,\mathcal{D}}$, if there exists a dyadic cube Q in $\mathcal{D}$ such that

(a1) supp $(a(x))\subseteq Q$.
(a2) $\int_X a(x)d\mu(x)=0$.
(a3) $\|a\|_{L^q(X,\mu)}\leq(\mu(Q))^{\frac{1}{q}-1}$ if $q<\infty$ and $\|a\|_{L^\infty(X,\mu)}\leq\mu(Q))^{-1}$ if $q=\infty$.

The spaces $H_1^{q,\mathcal{D}}$ on (X,d,μ) are defined as follows.

Definition 2.7. Let $\mathcal{D}$ be a dyadic family in the class $\mathfrak{D}(\delta)$. For $1<q\leq\infty$ we define the space $H_1^{q,\mathcal{D}}$ as the lineal space of all functions f on X, identifying those that are equal almost everywhere with respect to μ, that can be written as

$$f=\sum_{n\in\mathbb{Z}^+}\lambda_n\, a_n \quad \text{with} \quad \sum_{n\in\mathbb{Z}^+}|\lambda_n|<\infty, \tag{2.2}$$

where $a_n\in\mathcal{A}_{q,\mathcal{D}}$ for each n and the convergence is in the $L^1(X,\mu)$ norm.

For each function f in $H_1^{q,\mathcal{D}}$ we define the number

$$|||f|||_{1,q,\mathcal{D}}=inf\left\{\sum_{n\in\mathbb{Z}^+}|\lambda_n|<\infty:\ f=\sum_{n\in\mathbb{Z}^+}\lambda_n\, a_n,\ a_n\in\mathcal{A}_{q,\mathcal{D}}\right\}.$$

The following result is an easy consequence of Definition 2.6.

Proposition 2.8. *Let $\mathcal{D}$ be a dyadic family in the class $\mathfrak{D}(\delta)$.*

(1) If $1<q_1<q_2\leq\infty$, then $\mathcal{A}_{q_2,\mathcal{D}}\subseteq\mathcal{A}_{q_1,\mathcal{D}}$. Moreover, if $1<q\leq\infty$ and $a=a(x)$ is a q-dyadic atom, then $\|a\|_{L^1(X,\mu)}\leq 1$.
(2) For each $1<q_1<q_2\leq\infty$ we have that $H_1^{q_2,\mathcal{D}}(X,d,\mu)\subseteq H_1^{q_1,\mathcal{D}}(X,d,\mu)$ and $\|f\|_{L^1(X,\mu)}\leq|||f|||_{1,q_1,\mathcal{D}}\leq|||f|||_{1,q_2,\mathcal{D}}$.
(3) $|||.|||_{1,q,\mathcal{D}}$ is a norm and $\left(H_1^{q,\mathcal{D}},|||.|||_{1,q,\mathcal{D}}\right)$ is a Banach space, for each $1<q\leq\infty$.

Next, we need the definition of a string of spaces in duality with the spaces $H_1^{q,\mathcal{D}}$. The $BMO_p^{\mathcal{D}}$ spaces of all functions f of bounded p-mean oscillation, $1\leq p<\infty$, is defined by $BMO_p^{\mathcal{D}}=\{f:\|f\|_{*,p}<\infty\}$, where

$$\|f\|_{*,p}=\sup_{Q\in\mathcal{D}}\left(\frac{1}{\mu(Q)}\int_Q|f(x)-f_Q|^p d\mu(x)\right)^{1/p}$$

and $f_Q=\frac{1}{\mu(Q)}\int_Q f d\mu$. Since each dyadic cube Q in $\mathcal{D}$ is a space of homogeneous type with uniform doubling constant, following the lines in the proof of Theorem 6.16 in [14] we can prove the following dyadic version of John-Nirenberg inequality.

Theorem 2.9. *Let $\mathcal{D}$ be a dyadic family in the class $\mathfrak{D}(\delta)$. Then there exist two positive constants C_1 and C_2 such that for every function $f\in BMO_1^{\mathcal{D}}$, every dyadic cube $Q\in\mathcal{D}$ and every $t\geq 0$ we have the following inequality*

$$\mu\left(\{x\in Q:|f(x)-f_Q|>t\}\right)\leq C_1\mu(Q)e^{-\frac{C_2 t}{\|f\|_{*,1}}}.$$

Corollary 2.10. *Let $\mathcal{D}$ be a dyadic family in the class $\mathfrak{D}(\delta)$ and $1 \leq p < \infty$. Then there exists a positive constant C such that for each function $f \in BMO_p^{\mathcal{D}}$ we have*

$$\|f\|_{*,1} \leq \|f\|_{*,p} \leq C\|f\|_{*,1}.$$

Any of the equivalent norms $\|.\|_{*,p}$ will be denoted by $\|.\|_{BMO^{\mathcal{D}}}$. The proof of Theorem 6.18 in [14] can be adapted to our dyadic context on space of homogeneous type with the obvious changes to obtain the following result.

Theorem 2.11. *Let $\mathcal{D}$ be a dyadic family in the class $\mathfrak{D}(\delta)$. For $1 < q \leq \infty$ the spaces $H_1^{q,\mathcal{D}}$ coincide and the norms $|||.|||_{1,q,\mathcal{D}}$ are equivalent. This unique space will be denoted by $H_1^{\mathcal{D}}$ and any of the norms $|||.|||_{1,q,\mathcal{D}}$ will be denoted by $|||.|||_{1,\mathcal{D}}$. We have also that $(H_1^{\mathcal{D}})^*$, the dual of $H_1^{\mathcal{D}}$, is $BMO^{\mathcal{D}}$ in the sense that for each continuous linear functional φ on $H_1^{\mathcal{D}}$ there exists a unique (up to functions which are constant on each quadrant) function $b \in BMO^{\mathcal{D}}$ such that if f is any finite sum of atoms we have that $\varphi(f) = \int_X bf d\mu$ and that the $BMO^{\mathcal{D}}$ norm of b and the functional norm of φ are equivalent.*

Our main result in this note is contained in the next statement.

Theorem 2.12. *Let (X, d, μ) be a space of homogeneous type and let $\mathcal{H}$ be a Haar system associated to the dyadic family $\mathcal{D}$ in the class $\mathfrak{D}(\delta)$. Then the system $\mathcal{H}$ is an unconditional basis of $H_1^{\mathcal{D}}$.*

Now, Theorem 2.12, the L^2 theory for the system $\mathcal{H}$, interpolation and duality give another technique for the proof of the unconditionality of $\mathcal{H}$ in $\mathcal{L}^p(X,\mu)$, $1 < p < \infty$. In fact, as in the proof of Theorem 6.23 in [14] we can obtain an interpolation theorem from dyadic Hardy spaces in space of homogeneous type. Thus, for each $1 < p < 2$ and each finite set $F \subseteq \mathcal{H}$ we get, from Theorem 2.12 and interpolation that the following inequality

$$|| \sum_{h \in F} < f, h > h||_{\mathcal{L}^p(X,\mu)} \ \leq \ C \ ||f||_{\mathcal{L}^p(X,\mu)},$$

holds for every function $f \in \mathcal{L}^p(X,\mu)$, where $< f, h >= \int fh d\mu$.

3. Proof of Theorem 2.12

We must show the following three basic facts for $\mathcal{H}$.

(u1) The operators $\sum_{h \in F} < f, h > h$ are uniformly bounded on $H_1^{\mathcal{D}}$ with F varying on the finite subsets of $\mathcal{H}$.

(u2) Each $h \in \mathcal{H}$ defines, by $h^*(f) =< f, h >$, a continuous linear functional on $H_1^{\mathcal{D}}$ and for every h and $\tilde{h}$ in $\mathcal{H}$ holds that $h^*(\tilde{h}) = 0$ if $h \neq \tilde{h}$ and $h^*(\tilde{h}) = 1$ if $h = \tilde{h}$.

(u3) The linear span of $\mathcal{H}$ is dense in $H_1^{\mathcal{D}}$.

Let us start by showing (u1) for dyadic atoms.

Proposition 3.1. *Let $\mathcal{H}$ be a Haar system associated to a dyadic family $\mathcal{D}$ in the class $\mathfrak{D}(\delta)$. Then for each dyadic ∞-atom a and for each finite set $F \subseteq \mathcal{H}$ we have that*

$$\left|\left|\left| \sum_{h \in F} < a, h > h \right|\right|\right|_{1,\mathcal{D}} \leq 1.$$

Proof. Let F be a finite subset of $\mathcal{H}$ and let a be a dyadic ∞-atom. We shall write Q_a to denote the dyadic cube in Definition 2.6 for the dyadic atom a. Let $j_0 \in \mathbb{Z}$ such that $Q_a \in \mathcal{D}^{j_0}$. Set $F_1 = \{h \in F : Q(h) \in \tilde{\mathcal{D}}^j, j \leq j_0\}$ and $F_2 = \{h \in F : Q(h) \in \tilde{\mathcal{D}}^j, j > j_0\}$, where $Q(h)$ is the dyadic cube in $\tilde{\mathcal{D}}$ given in *(h.1)*. Let us first consider $h \in F_1$. Since Q_a and $Q(h)$ are dyadic cubes in the dyadic family $\mathcal{D}$, from *(d.4)* and the definition of F_1 we get that $Q_a \subseteq Q(h)$ or $Q_a \bigcap Q(h) = \emptyset$. Clearly, if $Q_a \bigcap Q(h) = \emptyset$ then $< a, h >= 0$. If $Q_a \subseteq Q(h)$ then, from *(h.4)*, we have that h is constant in the dyadic cube Q_a. Thus, from (a2), we get that

$$\begin{aligned} < a, h > &= \int_{Q_a} a(x)h(x)d\mu(x) \\ &= c\int_{Q_a} a(x)d\mu(x) \\ &= c\int_X a(x)d\mu(x) = 0. \end{aligned}$$

Hence

$$\left|\left|\left|\sum_{h\in F_1} < a, h > h\right|\right|\right|_{1,\mathcal{D}} = 0.$$

Let us now estimate $\left|\left|\left|\sum_{h\in F_2} < a, h > h\right|\right|\right|_{1,\mathcal{D}}$. Set $f(x) = \sum_{h\in F_2} < a, h > h(x)$. From *(h.3)* we get that

$$\int_X f(x)d\mu(x) = \sum_{h\in F_2} < a, h > \int_X h(x)d\mu(x) \;=\; 0. \tag{3.1}$$

On the other hand, from Proposition 2.8, a is a dyadic 2-atom. We shall prove that f is also a dyadic 2-atom. In fact, from Theorem 2.5, Parseval identity, Bessel inequality and the definition of dyadic 2-atom we get

$$\begin{aligned} \|f\|^2_{L^2(X,\mu)} &= \left\|\sum_{h\in F_2} < a, h > h\right\|^2_{L^2(X,\mu)} \\ &= \sum_{h\in F_2} | < a, h > |^2 \\ &\leq \|a\|^2_{L^2(X,\mu)} \leq \frac{1}{\mu(Q_a)}. \end{aligned} \tag{3.2}$$

(3.3)

Notice that, from *(d.4)* and the definition of F_2 we have that $Q(h) \subseteq Q_a$ or $Q_a \bigcap Q(h) = \emptyset$, for any $h \in F_2$. As before, if $Q(h) \bigcap Q_a = \emptyset$ we get $< a, h >= 0$. If $Q(h) \subseteq Q_a$ then from *(h.1)* we have that $h(x) = 0$ for all $x \notin Q_a$. Thus

$$supp(f) \subseteq Q_a. \tag{3.4}$$

Therefore, from (3.1), (3.2) and (3.4) we obtain that f is a dyadic 2-atom and then $|||\ f\ |||_{1,\mathcal{D}} \leq 1$. Hence

$$\left|\left|\left|\sum_{h\in F} < a, h > h\right|\right|\right|_{1,\mathcal{D}} \leq 1.$$

$\square$

It is well known (see [4] and [13]) that in general it is not enough to verify that an operator is bounded on atoms to conclude that it extends boundedly to the whole Hardy space. However, as the following result shows, this is the situation in our case.

Theorem 3.2. *Let $\mathcal{H}$ be a Haar system associated to the dyadic family $\mathcal{D}$ in the class $\mathfrak{D}(\delta)$. Then there exists a positive constant C such that for each finite set $F \subseteq \mathcal{H}$ we have that*

$$\left|\left|\left| \sum_{h\in F} < f,h > h \right|\right|\right|_{1,\mathcal{D}} \leq C\, |||f|||_{1,\mathcal{D}}\,,$$

for every function $f \in H_1^{\mathcal{D}}$.

Proof. First notice that if $f \in H_1^{\mathcal{D}}$ then $< f,h >$ is well defined for every function $h \in \mathcal{H}$. In fact, $f \in L^1(X,\mu)$ and $h \in L^\infty(X,\mu)$ for each $h \in \mathcal{H}$. Thus, the result is a consequence of Proposition 3.1 and the two following statements.

(1) If $(a_n : n \in \mathbb{Z}^+) \subseteq \mathcal{A}_{\infty,\mathcal{D}}$ and $(\lambda_n : n \in \mathbb{Z}^+) \subseteq \mathbb{R}$ such that $\sum_{n\in\mathbb{Z}^+} |\lambda_n| < \infty$, then
$$< \sum_{n\in\mathbb{Z}^+} \lambda_n a_n, h >= \sum_{n\in\mathbb{Z}^+} < \lambda_n a_n, h >,$$
for all function $h \in \mathcal{H}$.

(2) For every function $f \in H_1^{\mathcal{D}}$ with $f = \sum_{n\in\mathbb{Z}^+} \lambda_n a_n$ and each finite subset F of $\mathcal{H}$ we have that
$$\sum_{n\in\mathbb{Z}^+} \lambda_n \left(\sum_{h\in F} < a_n, h > \right) h = \sum_{h\in F} \left(\sum_{n\in\mathbb{Z}^+} \lambda_n < a_n, h > \right) h,$$
where the convergence is in the sense of the $L^1(X,\mu)$ norm.

Take $f \in H_1^{\mathcal{D}}$, and suppose that (1) and (2) hold. For each $\varepsilon > 0$, from the definition of $H_1^{\mathcal{D}}$, we have that $f = \sum_{n\in\mathbb{Z}^+} \lambda_n(\varepsilon) a_n(\varepsilon)$, where $(a_n(\varepsilon) : n \in \mathbb{Z}^+) \subseteq \mathcal{A}_{\infty,\mathcal{D}}$, $(\lambda_n(\varepsilon) : n \in \mathbb{Z}^+) \subseteq \mathbb{R}$ with $\sum_{n\in\mathbb{Z}^+} |\lambda_n(\varepsilon)| < \infty$ and $|||f|||_{1,\mathcal{D}} + \varepsilon \geq \sum_{n\in\mathbb{Z}^+} |\lambda_n(\varepsilon)|$. Let F be a finite subset of $\mathcal{H}$. From (1) and (2) we have that

$$\begin{aligned} \sum_{h\in F} < f,h > h &= \sum_{h\in F} \sum_{n\in\mathbb{Z}^+} < \lambda_n(\varepsilon) a_n(\varepsilon), h > h \\ &= \sum_{n\in\mathbb{Z}^+} \sum_{h\in F} < \lambda_n(\varepsilon) a_n(\varepsilon), h > h, \end{aligned}$$

in the $L^1(X,\mu)$ sense. Thus, from Proposition 3.1 we get that

$$\begin{aligned} \left|\left|\left| \sum_{h\in F} < f,h > h \right|\right|\right|_{1,\mathcal{D}} &= \left|\left|\left| \sum_{n\in\mathbb{Z}^+} \sum_{h\in F} < \lambda_n(\varepsilon) a_n(\varepsilon), h > h \right|\right|\right|_{1,\mathcal{D}} \\ &\leq \sum_{n\in\mathbb{Z}^+} |\lambda_n(\varepsilon)| \left|\left|\left| \sum_{h\in F} < a_n(\varepsilon), h > h \right|\right|\right|_{1,\mathcal{D}} \\ &\leq C \sum_{n\in\mathbb{Z}^+} |\lambda_n(\varepsilon)| \\ &\leq C\, |||f|||_{1,\mathcal{D}} + C\varepsilon, \end{aligned}$$

for each $\varepsilon > 0$. Hence,

$$\left|\left|\left|\sum_{h\in F} < f, h > h\right|\right|\right|_{1,\mathcal{D}} \leq C\,|||f|||_{1,\mathcal{D}}\,.$$

Now we shall prove (1) and (2). We first show (1). Let $f \in H_1^{\mathcal{D}}$, $f = \sum\limits_{n\in\mathbb{Z}^+} \lambda_n a_n$ where $a_n \in \mathcal{A}_{\infty,\mathcal{D}}$ for each n. Set $S_N = \sum_{n=1}^N \lambda_n a_n$. Then, $\|f - S_N\|_{L^1(X,\mu)} \longrightarrow 0$ when $n \longrightarrow \infty$. Thus,

$$\begin{aligned}\left|\sum_{n=1}^{N} < \lambda_n a_n, h > - < \sum_{n\in\mathbb{Z}^+} \lambda_n a_n, h >\right| &= \left|< \sum_{n>N} \lambda_n a_n, h >\right| \\ &\leq \int_X \left|\sum_{n>N} \lambda_n a_n\right| |h| d\mu \\ &\leq \|f - S_N\|_{L^1(X,\mu)} \|h\|_{L^\infty(X,d\mu)}.\end{aligned}$$

Hence, since $h \in L^\infty(X, d\mu)$ for each $h \in \mathcal{H}$, (1) holds. For the proof of (2), take $f \in H_1^{\mathcal{D}}$, $f = \sum\limits_{n\in\mathbb{Z}^+} \lambda_n a_n$ where $a_n \in \mathcal{A}_{\infty,\mathcal{D}}$ for each n. Set $Z_N = \sum_{n=0}^N \lambda_n \left(\sum_{h\in F} < a_n, h >\right) h$ and $S_N = \sum_{n=0}^N \lambda_n a_n$. We shall show that if N tends to ∞ then $Z_N \longrightarrow \sum_{h\in F} < f, h > h$ in the sense of $L^1(X, \mu)$. In fact,

$$\begin{aligned}\int_X |Z_N - \sum_{h\in F} < f, h > h| d\mu &\leq \sum_{h\in F} \int_X |< S_N - f, h >|\,|h| d\mu \\ &\leq \sum_{h\in F} \|h\|_\infty\ \mu(Q(h))\,|< S_N - f, h >| \\ &\leq \|S_N - f\|_{L^1(X,\mu)} \sum_{h\in F} \|h\|_\infty^2\ \mu(Q(h)).\end{aligned}$$

□

Now we shall prove (u2). Fixed $h \in \mathcal{H}$ and $f = \sum_{n\in\mathbb{Z}^+} \lambda_n a_n \in H_1^{\mathcal{D}}$, with $a_n \in \mathcal{A}_{\infty,\mathcal{D}}$ for each n. The linearity of h^* is a trivial consequence of the linearity of the integral. Since $\int_X |a_n| d\mu \leq 1$ for every n and since $h \in L^\infty(X, \mu)$ for each $h \in \mathcal{H}$, we get that

$$\begin{aligned}| < f, h > | &= \left|\int_X h \left(\sum_{n\in\mathbb{Z}^+} \lambda_n a_n\right) d\mu\right| \\ &\leq \|h\|_\infty \sum_{n\in\mathbb{Z}^+} \int_X |\lambda_n||a_n| d\mu \\ &\leq \|h\|_\infty \sum_{n\in\mathbb{Z}^+} |\lambda_n|.\end{aligned}$$

Therefore $| < f, h > | \leq \|h\|_\infty |||f|||_{1,\mathcal{D}}$. On the other hand, from Theorem 2.5 we have that $h^*(\tilde{h}) = 0$ if $h \neq \tilde{h}$ and $h^*(\tilde{h}) = 1$ if $h = \tilde{h}$.

In order to prove (u3) we shall use the following result. In the sequel we shall denote with $\mathbb{V}$ the set of all those functions g in $H_1^{\mathcal{D}}$ that are finite sum of dyadic ∞-atoms.

Lemma 3.3. *Let $\mathcal{D}$ be a dyadic family in the class $\mathfrak{D}(\delta)$. Then*

(a) $g-\sum_{h\in F}<g,h>h$ *belong to* $\mathbb{V}$ *for all finite subset* $F\subseteq\mathcal{H}$ *and every function* $g\in\mathbb{V}$;

(b) $\mathbb{V}$ *is dense in* $H_1^{\mathcal{D}}$.

Proof. Statement (b) is an easy consequence from the definition of $H_1^{\mathcal{D}}$. On the other hand, notice that from *(h.1)*, *(h.3)* and *(h.4)* we have that $h\mu(Q(h))^{1/2}\in\mathcal{A}_{2,\mathcal{D}}$ for every $h\in\mathcal{H}$, where $Q(h)$ is the dyadic cube in *(h.1)* for h. Then the function $\sum_{h\in F}<g,h>h\in\mathbb{V}$. From this fact and since $\mathcal{A}_{\infty,\mathcal{D}}\subseteq\mathcal{A}_{2,\mathcal{D}}$ we get (a). $\square$

We shall use also the well known fact that the norm of a point f in a Banach space $\mathbb{B}$ can be computed as the least upper bound of the evaluations $\varphi(f)$ for $\varphi\in\mathbb{B}^*$ with $\|\varphi\|=1$.

Now, we shall prove the density of the linear span of $\mathcal{H}$ in $H_1^{\mathcal{D}}$. For each positive integer M we define the family $\mathcal{F}_M=\{F\subseteq\mathcal{H}:\#(F)=M\}$ and the operators defined in $H_1^{\mathcal{D}}$ by $S_F(f)=\sum_{h\in F}<f,h>h$ for $F\in\mathcal{F}_M$ and $M\in\mathbb{Z}^+$. We shall show that for each $f\in H_1^{\mathcal{D}}$ and each $\varepsilon>0$ there exists a positive integer M_ε and a set $F_\varepsilon\in\mathcal{F}_{M_\varepsilon}$ such that $|||f-S_{F_\varepsilon}(f)|||_{1,\mathcal{D}}<\varepsilon$. From Theorem 3.2, we get that $|||S_F(f)|||_{1,\mathcal{D}}\leq C|||f|||_{1,\mathcal{D}}$ for some positive constant C independent of $F\in\mathcal{F}_M$ and of $M\in\mathbb{Z}^+$. Set $\tilde{C}=sup\{\|S_F\|:F\in\mathcal{F}_M,M\in\mathbb{Z}^+\}$. Let f be a function in $H_1^{\mathcal{D}}$ and let $\varepsilon>0$ be given. From (b) in Lemma 3.3 there exists a function $g\in\mathbb{V}$ such that

$$|||f-g|||_{1,\mathcal{D}}\leq\frac{\varepsilon}{\tilde{C}+3}. \tag{3.5}$$

Also, there exists a positive integer $M_{\varepsilon,g}$ and a set $F_{\varepsilon,g}\in\mathcal{F}_{M_{\varepsilon,g}}$ such that

$$|||g-S_{F_{\varepsilon,g}}(g)|||_{1,\mathcal{D}}<\frac{2\varepsilon}{\tilde{C}+3}. \tag{3.6}$$

In fact. From (a) in Lemma 3.3 we get that $g-S_F(g)$ belong to $\mathbb{V}$ for each finite set $F\subseteq\mathcal{H}$. Thus, from the above remark on the norm of an element in a Banach space via duality and Theorem 2.11 we have that for each $F\in\mathcal{F}_M$ and $M\in\mathbb{Z}^+$ there exists a function $\varphi_{g,F}\in BMO^{\mathcal{D}}$ such that $\|\varphi_{g,F}\|_{BMO^{\mathcal{D}}}=1$ and

$$|||g-S_F(g)|||_{1,\mathcal{D}}\leq\left|\int_X(g-S_F(g))\varphi_{g,F}d\mu\right|+\frac{\varepsilon}{\tilde{C}+3}. \tag{3.7}$$

Since $g-S_F(g)$ belong to $\mathbb{V}$ and $supp(h)\subseteq\overline{Q(h)}$, where $\overline{Q(h)}$ is the closure of $Q(h)$, it is clear that there exists a finite index set $I\subseteq\mathbb{Z}^+$ and a family $\{Q_n:n\in I\}$ of disjoint dyadic cubes in $\mathcal{D}$ such that

$$\left|\int_X(g-S_F(g))\varphi_{g,F}d\mu\right|=\left|\int_X(g-S_F(g))\phi_{g,F}d\mu\right|, \tag{3.8}$$

where $\phi_{g,F}=(\chi_{\underset{n\in I}{\cup}Q_n})\varphi_{g,F}$. Notice that since $\varphi_{g,F}\in BMO^{\mathcal{D}}$, then for some constants C_n with $n\in I$ we get that

$$\begin{aligned}\|\phi_{g,F}\|_{L^2(X,\mu)} &= \sum_{n\in I}\left(\int_{Q_n}|\varphi_{g,F}|^2d\mu\right)^{1/2}\\ &\leq \sum_{n\in I}\left(\left(\int_{Q_n}|\varphi_{g,F}-C_n|^2d\mu\right)^{1/2}+\left(\int_{Q_n}|C_n|^2d\mu\right)^{1/2}\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\in I}\left(\mu(Q_n)^{1/2}\left(\frac{1}{\mu(Q_n)}\int_{Q_n}|\varphi_{g,F}-C_n|^2 d\mu\right)^{1/2} + C_n\mu(Q_n)^{1/2}\right)\\
&\leq \sum_{n\in I}\left(\|\varphi_{g,F}\|_{BMO^{\mathcal{D}}}+C_n\right)\mu(Q_n)^{1/2} < \infty.
\end{aligned}
$$

Hence $\phi_{g,F} \in L^2(X,\mu)$. Thus, since $\int_X S_f(g)\phi_{g,F}d\mu = \int_X gS_f(\phi_{g,F})d\mu$, from Schwartz inequality we get that

$$
\begin{aligned}
\left|\int_X (g - S_F(g))\phi_{g,F}d\mu\right| &= \left|\int_X g(\phi_{g,F} - S_F(\phi_{g,F})d\mu\right|\\
&\leq \|g\|_{L^2(X,\mu)}\|\phi_{g,F} - S_F(\phi_{g,F}\|_{L^2(X,\mu)},
\end{aligned}
$$

for each $F \in \mathcal{F}_M$ and each $M \in \mathbb{Z}^+$. Therefore, since $\mathcal{H}$ is a orthonormal basis in $L^2(X,\mu)$, there exists a positive integer $M_{\varepsilon,g}$ and a set $F_{\varepsilon,g} \in \mathcal{F}_{M_{\varepsilon,g}}$ such that

$$
\|\phi_{g,F} - S_{F_{\varepsilon,g}}(\phi_{g,F})\|_{L^2(X,\mu)} \leq \frac{\varepsilon}{\|g\|_{L^2(X,\mu)}(\tilde{C}+3)},
$$

which prove that (3.6) holds. Thus, from (3.5) and (3.6) we obtain that

$$
\begin{aligned}
|||f - S_{F_{\varepsilon,g}}(f)|||_{1,\mathcal{D}} &\leq |||f-g|||_{1,\mathcal{D}} + |||g - S_{F_{\varepsilon,g}}(g)|||_{1,\mathcal{D}} + |||S_{F_{\varepsilon,g}}(g-f)|||_{1,\mathcal{D}}\\
&\leq \frac{3\varepsilon}{\tilde{C}+3} + |||S_{F_{\varepsilon,g}}(g-f)|||_{1,\mathcal{D}}\\
&\leq \frac{3\varepsilon}{\tilde{C}+3} + \|S_{F_{\varepsilon,g}}\|\ |||g-f|||_{1,\mathcal{D}} \leq \varepsilon.
\end{aligned}
$$

This proves that the linear span of $\mathcal{H}$ is dense in $H_1^{\mathcal{D}}$ which concludes the proof of Theorem 2.12.

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