ON RIESZ TRANSFORMS AND MAXIMAL FUNCTIONS IN THE CONTEXT OF GAUSSIAN HARMONIC ANALYSIS

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Abstract. The purpose of this paper is twofold. We introduce a general maximal function on the Gaussian setting which dominates the Ornstein-Uhlenbeck maximal operator and prove its weak type $(1,1)$ by using a covering lemma which is halfway between Besicovitch and Wiener. On the other hand, by taking as starting point the generalized Cauchy-Riemann equations, we introduce a new class of Gaussian Riesz Transforms. We prove, using the maximal function defined in the first part of the paper, that unlike the ones already studied, these new Riesz Transforms are weak type $(1,1)$ independently of their orders.

1. Introduction and main results.

Hermite polynomials play a central role in the context of Gaussian Harmonic Analysis. They are also the building blocks for the eigenfuctions of the harmonic oscillator in Quantum Mechanics. In this context (see [17]), let us denote by $\mathcal{P}$ the one-dimensional momentum operator defined on a test function u as $\mathcal{P}u = -i\frac{\partial u}{\partial x}$ and by $\mathcal{Q}$ the position operator defined by $\mathcal{Q}u = xu$. When solving the harmonic oscillator the underlying Hamiltonian is essentially given by

$$\frac{1}{2}(\mathcal{P}^2 + \mathcal{Q}^2),$$

the quantum mechanical problem is then to find all the eigenvalues and eigenfunctions of the differential operator

$$\frac{1}{2}\left[-\frac{d^2}{dx^2} + x^2\right].$$

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Its eigenfunctions are $e^{-x^2/2}H_k$, where H_k denotes the Hermite polynomial of degree k. They can be defined through the Rodrigues formula as follows

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$$

for $x \in \mathbb{R}$ and $k = 0, 1, \ldots$. They are also the eigenfunctions of the differential operator

$$\frac{1}{2}\frac{d^2}{dx^2} - x\frac{d}{dx} = \frac{1}{2}e^{x^2}\frac{d}{dx}(e^{-x^2}\frac{d}{dx}).$$

The n-dimensional Hermite polynomial of order $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and degree $|\alpha| = \sum_{j=1}^n \alpha_j$, denoted by H_α is defined as the tensor product of the one-dimensional ones,

$$H_\alpha(x) = \bigotimes_{j=1}^n H_{\alpha_j}(x_j)$$

with $x \in \mathbb{R}^n$. They are orthogonal with respect to the Gaussian measure

$$d\gamma(x) = e^{-|x|^2} dx.$$

Let us consider the normalization h_α of H_α given by $h_\alpha(x) = \frac{H_\alpha(x)}{(\sqrt{\pi}\alpha!2^{|\alpha|})^{1/2}}$, then the set $\mathcal{F} = \{h_\alpha\}_{\alpha\in\mathbb{N}_0^n}$ turns out to be an orthonormal basis in $L^2(d\gamma)$.
Let $f \in L^2(d\gamma)$, then $f = \sum_\alpha a_\alpha h_\alpha$ with $a_\alpha = \int f h_\alpha d\gamma$. It can be proved that the Abel's expansion $\sum_\alpha e^{-|\alpha|t} a_\alpha h_\alpha$ converges absolutely to the Ornstein-Uhlenbeck semigroup

$$T^t f(x) = \int_{\mathbb{R}^n} M(t, x, y) f(y)\, dy$$

for almost every $x \in \mathbb{R}^n$, where

$$\begin{aligned} M(t, x, y) &= \sum_\alpha e^{-|\alpha|t} h_\alpha(x) h_\alpha(y) \\ &= \pi^{-n/2}(1 - e^{-2t})^{-n/2} e^{-\frac{|e^{-t}x - y|^2}{1-e^{-2t}}}; \quad t > 0, \end{aligned}$$

$M(t, x, y)$ is called Mehler kernel (see [15]).
By writing $u(x, t) = T^t f(x)$, u turns out to be the solution of the parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - x \cdot \nabla u$$

with initial data $f \in L^2(d\gamma)$.
The Ornstein-Uhlenbeck differential operator is defined by $L = \frac{1}{2}\Delta - x \cdot \nabla$, with Δ the Laplace operator and $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_1})$ the gradient. Thus $T^t = e^{Lt}$. The

n-dimensional Hermite polynomials are the eigenfunctions of L, ie $Lh_\alpha = -|\alpha| h_\alpha$. The Ornstein-Uhlenbeck semigroup T^t as well as L are selfadjoint operators with respect to the Gaussian measure.

If we re-parametrize T^t with $r = e^{-t}$ and use the same notation for the re-parametrized operator, then we have

$$T^r f(x) = \frac{e^{|x|^2}}{\pi^{1/2}(1-r^2)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-r\,y|^2}{1-r^2}} f(y)\, d\gamma(y).$$

In 1969, C. Calderón [2] proved that the multiparametric maximal operator

$$\mathcal{T}^* f(x) = \sup_{\substack{0<r_1<1 \\ \vdots \\ 0<r_n<1}} \left| \frac{e^{|x|^2}}{\pi^{n/2}} \prod_{i=1}^{n} \frac{1}{(1-r_i^2)^{1/2}} \int_{\mathbb{R}^n} e^{-\sum_{i=1}^{n} \frac{(x_i - r_i\, y_i)^2}{1-r_i^2}} f(y)\, d\gamma(y) \right|$$

is bounded in $L^p(\mathbb{R}^n, d\gamma)$, $p > 1$. From this result, the $L^p(\mathbb{R}^n, d\gamma)$ strong type property $p > 1$ for the one-parameter maximal operator

$$T^* f(y) = \sup_{0<r<1} |T^r f(y)|$$

follows. It is worth mentioning that this result also follows from the general theory of symmetric diffusion semigroups and in this case the L^p constant obtained is independent of dimension. It is known that, for $n > 1$, $\mathcal{T}^*$ is not weak type (1,1) with respect to the Gaussian measure. The same question for T^*, with $n > 1$, was an open problem until 1984, when P. Sjögren, [23], proved that T^* is γ-weak type (1,1). Sjögren's proof does not give pointwise estimates by means of average maximal operators on the global part; covering results, such as Besicovitch or Wiener Lemmas, are not used either. These are the basic classical tools used on the approximations of the identity with the Lebesgue measure and with doubling measures.

The ad hoc method developed by Sjögren is very useful and was used by other people in order to prove weak type inequalities of certain singular integral operators associated with this semigroup. But there are operators which cannot be handled likewise since their kernels exceed the bounds necessary to apply his "forbidden region" technique.

S. Pérez in [20], whose goal was to study the operators which could not be handled by Sjögren's technique, came back to the Ornstein-Uhlenbeck semigroup and gave

an explicit formula for the maximal kernel of this semigroup and with that she associated the right geometry to get the weak type inequality.
Later on, P. Sjögren with this explicit formula of the maximal kernel gave a very simple and elegant proof of the weak-type $(1,1)$ of T^* which can be found in [25].

In 1988 C. Gutiérrez and W. Urbina [14] came back to the problem of pointwise estimates for T^* and proved that

$$T^* f(y) \leq M_\gamma f(x) + \max(2\,,\, |x|^n) e^{|x|^2}\ ||f||_{1,\gamma},$$

where

$$M_\gamma f(x) = \sup_{r>0} \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} |f|\, d\gamma$$

is the centered Gaussian Hardy-Littlewood maximal function. By using Besicovitch covering Lemma, the γ-weak type (1,1) of M_γ follows. Nevertheless this estimate does not give the weak type (1,1) inequality with respect to the Gaussian measure for T^* except for $n = 1$.
Since for r and x fixed, the maximum of the kernel of the operator T^r is attained at $y = x/r$, the centered maximal operator does not seem to be the best average maximal function to be used in order to get the γ-weak type (1,1) inequality.
We can estimate T^* by the non-centered Gaussian Hardy-Littlewood maximal function, but P. Sjögren proved in [24] that this maximal function is not weak type (1,1). The main difficulty with this maximal operator is that we can not use Wiener covering Lemma since γ is far from being a doubling measure.
The first of the two basic goals of this article is to prove the weak type (1,1) inequality for T^* with respect to the Gaussian measure by using a covering lemma which is halfway between Besicovitch and Wiener and whose origin goes back to the Doctoral Dissertation of L. Forzani in [4]. We will prove something stronger than the weak type $(1,1)$ inequality for T^*. First let us define

$$M_\Phi f(x) = \sup_{0<r<1} \frac{1}{\gamma((1+\delta)B(\frac{x}{r}, \frac{|x|}{r}(1-r)))} \int_{\mathbb{R}^n} \Phi\left(\frac{|x - r\,y|}{\sqrt{1-r^2}}\right)\ |f(y)|\ d\gamma(y)$$

where $\Phi : \mathbb{R}^+_o \to \mathbb{R}^+_o$ is a non-increasing function such that $S = \sum_{\nu\geq 1} \Phi(\frac{1}{2}(\nu - 1))\, \nu^{2n} < \infty$ and $\delta = \delta_{r,x} = \frac{r}{|x|(1-r)} \min\{\frac{1}{|x|}, \sqrt{1-r}\}$.

By taking now $\Phi(t) = \frac{1}{\pi^{n/2}} \exp(-t^2)$, it will be proved that

$$T^* f(x) \leq C M_\Phi f(x)$$

(see proof of Corollary 1.1 in § 4).

In § 2 we will prove the following theorem and its corollary in § 4:

Theorem 1.1. *There exists a constant C depending only on S and n, such that for all $f \in L^1(d\gamma)$, $\lambda > 0$, we have*

$$\gamma\{x \in \mathbb{R}^n : \ M_\Phi f(x) > \lambda\} \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| \, d\gamma(y),$$

i.e. $M_\Phi f$ is γ-weak type (1,1).

Corollary 1.1. *T^* is γ-weak type (1,1).*

This result relies strongly on the following subtle covering lemma which will be proved in section § 4, where a polynomial growth for the overlapping of an special family of dilations for the covering balls is obtained.

Lemma 1.1. *Let $A = \{x_\alpha : \ \alpha \in I\}$ be a subset of $\mathbb{R}^n \setminus \bar{B}(0, 2\zeta)$, with $\zeta > 2$ fixed and I a finite set of indices. For each $x \in A$ a number $r = r(x) \in (\frac{3}{4}, 1 - \frac{\zeta^2}{|x|^2})$ is given. Let B_j and B_j^ν be the balls $B(\frac{x_j}{r_j}, \frac{|x_j|}{r_j}(1 - r_j))$ and $B(\frac{x_j}{r_j}, \nu\rho_j)$ respectively, with $\nu \geq 1$ and $\rho_j = \sqrt{1 - r_j}$, and $\delta_j = \frac{r_j}{|x_j|(1-r_j)} \min\{\frac{1}{|x_j|}, \sqrt{1 - r_j}\} = \frac{r_j}{|x_j|^2 \rho_j^2}$. Then there exist a positive constant C, depending only on n, and a subset J of I such that*

i) $A \subset \bigcup_{j \in J} (1 + \delta_j) \, B_j$;

ii) $\sum_{j \in J} \chi_{B_j^\nu}(z) \leq C \, \nu^{2\,n}$.

On the other hand the proof of Corollary 1.1 is based on the following lemma where we compute explicitly the Gaussian measure of a ball:

Lemma 1.2. *There exists a constant C depending on n such that for all $x \in \mathbb{R}^n \setminus \{0\}$, $r \in (1/2, 1)$ and $s \in (0, 1/2)$ the following inequality holds*

$$\gamma\left(B\left(\frac{x}{r}, \frac{|x|}{r}\, s\right)\right) \leq C \, s^{\frac{n-1}{2}} \, \exp\left(-\frac{|x|^2}{r^2}(1-s)^2\right) \min\left\{\frac{1}{|x|^2}, s^{\frac{1}{2}}\right\}.$$

Let us now introduce the second problem of this paper.

At the end of the last century great efforts have been made in order to get a general singular integral theory in the context of the Ornstein-Uhlenbeck differential operator L. By analogy with the classical harmonic analysis the Gaussian Riesz Potentials were defined as $I_\eta = (-L)^{-\eta}$ with $\eta > 0$ over the orthogonal complement of the eigenspace associated with the eigenvalue 0. Formally,

$$I_\eta = \frac{1}{\Gamma(\eta)} \int_0^\infty t^{\eta-1} T^t \, dt.$$

These operators turned out to be not weak type $(1,1)$ (see [9]). They are indeed bounded on $L^p(d\gamma)$ for $1 < p < \infty$ but unlike the classical Riesz Potentials these do not improve integrability on the L^p scale. They do though on the $L^p LogL$ scale (see [7]).

By following [26] it is possible to define the higher order Riesz Transforms as

$$\mathcal{R}_\alpha f(x) = (-1)^{|\alpha|} d^\alpha I_{|\alpha|/2} f(x)$$

with $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| = \sum_{j=1}^n \alpha_j$, and $d^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}$. These operators were proved to be bounded on $L^p(d\gamma)$ by several people from different points of view, see [18], [11], [22], [27], [12], [13], [20] & [8]. But surprisingly the weak type $(1,1)$ case of these operators need not be true for all α. These operators are weak type $(1,1)$ if and only if $|\alpha| \leq 2$, see [19], [6], [3], [20], [10], [5] & [1].

Let us go back and review the relationship between L the infinitesimal generator of T^t and the derivative operator d which defines the higher order Riesz Transforms $\mathcal{R}_\alpha$ through the potentials $I_{\frac{|\alpha|}{2}}$.

B. Muckenhoupt in [19] defined in this context the Poisson integral u and its conjugate function v in L^2 through Hermite expansions, and as integral operators otherwise, so that they satisfy the following generalized Cauchy-Riemann equations

$$\begin{cases} \frac{\partial u}{\partial x} & = & -\frac{\partial v}{\partial t} \\ \frac{\partial u}{\partial t} & = & \frac{1}{2} e^{x^2} \frac{\partial}{\partial x}(e^{-x^2} v) \end{cases}$$

with u verifying the following second order elliptic differential equation

$$\frac{\partial^2 u}{\partial t^2} + Lu = 0.$$

In [16], K. Itô factors L out in terms of two derivative operators which are in duality with respect to the Gaussian measure: $L = \ \delta \ d$, which in the finite dimensional case d is just the usual gradient and $\delta = \frac{1}{2}e^{|x|^2} d\ e^{-|x|^2}$ is the *Gaussian gradient*. If we use δ instead of d in Muckenhoupt's approach, we get the new generalized Cauchy-Riemann system

$$\begin{cases} \delta\bar{u} &= -\frac{\partial\bar{v}}{\partial t} \\ \frac{\partial\bar{u}}{\partial t} &= d\bar{v}, \end{cases}$$

with the function $\bar{u}$ satisfying the second order partial differential equation

$$\frac{\partial^2\bar{u}}{\partial t^2} + \bar{L}\bar{u} = 0,$$

where $\bar{L} = L - I = d\,\delta$.

From the Quantum Mechanics point of view, what we are doing is to substitute the pair of operators $(\mathcal{P}, \mathcal{Q})$ by the real one $(i\mathcal{P} - 2\mathcal{Q}, i\mathcal{P}) = (\delta, d)$. Since the commutator $[\delta, d]$ is the identity operator, we again have that $\bar{L} = d\,\delta$.

If in the construction of the Gaussian Riesz transforms we use δ instead of d and $\bar{L}$ instead of L we obtain an awesome result: the Riesz transforms associated with these new operators are all weak type $(1,1)$ independently of their orders.

For $1 < p < \infty$, the $L^p(d\gamma)$ boundedness of these new operators follows from P. A. Meyer Multiplier Theorem in [18] which cannot be applied to prove the weak type $(1,1)$ inequality.

If in $\mathbb{R}^n$ we use the following gradient

$$\delta^\alpha = \frac{1}{2^{|\alpha|}} e^{|x|^2} d^\alpha e^{-|x|^2}$$

and the Riesz potentials associated with $\bar{L}$, then these new singular integral operators are defined by

$$\bar{\mathcal{R}}_\alpha f(x) = (-1)^{|\alpha|} \delta^\alpha (-\bar{L})^{-|\alpha|/2} f(x).$$

The action of one of these operators over a Hermite polynomial is as follows

$$\begin{aligned} \bar{\mathcal{R}}_\alpha H_\beta &= \frac{(-1)^{|\alpha|}}{2^{|\alpha|}(|\beta|+1)^{|\alpha|/2}} e^{|x|^2} d^\alpha (e^{-|x|^2} H_\beta(x)) \\ &= \frac{(-1)^{|\alpha+\beta|}}{2^{|\alpha|}(|\beta|+1)^{|\alpha|/2}} e^{|x|^2} d^{\alpha+\beta} (e^{-|x|^2}) \qquad (1.1) \\ &= \frac{1}{2^{|\alpha|}(|\beta|+1)^{|\alpha|/2}} H_{\alpha+\beta}(x) \end{aligned}$$

On the first line we use that

$$\bar{L}H_\beta = -(|\beta|+1)H_\beta \ \text{ and } (-\bar{L})^{-|\alpha|/2} H_\beta = \frac{1}{(|\beta|+1)^{|\alpha|/2}} H_\beta.$$

The singular integral operators $\bar{\mathcal{R}}_\alpha$ are weak-type $(1,1)$ for all α. More precisely in § 3 we will prove the following theorem:

Theorem 1.2. *There exists a constant C depending only on n and α such that for all $f \in L^1(d\gamma)$, $\lambda > 0$, we have*

$$\gamma\{x \in \mathbb{R}^n : \bar{\mathcal{R}}_\alpha f(x) > \lambda\} \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)|\, d\gamma(y).$$

i.e. $\bar{\mathcal{R}}_\alpha f$ is $\gamma-$ weak type (1,1).

The main feature in order to prove this theorem will be to apply Theorem 1.1 with an special Φ.

2. A new maximal function M_Φ

In this section we will prove the $\gamma-$weak type of the operator

$$M_\Phi f(x) = \sup_{0<r<1} \frac{1}{\gamma((1+\delta)B(\frac{x}{r}, \frac{|x|}{r}(1-r)))} \int_{\mathbb{R}^n} \Phi\left(\frac{|x - r\,y|}{\sqrt{1-r^2}}\right) |f(y)|\, d\gamma(y)$$

where $\Phi : \mathbb{R}_o^+ \to \mathbb{R}_o^+$ is a non-increasing function such that $S = \sum_{\nu \geq 1} \Phi(\frac{1}{2}(\nu - 1))\, \nu^{2n} < \infty$ and $\delta = \delta_{r,x} = \frac{r}{|x|(1-r)} \min\{\frac{1}{|x|}, \sqrt{1-r}\}$.

Proof of Theorem 1.1 We consider only $r > \frac{3}{4}$, since the maximal operator is trivially γ-weak type (1,1) for $0 < r \leq \frac{3}{4}$ (see [4]). Let us denote with the same letter M_Φ the maximal operator restricted to the interval $\frac{3}{4} < r < 1$, with M_Φ^1 the maximal operator for $\frac{3}{4} < r < 1 - \frac{\zeta^2}{|x|^2}$ and M_Φ^2 the corresponding one for $1 - \frac{\zeta^2}{|x|^2} < r < 1$. (ζ is the constant chosen in Lemma 1.1).
First we will prove that for $|x| \leq 2\zeta$, $M_\Phi f(x) \leq C M_\gamma f(x)$, where M_γ is the centered Gaussian Hardy-Littlewood maximal function and which is known to be γ-weak type $(1,1)$. Indeed, let us call $R_{x,r} = \frac{|x|}{r}(1-r) + \min\{\frac{1}{|x|}, \sqrt{1-r}\}$, then

for $|x| \leq 2\zeta$,

$$\begin{aligned}
M_\Phi f(x) &= \sup_{3/4<r<1} \frac{1}{\gamma(B(\frac{x}{r}, R_{x,r}))} \int_{\mathbb{R}^n} \Phi\left(\frac{|x-r\,y|}{\sqrt{1-r^2}}\right) |f(y)|\, d\gamma(y) \\
&\leq C \sup_{3/4<r<1} \frac{e^{|x|^2}}{|B(x,R_{x,r})|} \sum_{\nu=0}^{\infty} \int_{\nu R_{x,r} \leq |y-\frac{x}{r}| \leq (\nu+1)R_{x,r}} \Phi\left(\frac{r|y-\frac{x}{r}|}{\sqrt{1-r^2}}\right) \\
&\qquad |f(y)| d\gamma(y) \\
&\leq C \sum_{\nu=0}^{\infty} \Phi(\nu/8\zeta)(\nu+2)^n \\
&\qquad \sup_{3/4<r<1} \frac{1}{\gamma(B(x,(\nu+2)R_{x,r}))} \int_{B(x,(\nu+2)R_{x,r})} |f(y)|\, d\gamma(y) \\
&\leq C M_\gamma f(x).
\end{aligned}$$

For $|x| \geq 2\zeta$, $M_\Phi f(x) \leq M_\Phi^1 f(x) + M_\Phi^2 f(x)$ and the γ-weak type $(1,1)$ of M_Φ will follow once we prove that both M_Φ^1 and M_Φ^2 are γ-weak type $(1,1)$.
In order to prove the weak type (1,1) of M_Φ^1 it is enough to prove that

$$\gamma(E_N^{1,\lambda}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)|\, d\gamma(y),$$

with that constant C independent of N and f, where

$$E_N^{1,\lambda} = \{x \in \mathbb{R}^n : \ |x| \geq 2\zeta \ \text{ and } \ M_\Phi^1 f(x) > \lambda\} \cap B(0,N).$$

For each $x \in E_N^{1,\lambda}$ there exists a $r = r(x) \in (\frac{3}{4}, 1 - \frac{\zeta^2}{|x|^2})$ such that

$$\frac{1}{\gamma((1+\delta)B(\frac{x}{r}, \frac{|x|}{r}(1-r)))} \int_{\mathbb{R}^n} \Phi(\frac{|r\,y-x|}{\sqrt{1-r^2}}) |f(y)| d\gamma(y) \geq \lambda. \tag{2.1}$$

For every $x \in E_N^{1,\lambda}$, we have that $|x|\frac{1-r}{r}$ is bounded above and below by positive numbers, and the centers $\frac{x}{r}$ are a bounded subset of $\mathbb{R}^n$. Hence, there exists $\epsilon > 0$ such that for all $0 < \alpha < 1$

$$\gamma\left(B(\frac{x}{r}, (1+\alpha)|x|\frac{1-r}{r} + \epsilon)\right) \leq 2\gamma\left(B(\frac{x}{r}, (1+\alpha)|x|\frac{1-r}{r})\right)$$

for all $x \in E_N^{1,\lambda}$. Let A be a subset of $E_N^{1,\lambda}$ which is a maximal set with the property $|x - \bar{x}| > \frac{\epsilon}{2}$ for $x \neq \bar{x}$, $x \in A$, $\bar{x} \in A$. Since $E_N^{1,\lambda}$ is bounded, A is a finite set $A = \{y_1, \ldots, y_L\}$. If we apply Lemma 1.1 to the set A we get a family of balls $\left\{B_j = B(\frac{x_j}{r_j}, |x_j|\frac{1-r_j}{r_j})\right\}_{j \in J \subset \{1,\ldots,L\}}$ such that $A \subset \cup_{j\in J}(1+\delta_j)B_j$ and *ii)* of Lemma 1.1 also holds. Thus

$$E_N^{1,\lambda} \subset \bigcup_{j\in J} B\left(\frac{x_j}{r_j}, (1+\delta_j)\frac{|x_j|}{rj}(1-r_j) + \epsilon\right).$$

Then

$$\gamma(E_N^{1,\lambda}) \leq 2\sum_{j\in J}\gamma\left(B(\frac{x_j}{r_j},(1+\delta_j)|x_j|\frac{1-r_j}{r_j})\right).$$

From (2.1), since Φ is a non-increasing function such that

$$\sum_{v\geq 1}\Phi(\frac{1}{2}(v-1))v^{2n}<\infty,$$

we have, using (ii) of Lemma 1.1, that

$$\begin{aligned}
\gamma(E_N^{1,\lambda}) &\leq 2\sum_{j\geq 1}\gamma((1+\delta_j)B_j)\\
&\leq \frac{C}{\lambda}\sum_{j\geq 1}\int_{\mathbb{R}^n}\Phi\left(\frac{|r_jy-x_j|}{(1-r_j^2)^{1/2}}\right)|f(y)|\,d\gamma(y)\\
&\leq \frac{C}{\lambda}\sum_{j\geq 1}\sum_{\nu\geq 1}\int_{B(\frac{x_j}{r_j},\nu\rho_j)-B(\frac{x_j}{r_j},(\nu-1)\rho_j)}\Phi\left(\frac{|r_jy-x_j|}{(1-r_j^2)^{1/2}}\right)|f(y)|\,d\gamma(y)\\
&= \frac{C}{\lambda}\sum_{j\geq 1}\sum_{\nu\geq 1}\Phi(\frac{1}{2}(\nu-1))\int_{B_j^\nu}|f(y)|\,d\gamma(y)\\
&\leq \frac{C}{\lambda}\int_{\mathbb{R}^n}\sum_{\nu\geq 1}\Phi(\frac{1}{2}(\nu-1))\sum_j\chi_{B_j^\nu}(y)|f(y)|\,d\gamma(y)\\
&\leq \frac{C}{\lambda}\int_{\mathbb{R}^n}\sum_{\nu\geq 1}\Phi(\frac{1}{2}(\nu-1))\nu^{2n}|f(y)|\,d\gamma(y)\\
&\leq \frac{C}{\lambda}\int_{\mathbb{R}^n}|f(y)|\,d\gamma(y).
\end{aligned}$$

Now, we will prove that M_Φ^2 is weak type (1,1). First, let us observe that, if $r>1-\frac{\zeta^2}{|x|^2}$ then, for all $y\in(1+\delta)B(\frac{x}{r},\frac{|x|}{r}(1-r))=B(\frac{x}{r},\frac{|x|}{r}(1-r)+\sqrt{1-r})$, the values of $e^{-|y|^2}$ are equivalent.

Now, let us define

$$E_N^{2,\lambda}=\{x\in\mathbb{R}^n:\ |x|\geq 2\zeta \text{ and } M_\Phi^2f(x)>\lambda\}\cap B(0,N).$$

The weak type (1,1) for M_Φ^2 follows once we prove the following inequality

$$\gamma(E_N^{2,\lambda})\leq\frac{C}{\lambda}||f||_{1,\gamma} \tag{2.2}$$

with C being a constant indepedent of N and f.

For each $x \in E_N^{2,\lambda}$, we have $\gamma((1+\delta)B(\frac{x}{r}, \frac{|x|}{r}(1-r))) \simeq e^{-|x|^2}(1-r)^{\frac{n}{2}}$. To prove (2.2), we will divide the integral in $M_\Phi^2 f$ in two parts: one given by $|y-x| < 2\frac{C}{|x|r}$ and the other one by $|y-x| > 2\frac{C}{|x|r}$.

For the first region we have

$$\begin{aligned}
&\frac{e^{|x|^2}}{(1-r)^{\frac{n}{2}}} \int_{|y-x|<2\frac{C}{|x|r}} \Phi\left(\frac{|ry-x|}{\sqrt{1-r^2}}\right) |f(y)|\, d\gamma(y) \\
\leq \quad & C\, \frac{e^{|x|^2}}{(1-r)^{\frac{n}{2}}} \int_{|y-x|<C\frac{\sqrt{1-r}}{r}} |f(y)|\, d\gamma(y) + \\
& C \frac{e^{|x|^2}}{(1-r)^{\frac{n}{2}}} \int_{C\frac{\sqrt{1-r}}{r}<|y-x|<2\frac{c}{r|x|}} \Phi\left(c\frac{|y-x|}{\sqrt{1-r^2}}\right) |f(y)|\, d\gamma(y) \\
\leq \quad & C M_T f(x),
\end{aligned} \tag{2.3}$$

with M_T the truncated non-centered Gaussian maximal function defined by

$$M_T f(x) = \sup_{\substack{x\in B(y,t)\\ 0<t<\min\{1, \frac{1}{|x|}\}}} \frac{e^{|x|^2}}{\omega_n t^n} \int_{B(y,t)} |f(z)|\, d\gamma(z),$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. The first inequality follows from the fact that Φ is bounded and $|ry-x| \geq r|y-x| - (1-r)|x| \geq \frac{r}{2}|y-x|$. The second inequality follows from the fact that Φ is a Lebesgue integrable, non-increasing function and hence, it is a good approximation of the identity.

The truncated non-centered Gaussian maximal function is bounded by the centered Gaussian Hardy-Littlewood maximal function and therefore is γ-weak type (1,1).

For the second region, we have that $|ry-x| > C|y-x|$. Therefore

$$\begin{aligned}
\frac{1}{(1-r^2)^{\frac{n}{2}}} \Phi\left(\frac{|ry-x|}{\sqrt{1-r^2}}\right) \quad &\leq \quad \frac{1}{|y-x|^n}\left(\frac{|y-x|}{\sqrt{1-r^2}}\right)^n \Phi\left(\frac{c|y-x|}{\sqrt{1-r^2}}\right) \\
&\leq \quad C\frac{(\sqrt{1-r^2})^n}{|y-x|^{2n}} \\
&\leq \quad \frac{C}{|x|^n|y-x|^{2n}},
\end{aligned}$$

since $\Phi(\frac{1}{2}(\nu-1)) \leq S$.

Then,

$$\begin{aligned}
&\frac{e^{|x|^2}}{(1-r)^{\frac{n}{2}}} \int_{|y-x|>2\frac{c}{|x|r}} \Phi\left(\frac{|ry-x|}{\sqrt{1-r^2}}\right) |f(y)|\, d\gamma(y) \\
\leq \quad & C\frac{e^{|x|^2}}{|x|^n} \int_{|y-x|>2\frac{c}{|x|}} \frac{|f(y)|}{|y-x|^{2n}}\, d\gamma(y),
\end{aligned}$$

but

$$\frac{e^{|x|^2}}{|x|^n}\int_{|y-x|>2\frac{c}{|x|}}\frac{|f(y)|}{|y-x|^{2n}}\,d\gamma(y)\in L^1(d\gamma).$$

So, inequality (2.2) follows.

3. New higher order Gaussian Riesz Transforms

The new higher order Gaussian Riesz Transforms are defined as

$$\bar{\mathcal{R}}_\alpha f(x) = \text{p.v. } e^{|x|^2}\int_{\mathbb{R}^n}\bar{\mathcal{K}}_\alpha(x,y)f(y)\,d\gamma(y)$$

where

$$\bar{\mathcal{K}}_\alpha(x,y) = C_\alpha\int_0^1\left(\frac{-\log r}{1-r^2}\right)^{\frac{|\alpha|-2}{2}} H_\alpha\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}}\,dr.$$

Formally $\bar{\mathcal{K}}_\alpha$ is obtained by differentiating with the dual derivative the kernel corresponding to the Riesz potentials associated with $\bar{L}$

$$\begin{aligned}(-\bar{L})^{-|\alpha|/2}f(x) &= \frac{1}{\Gamma(|\alpha|/2)}\int_0^1(-\log r)^{\frac{|\alpha|}{2}-1}T^r f(x)dr\\ &= \tilde{C}_\alpha e^{|x|^2}\int_{\mathbb{R}^n}\left(\int_0^1(-\log r)^{\frac{|\alpha|-2}{2}}\frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}}}\,dr\right)\,f(y)\,d\gamma(y).\end{aligned}$$

Proof of Theorem 1.2 For each $x\in\mathbb{R}^n$ we view this operator as the sum of two ones which are obtained, as it is usual in this context, by splitting $\mathbb{R}^n$ into a local part, B_x, the Euclidean ball centered at x and radius min $(1,\frac{1}{|x|})$, and its complement called the global part. Thus

$$\bar{\mathcal{R}}_\alpha f(x) = \bar{\mathcal{R}}_{\alpha,l}f(x)+\bar{\mathcal{R}}_{\alpha,g}f(x)$$

where $\bar{\mathcal{R}}_{\alpha,l}f(x)=\bar{\mathcal{R}}_\alpha(f\mathcal{X}_{B_x})$ and

$$\bar{\mathcal{R}}_{\alpha,g}f(x)=\bar{\mathcal{R}}_\alpha(f(1-\mathcal{X}_{B_x}))$$

We will prove that these two operators are γ-weak type (1,1) and so will be $\bar{\mathcal{R}}_\alpha$. In order to prove that $\bar{\mathcal{R}}_{\alpha,l}$ is γ-weak type (1,1) we state the following theorem whose proof can be found in either [7] or [21].

Theorem 3.1. *Let $\mathcal{K}(x,y)$ be a C^1 function off the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$ which satisfies*

$$|\mathcal{K}(x,y)| \leq \frac{C}{|x-y|^n} \text{ and } |D_y\mathcal{K}(x,y)| \leq \frac{C}{|x-y|^{n+1}}$$

for $|x-y| \leq min\ (1, \frac{1}{|x|})$, and the principal value of the integral operator T with kernel $\mathcal{K}$ is bounded on $L^p(d\gamma)$ for some $1 < p < \infty$, then T_l, defined as $T_l(f)(x) = T(fB_x)(x)$, is γ-weak type (1,1)

In our case

$$Tf(x) = p.v. \int \mathcal{K}(x,y)f(y)dy$$

with

$$\begin{aligned}\mathcal{K}(x,y) &= e^{|x|^2}\bar{\mathcal{K}}_\alpha(x,y)e^{-|y|^2} \\ &= C_\alpha \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\alpha|-2}{2}} H_\alpha\left(\frac{x-ry}{\sqrt{1-r^2}}\right) \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}}\, dr\end{aligned}$$

and therefore

$$\begin{aligned}\frac{\partial\mathcal{K}}{\partial y_j}(x,y) &= 2C_\alpha \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\alpha|-2}{2}} \left[\frac{-r\alpha_j}{\sqrt{1-r^2}} H_{\alpha-e_j}\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\frac{(x-ry)}{\sqrt{1-r^2}} + \right. \\ &\left. H_\alpha\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\frac{(rx_j-y_j)}{1-r^2}\right] \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}}\, dr.\end{aligned}$$

In the following two claims we will show that the hypotheses of Theorem 3.1 are fulfilled for this operator.

Claim 1: on B_x, $|\mathcal{K}(x,y)| \leq \frac{C}{|x-y|^n}$ and $|\frac{\partial\mathcal{K}}{\partial y_j}(x,y)| \leq \frac{C}{|x-y|^{n+1}}$.

PROOF: For every $y \in B_x$ there exists a constant $C > 0$ such that $C^{-1} \leq e^{|y|^2-|x|^2} \leq C$, then

$$|\mathcal{K}(x,y)| \leq C|e^{-|x|^2+|y|^2}\mathcal{K}(x,y)| = C|\mathcal{K}_\alpha(x,y)|$$

and

$$\left|\frac{\partial\mathcal{K}}{\partial y_j}(x,y)\right| \leq C\left|e^{-|x|^2+|y|^2}\frac{\partial\mathcal{K}}{\partial y_j}(x,y)\right|.$$

On the other hand on B_x,

$$e^{-c\frac{|x-ry|^2}{1-r^2}} = e^{-c\frac{|x-y|^2}{1-r^2}}e^{-c\frac{1-r}{1+r}|y|^2}e^{-c\frac{(x-y)\cdot y}{1-r}} \leq Ce^{-c\frac{|x-y|^2}{1-r}},$$

and thus with this inequality and taking into account that $t^m e^{-ct^2} \leq C,\ \forall t \geq 0$, we get

$$\left|H_\alpha\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\right| e^{-\frac{|x-ry|^2}{1-r^2}} \leq C\sum_{m=0}^{|\alpha|}\left|\frac{x-ry}{\sqrt{1-r^2}}\right|^m e^{-\frac{|x-ry|^2}{2(1-r^2)}} e^{-\frac{|x-ry|^2}{2(1-r^2)}} \leq Ce^{-c\frac{|x-y|^2}{1-r}}.$$

Therefore, by combining all the above remarks, on B_x we have

$$\begin{aligned}
|\mathcal{K}(x,y)| &\leq C\int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\alpha|-2}{2}} \frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{n}{2}+1}}\,dr \\
&\leq C\left[\int_0^{\frac{1}{2}} (-\log r)^{\frac{|\alpha|-2}{2}}\,dr + \int_{\frac{1}{2}}^1 \frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{n}{2}+1}}\,dr\right] \\
&\leq C\left(1+\frac{1}{|x-y|^n}\right) \leq \frac{C}{|x-y|^n}
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\partial\mathcal{K}}{\partial y_j}(x,y)\right| &\leq C\int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\alpha|-2}{2}} \frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{n+3}{2}}}\,dr \\
&\leq C\left[\int_0^{\frac{1}{2}} (-\log r)^{\frac{|\alpha|-2}{2}}\,dr + \int_{\frac{1}{2}}^1 \frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{n+3}{2}}}\,dr\right] \\
&\leq C\left(1+\frac{1}{|x-y|^{n+1}}\right) \leq \frac{C}{|x-y|^{n+1}}.
\end{aligned}$$

Claim 2: The operator $T = \bar{\mathcal{R}}_\alpha$ is bounded on $L^2(d\gamma)$.

Proof: Let $f \in L^2(d\gamma),\ f = \sum_\beta a_\beta h_\beta$ with $a_\beta = \int f h_\beta\, d\gamma$. From the action of $\bar{\mathcal{R}}_\alpha$ over Hermite polynomials (1.1)

$$\bar{\mathcal{R}}_\alpha h_\beta(x) = \left[\frac{\prod_{j=1}^n \prod_{k=0}^{\alpha_j-1}(\beta_j+\alpha_j-k)}{(2(|\beta|+1))^{|\alpha|}}\right]^{\frac{1}{2}} h_{\beta+\alpha}(x)$$

and therefore

$$\begin{aligned}
\|\bar{\mathcal{R}}_\alpha f\|^2_{L^2(d\gamma)} &= \sum_\beta \frac{\prod_{j=1}^n \prod_{k=0}^{\alpha_j-1}(\beta_j+\alpha_j-k)}{(2(|\beta|+1))^{|\alpha|}}|a_\beta|^2 \\
&\leq \sum_\beta \prod_{j=1}^n (1+\alpha_j)^{\alpha_j}|a_\beta|^2 \\
&\leq (1+|\alpha|)^{|\alpha|}\sum_\beta |a_\beta|^2 = C\|f\|^2_{L^2(d\gamma)}.
\end{aligned}$$

Now if we apply these two claims to Theorem 3.1, the γ-weak type (1,1) of $\bar{\mathcal{R}}_{\alpha,l}$ follows.

In order to prove that $\bar{\mathcal{R}}_{\alpha,g}$ is also γ-weak type (1,1) we will prove

Claim 3: on $\mathbb{R}^n \setminus B_x$, $|\bar{\mathcal{R}}_{\alpha,g}f(x)| \leq CM_\Phi f(x)$ with $\Phi(t) = e^{-ct^2}$.

This together with Theorem 1.1 give the weak type (1,1) inequality for $\bar{\mathcal{R}}_{\alpha,g}$.

Proof of Claim 3

$$
\begin{aligned}
|\bar{\mathcal{K}}_\alpha(x,y)| &= \left| \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\alpha|-2}{2}} H_\alpha\left(\frac{x-ry}{\sqrt{1-r^2}}\right) \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}}\, dr \right| \\
&\leq C \int_0^{\frac{3}{4}} (-\log r)^{\frac{|\alpha|-2}{2}} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{n}{2}}}\, dr + \\
&\quad C \int_{\frac{3}{4}}^{1-\zeta/|x|^2} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{n-1}{2}}} \left(|x| \vee (1-r^2)^{-\frac{1}{2}}\right) \frac{dr}{|x|(1-r^2)^{3/2}} + \\
&\quad C \int_{1-\zeta/|x|^2}^{1} \frac{e^{-c\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{n-1}{2}}} \left(|x| \vee (1-r^2)^{-\frac{1}{2}}\right) \frac{e^{-\bar{c}\frac{|x-y|^2}{1-r}}}{1-r}\, dr \\
&= C\left(\bar{\mathcal{K}}^1_\alpha(x,y) + \bar{\mathcal{K}}^2_\alpha(x,y) + \bar{\mathcal{K}}^3_\alpha(x,y)\right)
\end{aligned}
$$

where the inequality is obtained by annihilating the Hermite polynomial with part of the exponential, then splitting the unit interval of the integral into three subintervals $[0,3/4]$, $[3/4, 1-\zeta/|x|^2]$, and $[1-\zeta/|x|^2, 1]$ and taking into account that on the second one $|x| \vee (1-r^2)^{-1/2} \geq |x|$, on the third one $|x| \vee (1-r^2)^{-1/2} \geq (1-r^2)^{-1/2}$ and $|x-ry| \geq \bar{c}|x-y|$ and on the last two intervals $-\log r/(1-r^2)$ is bounded by a constant.

Thus, by using the definition of kernels $\bar{\mathcal{K}}^j_\alpha$ with $j=1,2,3$, interchanging the order of integration on each operator $\bar{\mathcal{R}}^j_{\alpha,g}$ with $j=1,2,3$, using Lemma 1.2 and setting in this context $\Phi(t) = e^{-ct^2}$, we get

$$
\begin{aligned}
\bar{\mathcal{R}}^1_{\alpha,g}f(x) &= e^{|x|^2} \int_{\mathbb{R}^n} \bar{\mathcal{K}}^1_\alpha(x,y)\ |f(y)|\ d\gamma(y) \\
&= e^{|x|^2} \int_{\mathbb{R}^n} \int_0^{\frac{3}{4}} (-\log r)^{\frac{|\alpha|-2}{2}} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{n}{2}}}\, dr\ |f(y)|\ d\gamma(y) \\
&= \int_0^{\frac{3}{4}} (-\log r)^{\frac{|\alpha|-2}{2}} e^{|x|^2} \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{n}{2}}} |f(y)|\ d\gamma(y)\, dr \\
&\leq C \int_0^{\frac{3}{4}} (-\log r)^{\frac{|\alpha|-2}{2}} dr \quad M_\Phi f(x) \\
&\leq C\ M_\Phi f(x),
\end{aligned}
$$

$$\begin{aligned}
\bar{\mathcal{R}}^2_{\alpha,g} f(x) &= e^{|x|^2} \int_{\mathbb{R}^n} \bar{\mathcal{K}}^2_\alpha(x,y)\ |f(y)|\ d\gamma(y) \\
&= e^{|x|^2} \int_{\mathbb{R}^n} \int_{\frac{3}{4}}^{1-\zeta/|x|^2} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{n-1}{2}}}\ (|x| \vee (1-r^2)^{-\frac{1}{2}})\ \frac{dr}{|x|(1-r^2)^{3/2}} \\
&\qquad |f(y)|\ d\gamma(y) \\
&= \int_{3/4}^{1-\zeta/|x|^2} e^{|x|^2} \int_{\mathbb{R}^n} \frac{e^{-c\frac{|x-ry|^2}{(1-r^2)}}}{(1-r^2)^{(n-1)/2}} (|x| \vee (1-r^2)^{-1/2}) |f(y)| d\gamma(y) \\
&\qquad \frac{dr}{|x|(1-r^2)^{3/2}} \\
&\le C \frac{1}{|x|} \int_{3/4}^{1-\zeta/|x|^2} \frac{dr}{(1-r)^{3/2}}\ \ M_\Phi f(x) \\
&\le C\ M_\Phi f(x),
\end{aligned}$$

and finally

$$\begin{aligned}
\bar{\mathcal{R}}^3_{\alpha,g} f(x) &= e^{|x|^2} \int_{\mathbb{R}^n} \bar{\mathcal{K}}^3_\alpha(x,y)\ |f(y)|\ d\gamma(y) \\
&= e^{|x|^2} \int_{\mathbb{R}^n} \int_{1-\zeta/|x|^2}^{1} \frac{e^{-c\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{n-1}{2}}}\ (|x| \vee (1-r^2)^{-\frac{1}{2}}) \\
&\qquad \frac{e^{-\bar{c}\frac{|x-y|^2}{1-r}}}{1-r}\ dr\ |f(y)|\ d\gamma(y) \\
&= \int_{1-\zeta/|x|^2}^{1} e^{|x|^2} \int_{\mathbb{R}^n} \frac{e^{-c\frac{|x-ry|^2}{(1-r^2)}}}{(1-r^2)^{(n-1)/2}} (|x| \vee (1-r^2)^{-1/2}) \\
&\qquad \frac{e^{-\bar{c}\frac{|x-y|^2}{1-r}}}{1-r}\ |f(y)|\ d\gamma(y)\ dr \\
&\le \int_{1-\zeta/|x|^2}^{1} e^{|x|^2} \int_{\mathbb{R}^n} \frac{e^{-c\frac{|x-ry|^2}{(1-r^2)}}}{(1-r^2)^{(n-1)/2}} (|x| \vee (1-r^2)^{-1/2}) \\
&\qquad \frac{1}{|x-y|^2}\ |f(y)|\ d\gamma(y)\ dr \\
&\le C|x|^2 \int_{1-\zeta/|x|^2}^{1} dr\ \ M_\Phi f(x) \\
&\le C\ M_\Phi f(x).
\end{aligned}$$

And since $|\bar{\mathcal{R}}_{\alpha,g} f(x)| \le C_\alpha \sum_{j=1}^{3} \bar{\mathcal{R}}^j_{\alpha,g} f(x)$, Claim 3 holds.

4. Proof of Lemmas & Corollary 1.1

Proof of Lemma 1.1. Let $I_1 = I$, $\alpha_1 \in I_1$ such that $|x_{\alpha_1}| = \min\{|x_\alpha| : \alpha \in I_1\}$. Let $x_1 = x_{\alpha_1}$ and $B_1 = B_{\alpha_1}$. Let $I_1, \dots I_{k-1}; x_1, \dots, x_{k-1}; B_1, \dots, B_{k-1}$ be chosen; we define $I_k = \{\alpha \in I : x_\alpha \notin \cup_{j=1}^{k-1}(1+\delta_j)B_j\}$, and we choose $\alpha_k \in I_k$ such that $|x_{\alpha_k}| = \min\,\{|x_\alpha| : \alpha \in I_k\}$. Let $x_k = x_{\alpha_k}$ and $B_k = B_{\alpha_k}$. Let $J = \{\alpha_1, \dots, \alpha_N\}$ where N is the first integer for which $I_{N+1} = \emptyset$. Then (i) is immediate. Before proving (ii) let us make some remarks.

(1) x_j was chosen so that $x_j \notin (1+\delta_s)B_s$ for all $s < j$. Hence

$$\begin{aligned}\left|\frac{x_s}{r_s} - x_j\right|^2 &= \frac{|x_s|^2}{r_s^2} + |x_j|^2 - 2|x_j|\frac{|x_s|}{r_s}\cos\langle\frac{x_s}{r_s}, x_j\rangle \\ &\geq R_s^2(1+\delta_s)^2;\end{aligned}$$

(2) $|x_j| \geq |x_s|$ for $s < j$; i.e. $|x_j|$ is increasing with j.

(3) $|\frac{x_s}{r_s} - \frac{x_j}{r_j}|^2 \geq \frac{1}{r_j}[\frac{|x_s|^2}{r_s^2 r_j}(r_j - r_s)^2 + 2\frac{(1-r_s)}{r_s}] \geq \theta^2\max^2(\rho_j, \rho_s)$ for $s < j$. In fact, using (1) and (2), and that $2R_s^2\delta_s = 2\frac{(1-r_s)}{r_s}$

$$\begin{aligned}\left|\frac{x_s}{r_s} - \frac{x_j}{r_j}\right|^2 &= \frac{|x_s|^2}{r_s^2} + \frac{|x_j|^2}{r_j^2} - 2\frac{|x_j|}{r_j}\frac{|x_s|}{r_s}\cos\langle\frac{x_s}{r_s}, \frac{x_j}{r_j}\rangle \\ &\geq \frac{|x_s|^2}{r_s^2} + \frac{|x_j|^2}{r_j^2} + \frac{1}{r_j}[R_s^2(1+\delta_s)^2 - |x_j|^2 - \frac{|x_s|^2}{r_s^2}] \\ &= \frac{1}{r_j}R_s^2(1+\delta_s)^2 - \frac{1}{r_j}[-|x_j|^2(\frac{1-r_j}{r_j}) + \frac{|x_s|^2}{r_s^2}(1-r_j)] \\ &\geq \frac{1}{r_j}[R_s^2(1+\delta_s)^2 + |x_s|^2(1-r_j)[\frac{1}{r_j} - \frac{1}{r_s^2}]] \\ &\geq \frac{1}{r_j}[\frac{|x_s|^2}{r_s^2}(1-r_s)^2 + 2\frac{(1-r_s)}{r_s} + |x_s|^2(1-r_j)[\frac{1}{r_j} - \frac{1}{r_s^2}]] \\ &= \frac{1}{r_j}[\frac{|x_s|^2}{r_s^2 r_j}(r_j - r_s)^2 + 2\frac{(1-r_s)}{r_s}] \\ &\geq \theta^2\max{}^2(\rho_j, \rho_s).\end{aligned}$$

To obtain the last inequality we consider two cases:

i) $\rho_s^2 \geq \frac{1}{2}\rho_j^2$. Because of $\rho_s^2 = (1-r_s)$ and the nonnegativity of the first term, the inequality follows.

ii) $\rho_j^2 \geq 2\rho_s^2$. We have that $(r_j - r_s)^2 = (\rho_j^2 - \rho_s^2)^2 \geq \frac{1}{4}\rho_j^4$. Using the fact that $|x_s|\rho_s \geq \zeta$ the inequality follows. (Recall that by hypothesis $r_s \leq 1 - \frac{\zeta^2}{|x_s|^2}$.)

Now in order to prove (ii) we define

$$\begin{aligned} I_1 = &\quad \{j : j \in J \text{ and } \nu\rho_j \geq \kappa\} \\ I_2 = &\quad \{j : j \in J \text{ and } \frac{R_j}{2} < \nu\rho_j < \kappa\} \\ I_3 = &\quad \{j : j \in J \text{ and } \nu\rho_j \leq \frac{R_j}{2}\} \end{aligned}$$

where $R_j = \frac{|x_j|}{r_j}(1 - r_j)$.

We will prove

$$\sum_{j \in I_i} \chi_{B_j^\nu}(z) \leq C\nu^{2n} \quad \text{for } i = 1, 2, 3, \tag{4.1}$$

from which (ii) follows.

Now we prove (4.1). Let us consider $I_1(z) = \{j \in I_1 : \quad z \in B_j^\nu\}$. In order to obtain the desired estimate all we need is to find a sequence of pairwise disjoint measurable sets $\{S_j\}_{j \in I_1(z)}$ such that

(4) $S_j \subset B(z, C\nu)$;

(5) $|S_j| \geq \frac{C}{\nu^n}$ for some constant C.

The case $i = 1$ in (4.1) follows from (4) and (5).

We define $S_j = B(\frac{x_j}{r_j}; \frac{\theta}{2}\rho_j)$; (5) is immediate since $j \in I_1$ which implies $\rho_j \geq \frac{\kappa}{\nu}$. In order to get (4) let us take $h \in S_j$. Since $z \in B_j^\nu$, we have

$$\begin{aligned} |h - z| &\leq |h - \frac{x_j}{r_j}| + |\frac{x_j}{r_j} - z| \\ &\leq \frac{\theta}{2}\rho_j + \nu\rho_j \\ &\leq C\nu. \end{aligned}$$

That $\{S_j\}_{j \in I_1(z)}$ is a family of pairwise disjoint sets follows from (3).

Now, consider $I_2(z) = \{j \in I_2 : \quad z \in B_j^\nu\}$. In order to obtain the desired estimate we just need to find a sequence of pairwise disjoint measurable sets $\{S_j\}_{j \in I_2(z)}$ such that

(6) $S_j \subset B(z, C\nu^2)$;

(7) $|S_j| \simeq 1$ for some constant C.

The case $i = 2$ in (4.1) follows from (6) and (7).

We define $S_j = B(z+(\frac{x_j}{r_j}-z)\frac{|x_j|}{r_j}, C)$, therefore (7) is immediate. Let us prove (6). Take $h \in S_j$, using the fact that $z \in B_j^\nu$ and $\frac{R_j}{2} \leq \nu\rho_j$ or equivalently $\rho_j \frac{|x_j|}{r_j} \leq 2\nu$, we get

$$\begin{aligned}
|h-z| &\leq C + |(\frac{x_j}{r_j}-z)|\frac{|x_j|}{r_j} \\
&\leq C + \nu\rho_j\frac{|x_j|}{r_j} \\
&\leq C + 2\nu^2 \\
&\leq C\nu^2.
\end{aligned}$$

To prove that the S_j are pairwise disjoint we will use consecutively the following facts:

i) $|\frac{x_s}{r_s} - \frac{x_j}{r_j}| \geq \theta\rho_s$;

ii) $|x_s|\rho_s \geq \sqrt{\zeta}$;

iii) $|\frac{x_j}{r_j} - z| \leq \nu\rho_j \leq \kappa$ $(j \in I_2)$; and

iv) $|\frac{|x_s|}{r_s} - \frac{|x_j|}{r_j}| \leq |\frac{x_s}{r_s} - \frac{x_j}{r_j} \leq |\nu\rho_s + \nu\rho_j \leq 2\kappa$.

So

$$\begin{aligned}
\left|(\frac{x_j}{r_j}-z)\frac{|x_j|}{r_j} - (\frac{x_s}{r_s}-z)\frac{|x_s|}{r_s}\right| &\geq \frac{|x_s|}{r_s}\left|\frac{x_j}{r_j}-\frac{x_s}{r_s}\right| - \left|\frac{x_j}{r_j}-z\right|\left|\frac{|x_s|}{r_s}-\frac{|x_j|}{r_j}\right| \\
&\geq \sqrt{\zeta}\,\frac{\theta}{r_s} - 2\,\kappa^2 \\
&\geq C
\end{aligned}$$

after choosing ζ and κ properly.

Finally, consider $I_3(z) = \{j \in I_3 : \; z \in B_j^\nu\}$. In order to obtain the desired estimate all we need is to find a sequence of pairwise disjoint measurable sets $\{S_j\}_{j\in I_3(z)}$ such that

(8) $S_j \subset B(z, C\nu\rho_\tau)$;

(9) $|S_j| \geq C\rho_\tau^n$ for some constant C, where $\rho_\tau = \rho_{\min\{j:\, j\in I_3(z)\}}$.

The case $i=3$ in (4.1) follows from (8) and (9).

We define $S_j = B(\frac{x_j}{r_j}, \frac{\theta}{2}\rho_j)$. We will prove that

$$\frac{1}{2}\rho_\tau \leq \rho_j \leq \frac{3}{2}\rho_\tau \quad \text{for all} \quad j \in I_3(z). \tag{4.2}$$

From (4.2) we have (8) and (9). That S_j are disjoint follows from (3).
Let us prove (4.2). From (3), $|\frac{x_\tau}{r_\tau}-\frac{x_j}{r_j}|^2 \geq \frac{1}{r_j}\left[\frac{|x_\tau|^2}{r_\tau^2 r_j}(r_j-r_\tau)^2+2\frac{(1-r_\tau)}{r_\tau}\right]$, then $|\frac{x_j}{r_j}-\frac{x_\tau}{r_\tau}|^2 \geq \frac{|x_\tau|^2}{r_\tau^2}(\rho_j^2-\rho_\tau^2)^2$ and since $\tau, j \in I_3(z)$, we have $\frac{|x_\tau|}{r_\tau} \geq 2\frac{\nu}{\rho_\tau}$. Therefore

$$
\begin{aligned}
(\nu\rho_j+\nu\rho_\tau)^2 &\geq |\frac{x_j}{r_j}-\frac{x_\tau}{r_\tau}|^2 \\
&\geq \frac{|x_\tau|^2}{r_\tau^2}(\rho_j^2-\rho_\tau^2)^2 \\
&\geq 4\frac{\nu^2}{\rho_\tau^2}(\rho_j^2-\rho_\tau^2)^2.
\end{aligned}
$$

Then $1 \geq 2\frac{|\rho_j-\rho_\tau|}{\rho_\tau}$. Now, this inequality is equivalent to $|\rho_j-\rho_\tau| \leq \frac{1}{2}\rho_\tau$ which in turn is equivalent to (4.2).

Proof of Lemma 1.2. We can write every $y \in \mathbb{R}^n$ as $y=(\xi+|x|\frac{1-s}{r})\frac{x}{|x|}+v$; with $\langle v, x\rangle = 0$. It is clear that $y \in B(\frac{x}{r}, \frac{|x|}{r}\, s)$ if and only if $\xi \in (0, 2s\frac{|x|}{r})$ and $|v| < \sqrt{2\frac{|y|}{r}\, s\,\xi-\xi^2}$. Then, using this fact we have that

$$
\begin{aligned}
\gamma\big(B(\frac{x}{r}, \frac{|x|}{r}\, s)\big) &= \int_{B(\frac{x}{r}, \frac{|x|}{r}\, s)} e^{-|y|^2}\, dz \\
&= e^{-\frac{|x|^2}{r^2}(1-s)^2}\int_0^{2s\frac{|x|}{r}} e^{-2\xi\frac{|x|}{r}(1-s)}\; e^{-|\xi|^2} \\
&\qquad \int_{\{v\in\mathbb{R}^{n-1}:\, |v|<\sqrt{2\frac{|x|}{r}\, s\,\xi-\xi^2}\}} e^{-|v|^2}\, dv\, d\xi \\
&\leq C_n e^{-\frac{|x|^2}{r^2}(1-s)^2}\int_0^{2s\frac{|x|}{r}} e^{-2\xi|x|\frac{1-s}{r}}\; (2\frac{|x|}{r}\, s\,\xi-\xi^2)^{\frac{n-1}{2}}\, d\xi \\
&\leq C_n e^{-\frac{|x|^2}{r^2}(1-s)^2}\, s^{\frac{n-1}{2}}\int_0^{2s\frac{|x|}{r}} e^{-2\xi|x|\frac{1-s}{r}}\; (2\frac{|x|}{r}\,\xi)^{\frac{n-1}{2}}\, d\xi \\
&\leq C_n e^{-\frac{|x|^2}{r^2}(1-s)^2}\, \frac{s^{\frac{n-1}{2}}}{|x|(1-s)^{\frac{n+1}{2}}}\int_0^{4s\frac{|x|^2}{r^2}(1-s)} e^{-t}\, t^{\frac{n-1}{2}}\, dt \\
&\leq C_n e^{-\frac{|x|^2}{r^2}(1-s)^2}\, \frac{s^{\frac{n-1}{2}}}{|x|}\min(1, s|x|^2) \\
&\leq C_n e^{-\frac{|x|^2}{r^2}(1-s)^2}\, s^{\frac{n-1}{2}}\min(\frac{1}{|x|}, s\,|x|) \\
&\leq C_n e^{-\frac{|x|^2}{r^2}(1-s)^2}\, s^{\frac{n-1}{2}}\min(\frac{1}{|x|}, s^{\frac{1}{2}}).
\end{aligned}
$$

Proof of Corollary 1.1. We choose $\Phi(t) = \frac{1}{\pi^{n/2}} e^{-t^2}$. From Lemma 1.2, $\gamma((1+\delta)B(\frac{x}{r}, \frac{|x|}{r}(1-r)) \leq C\, e^{-|x|^2}(1-r)^{\frac{n}{2}}$. Then $T^*f \leq CM_\Phi f(x)$, and therefore the γ-weak type (1,1) inequality for T^* follows from Theorem 1.1.

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