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A WELL BEHAVED QUASI-DISTANCE FOR SPACES OF HOMOGENEOUS TYPE.

by

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Abstract

For any space of homogeneous type a quasi-distance equivalent to the original one is obtained satisfying that, if B and B' are balls such that the center of B' belongs to B and the radius of B' is smaller than the radius of B then, the measure of $B \cap B'$ is smaller than a constant fraction of the measure of B'. An application to weighted norm inequalities for the Hardy-Littlewood maximal function, which extends a result of A. P. Calderón, is given.

Introduction.

A particular case of space of homogeneous type is R^n with the euclidean distance and the Lebesgue measure. Some familiar facts in R^n do not remain valid in general. However, whenever it is possible, it is desirable to recover them in some way or another. For instance in [5] it is shown that it can always be found a suitable quasi-distance, equivalent to the original one, havingthe property that the open balls are open subsets . A troublesome feature of spaces of homogeneous type is that, if B is a ball and B' is another ball with center in B and radius smaller than that of B, the measure of $B \cap B'$ is not, in general, greater than a constant fraction of the measure of B', as it is the case in R^n . This problem arises, for example , even in the rather simple case of the parabolic space (see [2]) induced in R^2 by the 2 x 2 diagonal matrix (a_{ij}) with $a_{11} = 3$ and $a_{22} = 1$. This handicap of the spaces of homogeneous type originates technical difficulties, for instance, when trying to solve problems involving weights. In this paper we give a method of constructing a quasi-distance equivalent to the original one, such that the balls defined by the new quasi-distance have the desired property (see Theorem (2.7)). In situations where the substitution of the original quasi-distance by an equivalent one does not affect the nature of the problem under study this method may be useful. We ilustrate this situation by giving in part 3 of this paper an application to the problem of weighted norm inequalities for the Hardy-Littlewood maximal function. This problem in spaces of homogeneous type was already considered by A.P. Calderón in [1] , where some additional restrictions on the space of homogeneous type are imposed. The use of Theorem (2.7) enables us to apply, in an almost literal manner, the method developped by R. R. Coifman and C. Fefferman for the case of R^n (see [3]). The result obtained in Theorem (3.2) extends that of [1] since less restrictions are imposed on the space of homogeneous type.

§ 1. Definitions and Notation.

Let X be a set. A real valued nonnegative function d(x, y) on $X \times X$ shall be called a quasi-distance on X, if the following conditions are satisfied:

(1.1) $$d(x, y) = 0 \text{ if and only if } x = y,$$

(1.2) $$d(x, y) = d(y, x) \text{ for every } x \text{ and } y \text{ in } X,$$

and there exists a finite constant K such that

(1.3) $$d(x, y) \leq K \, [d(x, z) + d(z, y)]$$

for every x, y and z in X. Let $U(r) = \{(x, y): d(x, y) < r\}$. The family $\{U(r)\}_{r>0}$ of subsets of $X \times X$ defines a basis of a metrizable uniformity for X. The balls

$B_d(x, r) = \{ y: d(x, y) < r \}$, $r > 0$, form a basis of neighborhoods for the topology induced by the uniformity. Let us consider a set X endowed with a quasi-distance d(x,y) and a positive measure μ, defined on a σ-algebra of subsets of X containing all the balls $B_d(x, r)$. We shall say that (X, d, μ) is a space of homogeneous type if there exists a finite constant A such that

$$(1.4) \qquad 0 < \mu(B_d(x, 2r)) \leq A\, \mu(B_d(x, r)) < \infty ,$$

hold for every x in X and $r > 0$. Using a Wiener type covering lemma (see, for instance, lemma 3 of [1]) we can see that every open subset is a countable union of balls. This shows that every open subset is measurable. As usual, if U and V are subsets of $X \times X$, $U \circ V$ shall stand for the composition of U and V, that is to say, the set of all the ordered pairs (x, y) such that there is an element z in X satisfying that $(x, z) \in U$ and $(z, y) \in V$.

§ 2. Construction of an equivalent quasi-distance.

Let n be a nonnegative integer, $0 < 2Ka < 1$ and $r > 0$. We define the subsets $U(r, n)$ of $X \times X$ as

$$U(r, 0) = U(r)$$

and, for $n > 0$,

$$U(r, n) = U(a^n r) \circ U(r, n-1) \circ U(a^n r).$$

Moreover, we shall denote by $V(r)$ the union

$$V(r) = \bigcup_{n=0}^{\infty} U(r, n).$$

It is easy to see that the family $\{V(r)\}_{r>0}$ also defines a uniformity for X. The following lemma shows that the uniformities for X defined by $\{U(r)\}_{r>0}$ and $\{V(r)\}_{r>0}$ coincide .

Lemma (2.1). For every $r > 0$, the following inclusions hold:

$$U(r) \subset V(r) \subset U(3k^2 r) .$$

Proof: It is clear from the definitions of $V(r)$ and $U(r, 0)$ that $U(r)$ is contained in $V(r)$. In order to show the remaining inclusion, let us consider $(x, y) \in V(r)$. Then, by definition of $V(r)$, there exists $n \geq 0$ such that $(x, y) \in U(r, n)$. Therefore, we can get a finite sequence $\{y_k\}$, $k = \pm 1, \pm 2, \ldots, \pm(n+1)$, satisfying the following conditions:

(2.2) $y_{n+1} = y$, $y_{-n-1} = x$, $(y_{-1}, y_1) \in U(r)$, and if $0 < k < n$, both (y_k, y_{k+1}) and (y_{-k-1}, y_{-k}) belong to $U(a^k r)$.

From this and the quasi-triangular property (1.3), we obtain

$$d(x, y) < K[d(x, y_1) + d(y_1, y)] < K^2 r + K^2[d(x, y_{-1}) + d(y_1, y)] .$$

By repeated application of property (1.3) it follows that

$$d(y_1, y) < \sum_{j=1}^{n} K^j d(y_j, y_{j+1}) \text{ and } d(x, y_{-1}) < \sum_{j=1}^{n} K^j d(y_{-j}, y_{-j-1}).$$

Taking into account the definition of $\{y_k\}$ and that $2Ka < 1$, we obtain

$$d(x, y) < K^2 r + 2K^2 \sum_{j=1}^{n} K^j a^j r < K^2 r (1 + Ka)(1 - Ka)^{-1} < 3K^2 r,$$

which proves that $V(r) \subset U(3K^2 r)$, as we wanted to show.

<u>Definition (2.3)</u>. Let x and y belong to X. We define $\delta(x, y)$ as

$$\delta(x, y) = \inf \{r : (x, y) \in V(r)\}$$

It is immediate that $\delta(x, y)$ is a nonnegative function on $X \times X$ and satisfies (1.2).

The following lemma shows that $\delta(x, y)$ is equivalent to $d(x, y)$.

<u>Lemma (2.4)</u>. For every x and y in X, we have

$$\delta(x, y) \leq d(x, y) \leq 3 K^2 \delta(x, y).$$

<u>Proof</u>. Given $r > d(x, y)$, from lemma (2.1), we get $(x, y) \in V(r)$. Therefore $\delta(x, y) < r$. This implies that $\delta(x, y) \leq d(x, y)$. On the other hand, if $s > \delta(x, y)$ then $(x, y) \in V(s)$. Applying again Lemma (2.1) we obtain $(x, y) \in U(3 k^2 s)$ or, equivalently, $d(x, y) < 3 k^2 s$. Hence $d(x, y) \leq 3 k^2 \delta(x, y)$∎

<u>Corollary (2.5)</u>. The function $\delta(x, y)$ is a quasi-distance on X which is equivalent to $d(x, y)$ and satisfy the quasi-triangular property (1.3), with a constant $K' = 3 K^3$.

In the next lemma we show the relationship between the quasi-distance $\delta(x, y)$ and the subsets $V(r)$ of $X \times X$.

<u>Lemma (2.6)</u>. Let x belong to X and $r > 0$. If $B_\delta(x, r) = \{y : \delta(x, y) < r\}$. Then

$$B_\delta(x, r) = \{y : (x, y) \in V(r)\}.$$

<u>Proof</u>. Take $y \in B_\delta(x, r)$. Then $\delta(x, y) < r$, implying that

$(x, y) \in V(r)$. To prove the converse, let us assume that $(x, y) \in V(r)$. As in the proof of Lemma (2.1), there exist $n > 0$ such that $(x, y) \in U(r, n)$ and a sequence $\{y_k\}$, $k = \pm 1, \pm 2, \ldots, \pm (n+1)$, satisfying (2.2). Since this sequence is finite it also satisfies the conditions (2.2) with some $s < r$ instead of r. This shows that $(x, y) \in U(s, n) \subset V(s)$. Therefore, $\delta(x, y) < s < r$. Hence $y \in B_\delta(x, r)$, as we wanted to prove ∎

The main result of this paper is stated in the next theorem.

Theorem (2.7). Let $\delta(x,y)$ be the quasi-distance defined in (2.3). There exists a constant $C > 0$ such that, if $x \in X$, $0 < r \leq 2KR$, and $y \in B_\delta(x,R)$, then

$$\mu(B_\delta(y, r) \cap B_\delta(x, R)) \geq C\, \mu(B_\delta(y, r)) .$$

Proof. Let p and m be the integers satisfying $a^p < 2K \leq a^{p-1}$, and $a^{m+1} R < r \leq a^m R$, respectively. Observe that $p \leq 0$. From the assumption $r \leq 2KR$, we get that $m + 1 > p - 1$. Then, if $j = m - p + 3$, it follows that $j \geq 2$. Since $y \in B_\delta(x, r)$, from Lemma (2.6) we obtain that $(x, y) \in V(r)$. Let n be the nonnegative integer such that $(x, y) \in U(R, n)$, and let us consider the sequence $\{y_k\}$, $k = \pm 1, \pm 2, \ldots, \pm (n+1)$ satisfying the conditions (2.2). We shall distinguish two cases, $n \geq j$ and $n < j$. Let us consider the case $n \geq j$. If z is a point in the ball

$B_d(y_j, a^j R)$, by repeated use of the quasi-triangular inequality, the definition of the sequence $\{y_k\}$ and the fact that $2Ka < 1$, we get

$$(2.8)\quad \begin{aligned} d(z,y) &\leq K\,[d(z,y_j) + d(y_j,y)] \\ &< K a^j R + K \sum_{i=0}^{n-j} K^{i+1}\, d(y_{j+i}, y_{j+i+1}) \\ &< a^{j-2} R \sum_{i=1}^{\infty} (Ka)^i \leq 2 K a^{j-1} R. \end{aligned}$$

This implies that $B_d(y_j, a^j R) \subset B_d(y, 2Ka^{j-1}R)$. From the definition of the integers p and j, we have $2Ka^{j-1} \leq a^{p+j-2} = a^{m+1}$. Therefore

$$B_d(y_j, a^j R) \subset B_d(y, a^{m+1}R) \subset B_d(y, r) \subset B_\delta(y, r).$$

On the other hand, by considering the sequence obtained from $\{y_k\}$ replacing y_{j+1} throughout y_{n+1} by z, it is clear that $B_d(y_j, a^j R) \subset B_\delta(x, R)$. Thus, we obtain

$$B_\delta(y_j, a^j R) \subset B_d(y, r) \cap B_\delta(x, R).$$

From Lemma (2.4) and (2.8) applied to $z = y_j$, we get

$$B_\delta(y, r) \subset B_d(y, 3K^2 r) \subset B_d(y_j, 6K^3 a^{p-3} a^j R).$$

The homogeneity of the measure (1.4) and these inclusions imply that there exists a constant C, depending only on K and A, such that

$$C\,\mu(B_\delta(y, r)) \leq \mu(B_d(y_j, a^j R)).$$

Summing up, we have

$$C\,\mu(B_\delta(y, r)) \leq \mu(B_d(y_j, a^j R)) \leq \mu(B_\delta(y, r) \cap B_\delta(x, R)),$$

as we wanted to prove ∎

Let us consider now the case $n < j$. In this case, we have

$B_d(y, a^j R) \subset B_d(y, a^{n+1} R)$. Let $z \in B_d(y, a^{n+1} R)$.

If we add the terms $y_{-n-2} = x$ and $y_{n+2} = z$ to the sequence $\{y_k\}$, $k = \pm 1, \pm 2, \ldots, \pm(n+1)$, the resulting sequence shows that $(x, z) \in V(R)$. By Lemma (2.1), we get $B_d(y, a^{n+1} R) \subset B_\delta(x, R)$. Hence

(2.9) $\quad B_d(y, a^j R) \subset B_\delta(x, R).$

On the other hand, using Lemma (2.4) and the facts that $p \leq 0$ and $a^{m+1} R < r$, we have

(2.10) $\quad B_d(y, a^j R) \subset B_\delta(y, a^j R) \subset B_\delta(y, a^{m+1} R) \subset B_\delta(y, r).$

From (2.9) and (2.10), we obtain

$$B_d(y, a^j R) \subset B_\delta(y, r) \cap B_\delta(x, R).$$

Finally, since $r \leqslant a^m R$ and using again Lemma (2.4), we get

$$B_\delta (y, r) \subset B_d(y, 3 K^2 a^m R) \subset B_d(y, 6 K^2 a^{p \cdot 3} a^j R).$$

Therefore, by homogeneity of the measure, it turns out that

$$C \quad \mu(B_\delta(y, r)) \leqslant \mu(B_\delta(y, r) \cap B_\delta(x, r))$$

holds with the same constant C obtained for the first case. This finishes the proof of the theorem ∎

From corollary (2.5) and theorem (2.7) we obtain.

Corollary (2.11). Let (X, d, μ) be a space of homogeneous type. Then the function $\delta(x, y)$, defined in (2.3), is a quasi-distance equivalent to $d(x, y)$. Moreover, the balls $B_\delta(x, R)$ endowed with the restrictions of the quasi-distance $\delta(x, y)$ and the measure μ become spaces of homogeneous type with constants K' and A' independent of $R > 0$ and $x \in X$.

§ 3. Application.

Let f be a locally integrable function defined on a space of homogeneous type (X, d, μ). Given a measurable set E, we denote by $m_E(f)$ the average $m_E(f) = \mu(E)^{-1}\int_E f\, d\mu$. The Hardy-Littlewood maximal function f^* of f is defined as $f^*(x) = \sup_{r>0} m_{B(x,r)}(|f|)$. As usual, we shall say that a non-negative function w(x) satisfies the condition A_p if there exists a constant C such that

$$(A_p) \qquad m_B(w) \cdot m_B(w^{-1/(p-1)}) \leq C < \infty$$

holds for any ball B.

B. Muckenhoup proves in [6] that in the case of the euclidean space R^n with the ordinary distance and the Lebesgue measure

$$\int_{R^n} f^*(x) \quad w(x)\, dx \quad \leq \quad C \int_{R^n} |f(x)|^p \; w(x) \quad dx$$

holds with a constant C independent of f if and only if w(x) satisfies A_p. Later R. R. Coifman and C. Fefferman gave in [3], among other things, a simplified demonstration of this fact. Imposing some restrictions on the space, A.P. Calderón (see [1]) generalized this result to spaces of homogeneous type. In all these proofs the crucial step is to show that if w(x) satisfies A_p, then it also satisfies a "reverse Hölder inequality". Namely, there exists a constant C, and q > 1 such that

$$m_B (w^q)^{1/q} < C m_B(w).$$

holds for every ball B. The core of the dificulty in the extension of the results above to spaces of homogeneous type lies in the proof of an adequate Calderón-Zygmund type lemma. A more careful analysis seems to indicate that the dificulty is due to the fact that in spaces of homogeneous type if B is a ball and B' is another ball with center in B and radius smaller than that of B then, the measure of $B \cap B'$ is not necessarily grater than a constant fraction of the measure of B'. The result obtained in Theorem (2.7) allows us to handle this problem, making possible to apply, almost without change, the simplified method given in [3], and eliminating some of the restrictions imposed in [1].

We proceed to give an outline of this application of the results of the preceding paragraph. Given a ball B = B(x, r) we denote by $\tilde{B}$ the ball $\tilde{B}$ = B (x, 5 K r).

We shall make use of the following version of the Calderón-Zigmund lemma:

<u>Lemma (3.1)</u>. Let (Y, d, μ) be a bounded space of homogeneous type, and let $f(y)$ be a nonnegative integrable function on Y. Then for any $\lambda \geqslant m_Y(f)$ there exists a sequence $\{B_i\}$ of disjoints balls such that

(i) $\lambda < m_{B_i}(f) < A\lambda$, where A is the constant satisfying (1.4) in Y.

(ii) For every y in the complement of $\cup_i \tilde{B}_i$ and every ball B centered at y

$$m_B(f) \leqslant \lambda .$$

<u>Proof</u>. Since Y is bounded, there exists R positive such that $Y = B(y, R)$ for every y in Y. Let Ω be the set of points $y \in Y$ such that there exists a ball B with center at y satisfying $m_B(f) > \lambda$. For any y in Ω the set $\{r : m_{B(y,r)}(f) > \lambda\}$ is not empty, and bounded above by R. Hence, there exists $r(y) > 0$ such that

$$m_{B(y,r(y))}(f) > \lambda \qquad \text{and } m_{B(y,2r(y))}(f) \leqslant \lambda$$

Applying a Wiener type lemma (see for instance Lemma 3 in [1]) to the family $\{B(y, r(y))\}_{y \in \Omega}$, we obtain a countable subfamily of disjoint balls $B_i = B(y_i, r_i)$ such that $\Omega \subset \cup_i \tilde{B}_i$

From the definiton of the B_i and (1.4) we have

$$\lambda < m_{B_i}(f) \leqslant \mu(B(x_i, 2r_i))\ \mu(B(x_i, r_i))^{-1} m_{B(y_i, 2r_i)}(f) \leqslant A\lambda.$$

This proves (i). To prove (ii) let y be such that $y \notin \cup \tilde{B}_i$.
Then $y \notin \Omega$. By definition of Ω , this implies that for every ball B centered at y $m_B(f) \leqslant \lambda$, as we wanted to show.

<u>Theorem (3.2)</u>. Let (X, d, μ) be a space of homogeneous type such that the continous functions with compact support are dense in $L^1(X, d\mu)$. If w(x) satisfies condition A_p , then there exist $\delta > 0$ and $C < \infty$ such that

$$m_B(w^{1+\delta}) \leqslant C\, m_B(w)^{1+\delta}$$

holds for every ball B.

<u>Proof</u>. It follows easily from condition A_p that $m_{B(x,2r)}(w) \leqslant C\, m_{B(x,r)}(w)$, where C is a finite constant independent of x and r. Using this property it is simple to see that if $d'(x, y)$ is a quasi-distance equivalent to $d(x, y)$ then the theorem is valid in (X, d, μ) if and only if it is valid in (X, d', μ). Accordingly, by lemma (2.4), it is enough to prove the theorem for the quasi-distance $\delta(x, y)$ defined in (2.3).

Proceeding exactly like in [3] it can be proved that the condition A_p implies the existence of two positive numbers α and β such that for every ball B_δ we have

$$(A'_\infty) \qquad \mu(B_\delta) < \alpha \cdot \mu(\{x : w(x) > \beta\, m_\delta(w)\} \cap B_\delta) .$$

We proceed to prove that there exists $C > 0$ satisfying that for every ball B_δ and $\lambda \geq m_{B_\delta}(w)$,

$$(3.3) \qquad \int_{\{x : w(x) > \lambda\} \cap B_\delta} w(x)\, d\mu(x) \leq C\, \lambda\, \mu(\{x : w(x) > \beta\lambda\} \cap B_\delta)$$

holds. Applying Corollary (2.11), we get that (B_δ, δ, μ) is a bounded space of homogeneous type with constants K' and A' independent of the ball B_δ under consideration. Applying Lemma (3.1) to (B_δ, δ, μ) and $f(x) = w(x)$ we get that there exists a sequence of disjoint balls B_i in this space such that

$$(3.4) \qquad \lambda < m_{B_i}(w) \leq A'\lambda .$$

Moreover, since we assume that the continuous functions are dense in $L^1(x)$ then, the Lebesgue differentiation theorem is valid (see f.i. [4], pp. 39-40). From part (ii) of Lemma (3.1) we obtain that $w(x) \leq \lambda$ for almost every $x \notin \cup \tilde{B}_i$. Therefore

$$\{x: w(x) > \lambda\} \cap B_\delta \subset \bigcup_i \tilde{B}_i .$$

Hence,

$$(3.5) \qquad \int_{\{x: w(x) > \lambda\} \cap B_\delta} w(x)\, d\mu(x) \leq \sum_i \int_{\tilde{B}_i} w(x)\, d\mu(x).$$

It can be easily deduced from Theorem (2.7) that if $w(x)$ satisfies A_p then its restriction to any ball B_δ also satisfies A_p in the space of homogeneous type (B_δ, δ, μ). Therefore, proceeding in the usual manner, it can be shown that

$$\int_{\tilde{B}_i} w(x)\, d\mu(x) \leq C \int_{B_i} w(x)\, d\mu(x) .$$

Thus, from (3.4), (3.5) and condition A'_∞ it follows

$$(3.6) \qquad \int_{\{x: w(x) > \lambda\} \cap B_\delta} w(x)\, d\mu(x) \leq C \sum_i \int_{B_i} w(x)\, d\mu(x) \leq A' \lambda \sum_i \mu(B_i)$$

$$\leq C A' \alpha \lambda \sum_i \mu(\{x: w(x) > \beta\, m_{B_i}(w)\} \cap B_i).$$

Since the sets B_i form a family of disjoint balls in the space of homogeneous type (B_δ, δ, μ), we have

$$\sum_i \mu(\{x: w(x) > \beta\, m_{B_i}(w)\} \cap B_i) = \mu(\{x: w(x) > \beta\, m_{B_i}(w)\} \cap B_\delta).$$

This equality and (3.6) imply (3.3), as was to be shown.

To finish the proof of the Theorem we just have to repeat the arguments given in [3, p. 248] .

REFERENCES

[1] A. P. Calderón, "*Inequalities for the maximal function relative to a metric*", Studia Math. 57 (1976), 207-306.

[2] A. P. Calderón and A. Torchinsky, "*Parabolic maximal functions associated with a distribution*", Advances in Math. 16 (1975), 1-63.

[3] R. R. Coifman and C. Fefferman , "*Weighted norm inequalities for maximal functions and singular integrals*", Studia Math. 51 (1974), 241-250.

[4] M. de Guzmán, "*Differentiation of integrals in* R^n ", Lecture Notes in Math. vol. 481, Springer-Verlag, Berlin and New York, 1975.

[5] R. A. Macías and C. Segovia, "*Lipschitz functions on spaces of homogeneous type*", Advances in Math, 33 (1979), 257-270.

[6] B. Muckenhoupt, "*Weighted norm inequalities for the Hardy maximal function*", Trans. Amer. Math. Soc. 167 (1972), 207-226.

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