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TOPOLOGY OPTIMIZATION OF CONTINUUM TWO-DIMENSIONAL STRUCTURES UNDER COMPLIANCE AND STRESS CONSTRAINTS

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Abstract. *This paper presents the problem of volume minimization of two-dimensional continuous structures with compliance and stress constraints. Problems are solved by a topology optimization technique, formulated as finding the best material distribution into the design domain. Discretizing the geometry into simpler pieces and approximating the displacement field, equilibrium equations are solved through the finite element method. A material parametrization method is used to represent the fictitious constant material distribution into each finite element. Sequential Linear Programming is used to solve the optimization problem. For both compliance and stress constrained problems, an analytical sensitivity analysis for elastic behavior is derived, and for this last problem, Von Mises equivalent stress is the failure criteria considered. A first neighborhood filter was implemented to minimize the effects of checkerboard patterns and mesh dependency, two common problems associated to topology optimization. Stress constrained problems have a further difficulty, the stress singularity, which may prevent the algorithm to reach a feasible solution. To overcome this problem, the feasible domain is modified using a mathematical perturbation technique, the epsilon-relaxation.*

1 INTRODUCTION

More and more, the human being is increasingly aware of the necessity of saving natural resources. This fact is the main motivation for researching optimum designs.

Structural engineers also adopted this trend in the design of new structures or modification of existent ones. In this context, a structure can be considered as an amount of distributed material over a design domain, in order to support loads (static or dynamic), absorb and distribute energy and transmit it to the supports. One of the goals of optimum design is best distribute the available material into the design domain.

Initially, only relatively simple optimization problems could be addressed, due to the difficulty in solving equilibrium equation for more complicated structures. In 1872, Maxwell[1] derived analytical solutions for the minimum volume problem in uniaxial structures subjected to several types of loads. Some years after, in 1904, Michell[2] developed analytical solutions for minimum weight trusses, also subjected to different load types and applying constraints on stresses in each bar (figure 1):

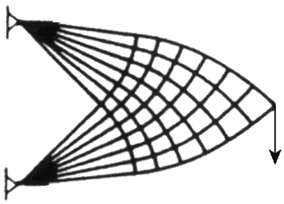

Figure 1: One load case Michell-like truss structure. It is the best possible truss for this specific load case. Notice that the bars crossing themselves forming 90°. This is an unstable structure for any other type of loads

However, structural optimization became practical only when numerical methods began to be used, specially for solving the equilibrium equations. One very popular method is the Finite Elements Method, where the continuum is approximated by an assembly of simpler geometric domains.

Mathematical programming is another important tool created to help the solution process in optimization problems. According to Rozvany et. al.[3], before the arise of the mathematical programming, the updating of the design parameters was based on analytical methods (many of them heuristically decided), known as optimality criteria.

In the early 60's, Schmit[4] published an important work, considered the dawn of the modern structural optimization. In his work with trusses, he combined the Finite Elements Method for the structural analysis with Linear Programming for the optimization.

With the development of 2-D and 3-D finite elements, new contributions to the structural optimization field were developed. An important result was obtained by Cheng and Olhoff[5], in 1981. Studying the problem of optimum thickness distribution in plates under compliance and natural frequency constraints, they concluded that the geometrical irregularities obtained in the thickness distribution could be interpreted as ribs (stiffeners). They also concluded that the exact solution for plate optimization contains an infinite number of ribs, so the finer the finite elements mesh more ribs will appear. This result showed the necessity of considering some kind of microstructure to find one valid macroscopic solid-void layout.

Addressing this problem, several authors worked with relaxed formulations, by relating the constitutive material properties with microstructural parameters. Works by Allaire and Kohn[6] (1993), Niordson[7] (1983) and Rossow and Taylor[8],(1973) present different methods of material parameterization.

Rossow and Taylor[8], for example, proposed a minimum compliance problem for membranes in plane stress behavior. Design variables were the thickness of each finite element, and they were related to the material stiffness according to the equation:

$$E = hE^0 \tag{1}$$

where h is the thickness and E^0 is the isotropic material stiffness. Imposing upper and lower bounds to the possible values of thickness, the obtained solution is a combination of elements with maximum, minimum and intermediary thickness.

However, topology optimization only received more attention after the introduction of Homogenization Method by Bendsøe and Kikuchi[9], in 1988. This theory is considered a natural extension of previous works, like Reiss[10] (1976) and Cheng[11] (1981).

This material parameterization model considers the existence of periodic microstructures (figure 2), from which composite material effective properties are computed. Mathematically, the different material scales are split using an asymptotic expansion (Sanchez-Hubert and Sanchez-Palencia[12]).

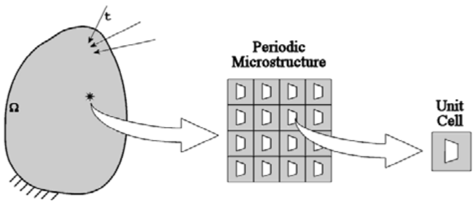

Figure 2: Representation of a composite material made of a periodic microstructure

Dealing either with isotropic or anisotropic material constitutive laws, this model considers the material stiffness as a function of microstructure and a density-like parameters, as follows:

$$E = E^0\left(\rho,\theta,\mu,\ldots\right) \tag{2}$$

In 1989, Bendsøe[13] proposed another type of material parameterization, nowadays named SIMP (Solid Isotropic Microstructure with Penalization). Differently from the homogenization, this approach considers the existence of only one design variable, a constant fictitious density (ρ) in each finite element. Therefore, the new stiffness parameterization is calculated as follows:

$$E = \rho^n E^0$$
$$0 \le \rho \le 1 \tag{3}$$

where n defines the amount of penalization. If n>1, material stiffness is penalized, avoiding the appearing of low stiffness elements (with intermediary densities). This parameterization is an extension of the work of Rossow and Taylor[8] (1973). If n=1, we have a very similar problem to that shown in equation (1), except that the upper bound is set to unity.

Besides the common problems associated to the topology optimization solution (checkerboard patterns and mesh dependency, for example), stress constraints (as considered in this work) bring further ones. Firstly, stress is a local constraint, then, each infinitesimal point of the structure should have its stress level under control. Moreover, the singular optimum phenomenon can arise.

Stress singularity was firstly pointed by Sved and Ginos[14], in 1968. Performing an analytical study on the 3 bar truss problem with 3 load cases and stress constraints, they have found out that only removing one structural bar the global optimum could be reached. This apparently simple problem defied all mathematical programming algorithms, causing stress constraint violation, or even non-convergence. Moreover, algorithmic difficulties appear because in most codes finite elements can not be simply removed from the mesh. The physical cause is easily understandable: in bars, the cross sectional area of each bar (or density, in SIMP model, according to Duysinx and Bendsøe[15] (1998)) is inversely proportional to the stress. Thus, when areas tend to vanish, stresses may increase unreasonably.

This problem was insolvable for many time, until 1997, when Cheng and Guo[16] proposed a perturbation technique called epsilon-relaxation. Thus, reformulating the stress constrained problem, the design space is modified, including new subdomains to the design space. This is a manner to modify the dimension of this space, without add or remove bars. Then, the design space is successively diminished, by decreasing this perturbation value, so that the solution of these series of modified sub-problems converge to the correct solution of the original problem.

Finally, in 1998 Duysinx and Bendsøe[15] extended this technique for two-dimensional continuum problems. In this work, they also developed an analytical solution for the sensitivity analysis of Von Mises equivalent stress.

The present paper presents some classical results obtained when solving the minimum volume problem with compliance and with Von Mises stress constraints. Both source codes were developed in MATLAB, including analysis and optimization sub-routines. Taylor non-conforming element is used to solve the equilibrium and Sequential Linear Programming is the chosen first order decision algorithm.

2 STRUCTURAL OPTIMIZATION

2.1 Basic concepts and definitions

Structural optimization aims to increase the structural performance of components and mechanical systems in a systematic way. Thus, firstly we need to identify which design variables best describes the features of some component. Then, by modifying these variables, following some criteria, we obtain the best solution, among a set of solutions.

Design variables for a typical structural optimization problem can be the elements size, structural configuration, mechanical or physical properties of materials, or other qualitative aspects for the project being analyzed. Cost function, also known as objective function is the scalar function to be minimized (or maximized) during optimization process. Constraints are conditions imposed to the physical problem, representing the limit of the admissible space. Any solution with a violated constraint represents an infeasible solution. More details concerning these and other basic concepts can be found in Arora[17] (1989), Haftka and Gürdal[18] (1992) and Bendsøe[19] (1995).

Sensitivity analysis is the computation of the derivatives of the objective function and the constraints with respect to the design variables. It points the direction where the optimization algorithm should follow in the design space. This work focus on the sensibility analysis of the compliance and Von Mises stress constraints with respect to the design variables, using the analytical method. This approach is simple to derive and allows the efficient use of mathematical programming. Figure 3 illustrates the iterative process:

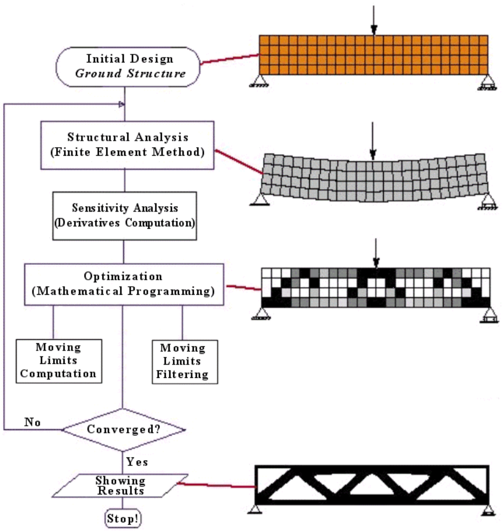

Figure 3: Basic algorithm for solving a topology optimization problem

Topology optimization aims to find the best stiffness distribution within an admissible domain while satisfying the constraints. Figure 4 better illustrates this concept:

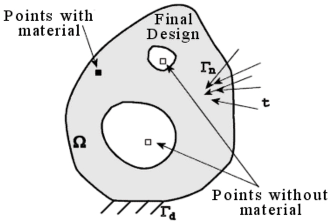

Figure 4: Representation of topology optimization problem, where the goal is to find the best material (or stiffness) distribution along the design domain

Mathematically, the material stiffness is formulated as follows:

$$E = 1_{\Omega^m} E^0$$
$$\text{where } 1_{\Omega^m} = \begin{cases} 1 \text{ if } x \in \Omega^m \\ 0 \text{ if } x \in \Omega \setminus \Omega^m \end{cases} \tag{4}$$

Solving this problem is a very hard task, due to its combinatorial nature. In 1997, Beckers[20] solved this problem in an efficient way. In her formulation, she considered compliance and perimeter constraints, in order to guarantee the existence of solution.

However, to make the solution process easier, this problem is usually relaxed, by making use of microstructure parameters as design variables. Homogenization Method (as represented in (2)) or SIMP Method (equation (3)) are commonly used.

Due to its relative simplicity, SIMP method was used in this work. The optimization problem is solved by using sequences of isotropic materials. Doing this, the only design variable is the constant density in each finite element, designed as ρ. The next picture shows the relation between ρ and different penalization levels concerning to the material stiffness (in normal direction, for example):

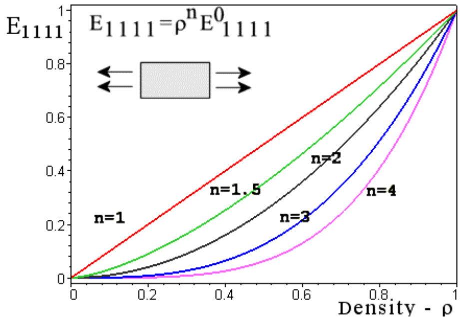

Figure 5: Relation between ρ and material stiffness

Intermediary densities represent an unknown isotropic microstructure with a known stiffness. In practical sense, these intermediary densities are not desirable, because their stiffness is too low and, at least nowadays, it is not possible to manufacture a stable composite microstructure formed by solid isotropic material and voids.

Penalizing intermediary densities make their stiffness low comparing to the stiffness of a solid-void structure. Thus, to respect equilibrium and constraint, the optimization decision algorithm makes the intermediary densities attain the upper and lower bounds.

Another important consideration arises when the mixtures theory is considered. One can see that n=1 represents an unattainable superior limit to the stiffness, i. e., no microstructure could be built with this stiffness and density. By the other hand, n=2 is an attainable superior limit for the stiffness.

2.2 Common problems associated to topology optimization

Nowadays, there are commercial programs to solve only simple topology optimization problem. When developing a new computer code, many computational and theoretical issues appear. The most common are the following:

a) checkerboard patterns;
b) mesh dependency;
c) local minima; and
d) singular topologies (for stress constrained problems).

Checkerboard pattern is one of the most common problems related to topology optimization. Figure 6 shows a typical example of this phenomenon, i. e., solid and void elements alternating themselves.

Figure 6: Example of checkerboard pattern

This phenomenon is a convergence problem caused by the incorrect evaluation of the strain energy by the finite element mesh. For example: if one is solving the problem of minimum compliance with volume constraint, the topology shown on figure 6 is really the minimum solution of the finite element problem, but not the continuum problem.

Therefore, this is not a desirable solution. To overcome this problem, we can use high order finite elements, perimeter constraints or filtering techniques. The first solution leads to a more expensive computer problem and, sometimes, can not even solve the problem if SIMP exponent higher than 3, for example (Jog and Haber[21] (1996)). Perimeter constraint is a good solution, because we are not only solving the checkerboard pattern but also the mesh dependency problem. Thus, constraining the perimeter, we can avoid the formation of several small holes (voids between two solid elements in a checkerboard pattern, for example).

Two drawbacks can be noted in this formulation. The first and more direct is that we are adding a new constraint in the optimization problem, and manage with many constraints usually is not an easy task. The second one is that, a priori, we have no idea about which amount of perimeter we have to constraint. This can lead to different final topologies.

In this work we have used the filtering strategy, by controlling the upper and lower moving limits gradients, according to the following equation:

$$x_i = w_1 x_{i-1} + w_2 x_i + w_3 x_{i+1} \tag{5}$$

where w_j is the filter weight and x_k is the upper or lower density bound (calculated through the moving limits) in the direction X_1 or X_2. A good setting for these weights is 0.02-0.96-0.02, i. e., to compute the density in the element i, 2% of the density from its neighbors is taken account, and “only” 96% from its density is considered.

Other two common related problems, according to Sigmund and Petersson[22] are the mesh dependency and local minima. Figure 7 illustrates these drawbacks:

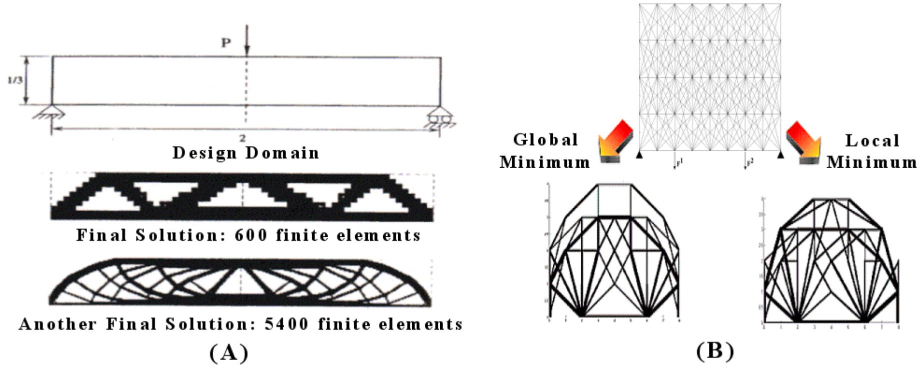

Figure 7: (A) Mesh dependency and; (B) local minimum

The mesh dependency problem comes from the fact that when the original discrete problem is relaxed (using finite elements, for example), each new mesh refinement leads to a new solution. There are several ways to overcome this problem. In fact, the same techniques used to avoid checkerboard can be used to control the mesh dependency.

Local minima can take place due to the non-convexity of the involved functions, as the case of penalized constitutive or objective function parameterization or stress constraints. A local minimum design is often impossible to avoid, unless constraints and objective function are both convex. In most practical cases, we do not know exactly the topology of design set. Thus, when a solution is found, we cannot guarantee that this solution is the global optimum. There are two very expensive possible solutions: the first one is starting the problem from several different initial designs, comparing the final value of the objective function. This means one have to solve several optimization problems. Another solution is using globally convergent algorithms, such as genetic or simulated annealing algorithms. Due to their combinatorial nature, using these algorithms can be prohibitive for practical designs.

The last problem, called stress singularity happens only when stress constraints are considered. As described by Cheng and Jiang[23], the cause of this problem is that the stress function is non-continuous when one element reaches the minimum value for cross section area (or density). A typical stress constraint is formulated as follows:

$$\sigma(\rho)-\bar{\sigma}\leq 0 \tag{6}$$

where $\bar{\sigma}$ is an equivalent stress measurement (such as Von Mises stress).

When this density tends toward zero, stress value tends to infinitum. Thus, one algorithm based on Karush-Kuhn-Tucker (KKT) optimality conditions (such as SLP) can not reach the actual optimum. Figure 8 shows a simple example where stress singularity happens (this example was taken from Hoback[24]):

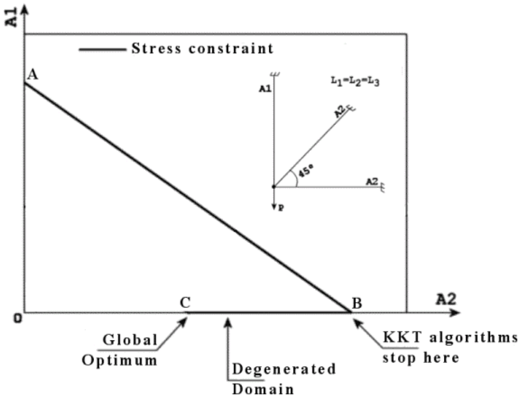

Figure 8: Qualitative effect of singular topologies phenomenon

The global optimum is located in a sub-domain whose dimension is smaller than the dimension of the whole space. In the above truss, this means to remove the bar whose cross section area is A_1 (point C). When solving this in a computer, we can not only remove that bar (or 2D finite element), because it makes the stiffness matrix loses its definite positiveness. One could make the cross sectional area of bar 1 goes toward one minimum value. But, from the picture we can see this leads to a different result, a local minimum (at point B). The proof can be found in Cheng and Guo[16] (1997).

In this same paper is proposed a solution for this problem, called epsilon-relaxation. The constraint showed in equation (6) is then modified:

$$\sigma(\rho) - \bar{\sigma} \leq \varepsilon \tag{7}$$

This parameter ε is basically a perturbation, where we change the design space by applying a large epsilon value (typically 10^{-1}). Gradually, we decrease this value towards zero, returning to the original problem. When ε is enough small, it can be proven that the final design is the correct optimum (point C, on figure 8).

3 FORMULATION AND SOLUTION STRATEGIES

In this paper, we have solved two types of problems: minimum volume considering compliance or stress constraints. In both, SIMP method without penalization (n=1) is used. Using this approach, the density has the same physical meaning of the membrane thickness problem. In order to diminishes checkerboard effects, we have made use of objective function (structural volume) penalization. The effects on the topology is very similar to the SIMP method, in spite its physical meaning is not. Considering the objective function is stated according to equation (8), figure 9 illustrates the effects of volume penalization (using FEM notation):

$$V = \sum_{i=1}^{ne} \left[\rho_i^p + \alpha \rho_i \left(1 - \rho_i\right) \right] V_i \tag{8}$$

where p and α are parameters that set the level of penalization and V_i is the volume of element i.

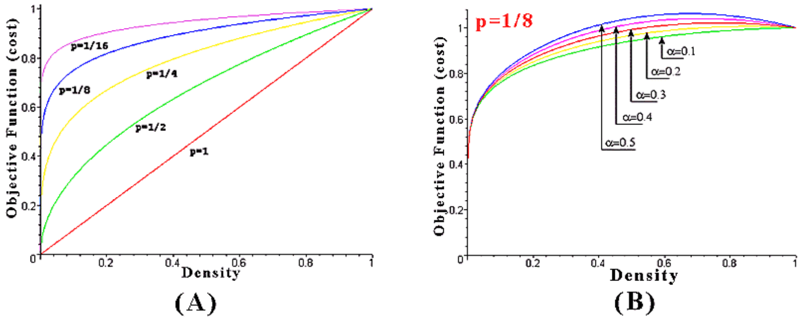

Figure 9: Different penalizations for equation (8): (A) varying p (α=0) and; (B) fixed p (=1/8) and different values for α

As one can see, differently from SIMP method, where the stiffness is penalized, the objective function is related to the cost, such that intermediary densities have a very high cost. Then, the optimization algorithm leads these densities to the maximum or minimum values. If p=1 and α=0, the objective function is the structural volume.

In order to avoid convergence problems, we commonly start without penalization. So, using a "continuation method" (Cardoso[25] (1999)), we continuously decrease the value of p (usually up to 1/8), obtaining a more and more cleaner final topology. If it is necessary, a stronger penalty function is used, by fixing p and varying α (figure 9 (B)).

Sequential Linear Programming (SLP), one type of mathematical programming algorithm was used to decide the new set of design variables in each new iteration. If the objective function and/or constraints are not linear, we still can use SLP by applying Taylor Series Expansion. A serious drawback of this algorithm is its strong dependency to the moving limits. Moving limits are parameters used to respect the linearization in an iteration. The adopted approach in this work is to start with a big value for the moving limits, reducing them next to the convergence.

3.1 Minimum volume with compliance constraints

$$\begin{aligned} \min \quad & V = \sum_{i=1}^{ne} \left[\rho_i^p + \alpha \rho_i \left(1 - \rho_i\right) \right] V_i \\ \text{s. t.:} \quad & W^k(\rho) \le W_{\lim} \\ & 0 < \rho_{\min} \le \rho \le 1 \end{aligned} \tag{9}$$

where V is the objective function, W^k is the compliance value for load case k and $W_{\lim}$ is a given compliance limit. W can be calculated as follows:

$$W = \underbrace{u^T K}_{\substack{\text{from discretized} \\ \text{equilibrium} \\ \text{equation: } Ku=f}} u = f^T u \tag{10}$$

where u is the nodal displacement vector and f is the nodal external loads vector. These vectors are linked through the discretized equilibrium equation:

$$Ku = f \tag{11}$$

In order to adapt to the standard form of linear programming, the above constraint is linearized by Taylor Series expansion:

$$\frac{\partial W^k}{\partial \rho_i} \rho_i \le W_{\lim}(\rho_0) - W^k(\rho_0) + \frac{\partial W^k}{\partial \rho_i} \rho_0 \tag{12}$$

3.2 Minimum volume with Von Mises equivalent stress

$$\begin{aligned} \min \quad & V = \sum_{i=1}^{ne} \left[\rho_i^p + \alpha \rho_i \left(1 - \rho_i\right) \right] V_i \\ \text{s. t.:} \quad & \rho \left(\frac{\sigma_i^k}{\sigma_{vm}} - 1 \right) \le \varepsilon \\ & \varepsilon^2 = \rho_{\min} \le \rho \le 1 \end{aligned} \tag{13}$$

where σ_i^k is the equivalent stress in the element i for load case k, σ_{vm} is the limit stress and ε is the parameter that defines the epsilon-relaxation perturbation. Epsilon-relaxation and equivalent stress formulation have already been applied to the constraint as shown in (13). The relationship with the density comes from the fact that each component from the stress tensor is inversely proportional to the density, when using SIMP method. Thus:

$$\sigma_{ij} = \frac{\langle \sigma_{ij} \rangle}{\rho^m} \quad \text{or} \quad \sigma_{vm} = \frac{\langle \sigma_{vm} \rangle}{\rho^m} \tag{14}$$

Von Mises equivalent stress can be calculated as follows, using a displacement notation:

$$\sigma_{vm} = \sqrt{u^T M u} \tag{15}$$

where u is the nodal displacement vector, $M = T^T V T$, $T = \underbrace{\rho^n E^0}_{SIMP} B$ and $V = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Again, applying Taylor Series expansion to stress constraint, we obtain (remember that σ_i is the Von Mises stress in element i):

$$\frac{\partial \sigma_i^k}{\partial \rho_j} \rho_j \leq \sigma_{vm} - \sigma_i(\rho_0) + \frac{\partial \sigma_i^k}{\partial \rho_j} \rho_0 + \sigma_{vm} \frac{\varepsilon}{\rho_j} \tag{16}$$

3.3 Sensitivity analysis

Sensitivity analysis is a very important stage when solving an optimization problem. This informs the directions the solver must follow during the search for the optimum. Physically, this represents how changes one given function when one changes the design variable. If a first order algorithm (such as SLP) is used, only first order derivatives are required. There are several methods to compute these sensitivities: finite differences method, adjoint method, semi-analytical method or analytical method. In this work we have chosen the last one, due to its simplicity to derive.

Thus, in this case, calculating the derivative of objective function is very easy. Differentiating equation (8) with respect to the design variable, we obtain:

$$\frac{\partial V}{\partial \rho_i} = \left[p\left(\rho_i^{p-1}\right) + \alpha\left(1 - 2\rho_i\right)\right] V_i \tag{17}$$

Calculating the compliance sensitivities is also easy. Differentiating equation (10) with respect to ρ_i, we obtain:

$$\frac{\partial W^k}{\partial \rho_i} = \left(f^k\right)^T \frac{\partial u^k}{\partial \rho_i} + \frac{\partial \left(f^k\right)^T}{\partial \rho_i} u^k \tag{18}$$

Using the discretized equilibrium equation (equation (11)), the orthogonality of the stiffness matrix and considering no existence of body loads exerting on the structure, we obtain:

$$\frac{\partial W^k}{\partial \rho_i} = -\left(u^k\right)^T \frac{\partial K}{\partial \rho_i} u^k \tag{19}$$

The derivative of global stiffness matrix with respect to the design variables is simple to be computed. In fact, it corresponds to the local stiffness matrix for element i.

To computing the stress sensitivity, Duysinx and Bendsøe[15] (1998) approach was used. Firstly, one have to remember that stress is a global constraint. Thus, we want to know what is happening with stress in a element i when we change the density in one element j. Therefore, differentiating equation (15) with respect to design variables (ρ_j), making use of the definition for stresses in porous environments (equation (14)), and again using the discretized equilibrium equation (equation (11)), we have (again, body forces are not considered):

$$\frac{\partial \sigma_i}{\partial \rho_j} = \frac{1}{\sigma_{vm}^0} u^T M_i^0 \underbrace{K^{-1}\left(-\frac{\partial K}{\partial \rho_j} u\right)}_{\frac{\partial u}{\partial \rho_j}} \tag{20}$$

4 RESULTS

4.1 Compliance constraint

As a first example, we have simulated the famous MBB beam. This represents a semi-long isostatic beam under bending loads. Geometry and boundary conditions are shown in figure 10:

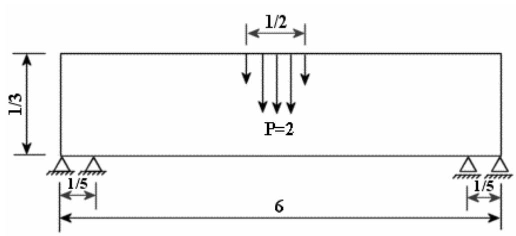

Figure 10: MBB beam under flexion (SI units)

This structure was discretized using **3330** Taylor elements. The used material has **E=1 N/m^2** and ν**=0.3.** Compliance limit was set to **W_{lim}=300.8 Nm** (50% higher than initial design). Thus, solving the problem and applying necessary penalizations, we obtain:

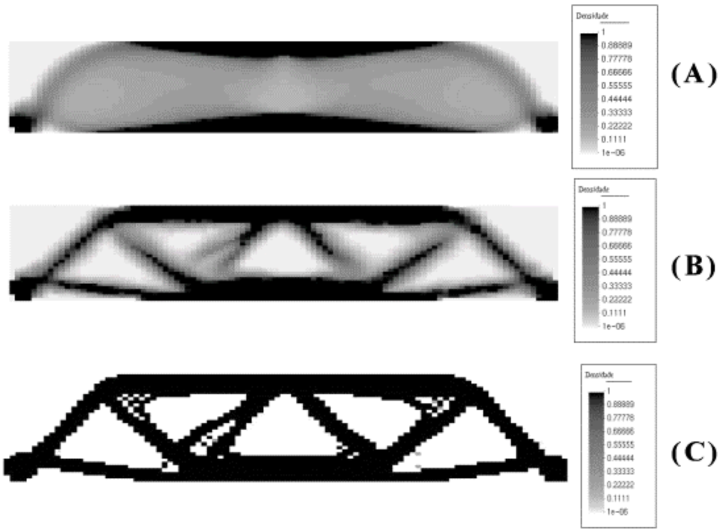

Figure 11: (A) Solution without penalization; (B) p=1/8, α=0 and; (C) p=1/8, α=0.3

Although the expected result is a symmetric structure, due to the symmetry of boundary conditions, not completely symmetrical structures can be obtained when solving topology optimization problems using linear programming.

We have solved this problem again, making some minor changes to the geometry and boundary conditions (the structure is not isostatic anymore):

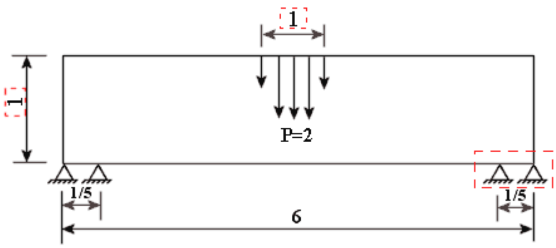

Figure 12: Another problem using MBB beam under flexion (SI units)

For this problem, we have **2400 finite elements, W_{lim}=147.7 Nm, E=2.1 x 10^{11} N/m^2** and ν=**0.3.** Applying all the possible filtering and penalizations, the obtained structure can be seen in the next picture:

Figure 13: Final topology

Although problems shown in pictures 10 and 12 are very similar, final topology in both problems is very different. The main reason for these discrepant results is the applied boundary conditions. While left support of first example is only constraining X_2 displacement, the second example has all directions constrained. Thus, in this last case, additional "bars" are not necessary in the supports region.

4.2 Von Mises equivalent stress constraint

For stress constrained problems, epsilon-relaxation must be applied for a correct convergence. Similarly to the continuation methods, we start with a big value for ε (0.1, for example), and solve this problem. When it converges, ε value is successively reduced by a factor of 10, up to the final convergence, when ε is small (typically 10^{-6} to 10^{-8}).

The first solved example is shown on figure 14:

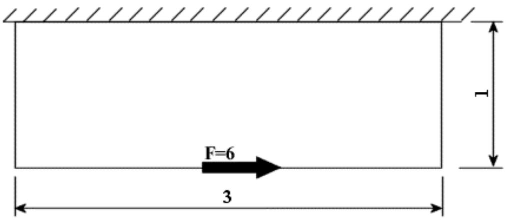

Figure 14: Geometry and boundary conditions

Since this a large scale problem, we have discretized this structure using **675 finite elements**. This means we have **675 constraints** and **675 design variables.** Material has **E=1N/m^2**, **ν=0.3** and **σ_{lim}=35 N/m^2**. After filtering, penalizing and applying ε-relaxation technique, we obtain the following result:

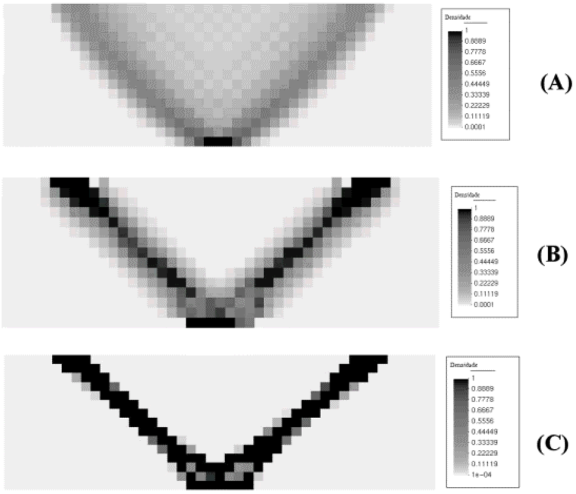

Figure 15: Final results for $\varepsilon=10^{-7}$; (A) p=1, α=0; (B) p=1/8, α=0; and (C) p=1/8, α=0.3

The stress field is represented on next picture, where can be seen that the stress limit is been respected:

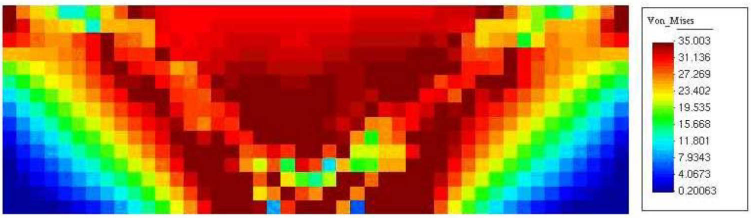

Figure 16: Stress field for the optimized problem. σ_{max}=35 N/m^2

Removing elements with minimum density and reanalyzing, we obtain a maximum stress of **49.6 N/m^2**. This means although we have solve the mathematical problem, this structure could not be manufactured, since it is not respecting the stress limit.

This method can be improved by adding more elements in the mesh, for a better representation of the stress field. But this is out of scope for this work.

Even though, we have analyzed another structure, whose geometry and boundary conditions is represented in picture 17:

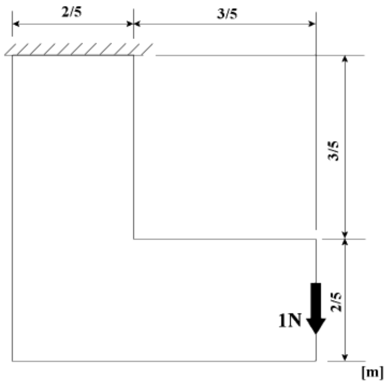

Figure 17: Geometry and boundary conditions for the L-shape beam

This problem has **576 design variables** and **constraints.** Material properties are **E=1 N/m^2, ν=0.3** and **σ_{vm}=50 N/m^2.** Performing the same procedures described in the previous examples, we obtain the following:

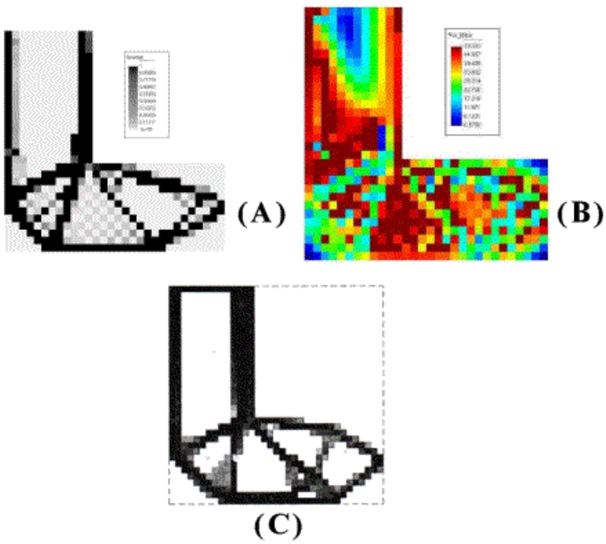

Figure 18: (A) Final topology for $\varepsilon=10^{-8}$, p=1/8 and α=0.3; (B) stress field, σ_{max}=50.5 N/m^2; and (C) Similar final topology obtained by Duysinx and Bendsøe[15] (1998) with 9-node quadrilateral elements

Again, we had the same problem, because only the mathematical problem was solved.

When removing low density elements, maximal stress was increased to **133.1 N/m^2**.

5 CONCLUSIONS

All the developed algorithms not only solved satisfactorily the proposed problems but also obtained results according to the literature.

Sequential linear programming algorithm can solve different types of constraints, but it is very sensitive to the initial parameters, such as moving limits.

All problems demand an extra post-processing in order to eliminate the jagged boundaries. Shape optimization after topology optimization is essential to obtain better results.

Problems considering stress constraints require a more refined finite element mesh to obtain better solutions for engineering problems.

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