Mecánica Computacional Vol. XXIV
A. Larreteguy (Editor)
Buenos Aires, Argentina, Noviembre 2005

STABILITY OF IMPLICIT METHODS ON STAGGERED GRIDS

Cassio M. Oishi and José A. Cuminato

Departamento de Matemática Aplicada e Estatística, ICMC,
Universidade de São Paulo,
Av. Trabalhador Saocarlense, 400, C.P. 668, 13251-900, São Carlos, Brasil
e-mail: oishi@icmc.usp.br jacumina@icmc.usp.br

Key Words: Implicit methods, Staggered grids, Navier-Stokes equations, Free surface flows.

Abstract. *The numerical solution of the Navier-Stokes equations with free surface when the Reynolds number is very small (*$Re \ll 1$*) requires the use of implicit time discretization. The parabolic stability condition of explicit methods imposes severe restrictions on the time-step making them too time consuming to be useful in practice. In the context of staggered grids, this work presents a study of the numerical stability of implicit methods. This stability is directly connected to the appropriate use of boundary conditions on the free surface and on rigid walls. The boundary conditions must be discretized with care so that the resulting method does not become conditionally stable. The stability results are derived for the model problem of the heat equation, and then applied to Navier-Stokes equations examples.*

1 INTRODUCTION

The MAC method (Marker-And-Cell)[1] is one of the first successful attempts to simulating viscous, incompressible, transient flows with free surfaces. The MAC method is derived from the discretization of the Navier-Stokes equations in primitive variables by finite differences on an uniform staggered mesh. In two dimensions, the velocity u is approximated on the right face of a grid cell, the velocity v is approximated on the top face of a grid cell, while the pressure is approximated at the cell center. The free surface is represented by massless particles that are passively carried by the flow velocities. Over the years several improvements on the MAC method have been proposed in the literature. Firstly the calculation of the pressure on the free surface have been improved, see,[2–4] and more accurate methods for tracking the free surface have been developed. Variations of the MAC method where those improvements have been implemented include: SMAC,[5] SUMMAC,[2] ALE,[6] SOLA-VOF,[7] TUMMAC,[8,9] GENSMAC[10,11] and SIMAC.[12] More recently other authors have used essentially the same MAC ideas: to simulate the impact drop problem,[13] for second order reconstruction of interfaces[14] and in a Lagrangian-Eulerian technique for the simulation of tridimensional flows in arbitrary domains.[15] Two common features to all these techniques are the explicit time discretization of the momentum equations by the Euler method and the use of a staggered grid. The use of an explicit method implies that the parabolic linear stability restriction on the time step applies. This restriction depends both on the Reynolds number and on the mesh spacing and is given by the expression

$$\delta t_{visc} \leq 0.5Re[(\delta x)^{-2} + (\delta y)^{-2}]^{-1}, \tag{1}$$

where δt_{visc} is the non-dimensional time-step resulting from the stability condition on the viscous terms. For inertial flows ($Re > 1$) the stability condition does not impose a severe restriction on the time step, and explicit methods will produce a numerical solution in a reasonable time. However, in some applications involving non-Newtonian fluids, very low Reynolds numbers flows can be encountered.[16,17] Reynolds numbers of order 10^{-1} to 10^{-4} are easily found in applications involving the flow of a polymer, eg. extrudate swell,[18] injection moulding,[19] jet buckling[20] and container filling.[20] For this class of problems an implicit method must be used in order to overcome the parabolic stability restriction. Implicit time discretization of the momentum equations can be derived, for instance, via the Euler implicit or the Crank-Nicolson methods. To gain full advantage of the unrestricted stability usually enjoyed by these methods the correct boundary conditions must be imposed on the free surface. In a previous work[21] the authors discussed how to impose implicit boundary conditions on the free surface and yet keep the linear system arising from the implicit discretization of the momentum equations decoupled from the system arising from the discretization of Laplace's equation for the pressure. In this paper we study the influence of the rigid walls boundary conditions on the stability of such implicit methods. As it is usually the case in stability studies the analysis is performed for a model problem. In this study our model problem will be the one dimension heat equation with Dirichlet boundary conditions. In the next section we present a detailed account of the problem and its discretization.

2 MODEL PROBLEM AND DISCRETIZATION

The model problem is the one dimension heat equation:

$$u_t = u_{xx}, \quad x \in [0, 1] \quad \text{and} \quad t > 0 \tag{2}$$

with initial and Dirichlet boundary conditions given by:

$$u(x, 0) = f(x), \quad u(0, t) = g(t) \quad \text{and} \quad u(1, t) = h(t). \tag{3}$$

As we are only interested in studying the stability of the numerical methods for the Navier-Stokes equations with boundary conditions on rigid walls where mainly the no-slip condition apply, we can assume that in (3) both functions $g(t)$ and $h(t)$ vanish identically. In fact the value of the boundary condition, do not affect in any way the stability of the method. In order to mimic the effects of the staggered grid discretization of the Navier-Stokes we shall approximate the heat equation and its boundary conditions on the interval $[0, 1]$ by the following three steps:

1. Discretize the interval $[0, 1]$ by a set of equally spaced points $x_i = (i - 1/2)\delta x,\ i = 0, 1, \ldots, m + 1$ where $\delta x = 1/m$;

2. At the internal points $x_1, x_2, \ldots, x_m$, approximate the heat equation (2) implicitly by either the Euler implicit or by the Crank-Nicolson methods;

3. As the discretization points x_0 and x_{m+1} do not coincide with the end points of the interval $[0, 1]$, interpolation must be used to eliminate the unknown values of u_0^n and u_{m+1}^n from the equations obtained in step 2. Here we are using the notation u_i^n to denote an approximation to $u(x_i, t_n)$.

Figure 1 illustrates the staggered mesh we are using for solving (2).

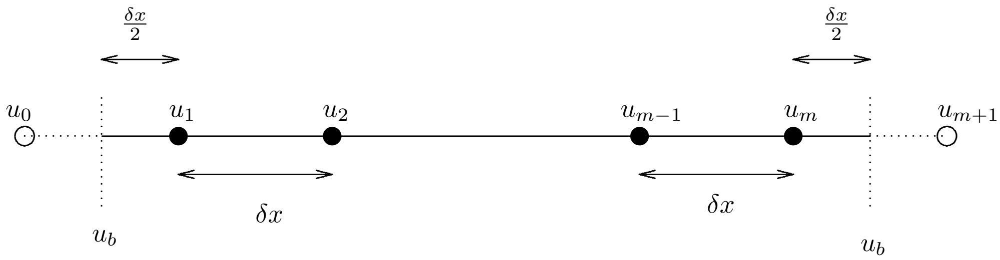

Figure 1: Staggered grid for solving (2) with boundary conditions (3), where $u(0, t) = u(1, t) = u_b$.

Linear interpolation is usually the choice for approximating the boundary values in step 3 above. The interpolation polynomial of degree one through the points (x_0, u_0^l) and (x_1, u_1^l), where l is a generic time level, is given by:

$$P_1(x) = \frac{1}{\delta x}\left((x - x_0)u_1^l - (x - x_1)u_0^l\right). \tag{4}$$

Using now the boundary condition $u(0,t) = 0$, we have that:

$$P_1(0) = 0 = \frac{1}{\delta x}\left(\frac{\delta x}{2}(u_1^l + u_0^l)\right) = \frac{1}{2}(u_0^l + u_1^l). \tag{5}$$

Analogous interpolation at the end $x = 1$ leads to the equation $\frac{1}{2}(u_{m+1}^l + u_m^l) = 0$. Hence the equations for the unknowns u_0^l and u_{m+1}^l become

$$u_0^n = -u_1^n \quad \text{and} \quad u_{m+1}^n = -u_m^n, \tag{6}$$

in the explicit case, and

$$u_0^{n+1} = -u_1^{n+1} \quad \text{and} \quad u_{m+1}^{n+1} = -u_m^{n+1}, \tag{7}$$

in the implicit case. The finite difference discretization of equation (2) by the explicit method, at the internal points $x_1, x_2, \dots x_m$ is,

$$u_i^{n+1} = \sigma u_{i+1}^n + (1-2\sigma)u_i^n + \sigma u_{i-1}^n, \tag{8}$$

by the Euler implicit method is

$$-\sigma u_{i-1}^{n+1} + (1+2\sigma)u_i^{n+1} - \sigma u_{i+1}^{n+1} = u_i^n, \tag{9}$$

and by Crank-Nicolson is

$$-\frac{\sigma}{2}u_{i-1}^{n+1} + (1+\sigma)u_i^{n+1} - \frac{\sigma}{2}u_{i+1}^{n+1} = \frac{\sigma}{2}u_{i-1}^n + (1-\sigma)u_i^n + \frac{\sigma}{2}u_{i+1}^n, \tag{10}$$

where $\sigma = \frac{\delta x}{\delta t^2}$. Observe that any one of the above methods (8), (9) or (10) with any one of the boundary conditions (6) or (7) can be written in matrix form as:

$$\mathsf{A}\mathbf{u}^{n+1} = \mathsf{B}\mathbf{u}^n + \mathbf{c} \tag{11}$$

where A, B are $m \times m$ matrices, $\mathbf{u} = (u_1, u_2, \dots, u_m)^T$ and $\mathbf{c} = (c_1, c_2, \dots, c_m)^T$ are $m \times 1$ vectors, all of them defined by the particular choice of the method and boundary condition. The numerical stability of the method will be determined by the eigenvalues λ of the iteration matrix $\mathsf{M} = \mathsf{A}^{-1}\mathsf{B}$. If those eigenvalues all have modulus smaller than one the method will be stable and if at least one eigenvalue has modulus greater or equal to one the method will be unstable. Note that the eigenvalues will depend on the parameter σ. When all the eigenvalues of the iteration matrix have modulus less than one for all $\sigma > 0$ we say that the method is unconditionally stable, if this is true only for σ on a finite interval we say the method is conditionally stable. In the next section we shall analyze the stability of 6 different choices of method and boundary condition, namely:

- Explicit method with explicit boundary conditions - conditionally stable for $\sigma \in [0, \frac{1}{2})$;

- Euler implicit with explicit boundary conditions - unconditionally stable;
- Euler implicit with implicit boundary conditions - unconditionally stable;
- Crank-Nicolson with explicit boundary conditions - conditionally stable for $\sigma \in [0, 2)$;
- Crank-Nicolson with implicit boundary conditions 1 - unconditionally stable;
- Crank-Nicolson with implicit boundary conditions 2 - unconditionally stable.

3 STABILITY ANALYSIS

The stability analysis of the above methods will be based on important theorems from linear algebra concerning the eigenvalues of a general matrix and tridiagonal matrices. Before we go on to derive the stability results we will present these theorems. For general matrices, important theorems to estimate the eigenvalues are the so called Gershgorin's theorem. These theorems give the geometrical location of the eigenvalues, i.e., the Gershgorin circle theorem identifies a region in the complex plane that contains all the eigenvalues of a complex square matrix. For tridiagonal matrices, we shall present a theorem that give an exact expression for the eigenvalues, in special cases.

Theorem 1. *Let* $\mathsf{A} \in \mathbb{C}^{m\times m}$*. Then*

$$\sigma(\mathsf{A}) \subseteq S_R = \bigcup_{i=1}^{m} R_i, \quad R_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j=1, j\neq i}^{m} |a_{ij}| \right\}, \tag{12}$$

where a_{ij} *are the elements of the matrix* A *for* $i, j = 1, \ldots, m$ *and* $\sigma(\mathsf{A})$ *is the set of the eigenvalues of* A *called the spectrum of* A*. The sets* R_i *are called Gershgorin circles.*

The proof of theorem 1 is presented in.[22]

Theorem 2. *Let* $\mathsf{A} \in \mathbb{R}^{m\times m}$ *with* $\mathsf{A} = \mathsf{E} + \mathsf{F}$*, and* E *and* F *symmetric matrices, then*

$$\lambda_m^{\mathsf{E}} + \lambda_i^{\mathsf{F}} \leq \lambda_i^{\mathsf{A}} \leq \lambda_1^{\mathsf{E}} + \lambda_i^{\mathsf{F}}, \tag{13}$$

where $\lambda_m^{\mathsf{E}} \leq \lambda_{m-1}^{\mathsf{E}} \leq \cdots \leq \lambda_2^{\mathsf{E}} \leq \lambda_1^{\mathsf{E}}$.

See the proof of the theorem 2 in.[23]

Theorem 3. *Consider the tridiagonal matrix of the form*

$$\mathsf{M} = \begin{bmatrix} -\alpha + b & c & 0 & 0 & \ldots \\ a & b & c & 0 & \ldots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \ldots & 0 & a & b & c \\ \ldots & 0 & 0 & a & -\beta + b \end{bmatrix}_{m\times m} \tag{14}$$

Case 1*: If $\alpha = \beta = 0$, then the eigenvalues λ_i^{M} of M are given by*

$$\lambda_i^{\mathsf{M}} = b + 2\sqrt{ac}\cos(\frac{i\pi}{m+1}), \quad i = 1, \dots, m. \tag{15}$$

Case 2*: If $\alpha = \beta = \sqrt{ac} \neq 0$, then the eigenvalues λ_i^{M} of M are given by*

$$\lambda_i^{\mathsf{M}} = b + 2\sqrt{ac}\cos(\frac{i\pi}{m}), \quad i = 1, \dots, m. \tag{16}$$

Case 3*: If $\alpha = \beta = -\sqrt{ac} \neq 0$, the eigenvalues λ_i^{M} of M are given by*

$$\lambda_i^{\mathsf{M}} = b + 2\sqrt{ac}\cos(\frac{(i-1)\pi}{m}), \quad i = 1, \dots, m. \tag{17}$$

Details of the proof of this theorem are presented by Yueh in.[24]

Definition 1. *Let λ_i be the eigenvalues of the matrix A. Then*

$$|\lambda_{max}^{\mathsf{A}}| = \max\{|\lambda_1^{\mathsf{A}}|, \dots, |\lambda_m^{\mathsf{A}}|\}, \tag{18}$$

and

$$|\lambda_{min}^{\mathsf{A}}| = \min\{|\lambda_1^{\mathsf{A}}|, \dots, |\lambda_m^{\mathsf{A}}|\}. \tag{19}$$

Explicit method with explicit boundary conditions

In this case the matrices in (11) are $\mathsf{A} = \mathsf{I}$ and

$$\mathsf{B} = \begin{bmatrix} 1-3\sigma & \sigma & 0 & 0 & \dots \\ \sigma & 1-2\sigma & \sigma & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & \sigma & 1-2\sigma & \sigma \\ \dots & 0 & 0 & \sigma & 1-3\sigma \end{bmatrix}_{m\times m} \tag{20}$$

The matrix B above can be rewritten as

$$\mathsf{B} = \mathsf{I} + \sigma\tilde{\mathsf{B}}, \tag{21}$$

where

$$\tilde{\mathsf{B}} = \begin{bmatrix} -3 & 1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & 1 & -2 & 1 \\ \dots & 0 & 0 & 1 & -3 \end{bmatrix}_{m\times m} \tag{22}$$

Theorem 4. *The eigenvalues of* $\mathsf{M} = \mathsf{A}^{-1}\mathsf{B}$ *all have modulus smaller than 1 if* $\sigma \in [0, \frac{1}{2})$.

Proof: The matrix $\tilde{\mathsf{B}}$ satisfies the assumptions of theorem 3, case 2 with $\alpha = \beta = 1$, $a = c = 1$ and $b = -2$. So its eigenvalues can be computed from (16) giving $\lambda_i^{\tilde{\mathsf{B}}} = -2 + 2\cos(\frac{i\pi}{m})$ for $i = 1, \ldots, m$. Hence, the eigenvalues of the B are given by

$$\lambda_i^{\mathsf{B}} = 1 - 2\sigma\left(1 - \cos(\frac{i\pi}{m})\right) = 1 - 2\sigma\left(2\sin^2(\frac{i\pi}{2m})\right) = 1 - 4\sigma\sin^2(\frac{i\pi}{2m}), \tag{23}$$

for $i = 1, \ldots, m$. As $\mathsf{A} = \mathsf{I}$, the eigenvalues of $\mathsf{M} = \mathsf{A}^{-1}\mathsf{B}$ are the same as in (23). The eigenvalue of maximum modulus occurs for $i = m$, i.e., $|\lambda_{max}^{\mathsf{M}}| = |\lambda_m^{\mathsf{M}}| = \left|1 - 4\sigma\sin^2(\frac{\pi}{2})\right| = |1 - 4\sigma|$. Therefore,

$$|\lambda_{max}^{\mathsf{M}}| = |1 - 4\sigma| < 1 \Rightarrow \sigma < \frac{1}{2}. \tag{24}$$

This turns out to be the well-known result, that the explicit method is stable for $\sigma \in [0, \frac{1}{2})$.

Euler implicit with explicit boundary conditions

For the case of Euler implicit with explicit boundary conditions the matrices A and B of (11) become:

$$\mathsf{A} = \begin{bmatrix} 1+2\sigma & -\sigma & 0 & 0 & \ldots \\ -\sigma & 1+2\sigma & -\sigma & 0 & \ldots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \ldots & 0 & -\sigma & 1+2\sigma & -\sigma \\ \ldots & 0 & 0 & -\sigma & 1+2\sigma \end{bmatrix}_{m\times m} \tag{25}$$

and

$$\mathsf{B} = \begin{bmatrix} 1-\sigma & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \ldots & 0 & 0 & 1 & 0 \\ \ldots & 0 & 0 & 0 & 1-\sigma \end{bmatrix}_{m\times m} \tag{26}$$

The matrix A can be rewritten as

$$\mathsf{A} = \mathsf{I} + \sigma\tilde{\mathsf{A}}, \tag{27}$$

where

$$\tilde{\mathsf{A}} = \begin{bmatrix} 2 & -1 & 0 & 0 & \ldots \\ -1 & 2 & -1 & 0 & \ldots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \ldots & 0 & -1 & 2 & -1 \\ \ldots & 0 & 0 & -1 & 2 \end{bmatrix}_{m\times m} \tag{28}$$

Thus the iteration matrix is $\mathsf{M} = \mathsf{A}^{-1}\mathsf{B} = (\mathsf{I} + \sigma\tilde{\mathsf{A}})^{-1}\mathsf{B}$. If λ_i^{M} are the eigenvalues of M, then $\frac{1}{\lambda_i^{\mathsf{M}}}$ are the eigenvalues of M^{-1}.

Now we may write $\mathsf{M}^{-1} = \mathsf{B}^{-1}(\mathsf{I} + \sigma\tilde{\mathsf{A}})$ giving,

$$\mathsf{M}^{-1} = \begin{bmatrix} \frac{1+2\sigma}{1-\sigma} & \frac{-\sigma}{1-\sigma} & 0 & 0 & \dots \\ -\sigma & 1+2\sigma & -\sigma & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & -\sigma & 1+2\sigma & -\sigma \\ \dots & 0 & 0 & \frac{-\sigma}{1-\sigma} & \frac{1+2\sigma}{1-\sigma} \end{bmatrix}_{m\times m} \tag{29}$$

Theorem 5. *The eigenvalues of* $\mathsf{M} = \mathsf{A}^{-1}\mathsf{B}$ *all have modulus smaller than 1.*

Proof: To prove this theorem we shall show that the eigenvalues of M^{-1} all have modulus greater than 1. For this purpose we shall use the Gershgorin theorem 1 to find bounds for the eigenvalues. From the form of the matrix M^{-1} we see that $R_1 = R_m$ and $R_2 = \ldots = R_{m-1}$, with

$$R_1 = \left\{ z \in \mathbb{C} : \left| z - \frac{1+2\sigma}{1-\sigma} \right| \le \left| \frac{-\sigma}{1-\sigma} \right| \right\}, \tag{30}$$

and

$$R_2 = \{ z \in \mathbb{C} : |z - (1+2\sigma)| \le |-\sigma| + |-\sigma| \} . \tag{31}$$

We know from theorem 1 that any eigenvalue of M^{-1} is in either R_1 or R_2. Let α be the real number closest to the origin, defined by intersection of the boundary of R_i with the real line. Consider first an eigenvalue λ which is in R_2 then $|\lambda - (1+2\sigma)| < r$.

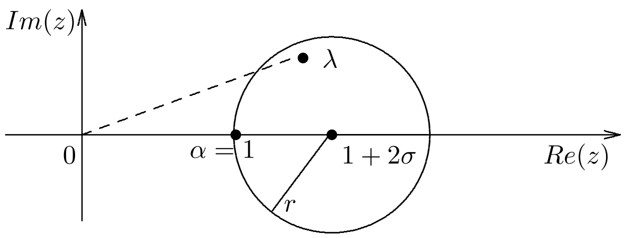

Figure 2: Gershgorin circle R_2.

As $\sigma > 0$, the real number α is given by $\alpha = 1 + 2\sigma - 2\sigma = 1$. Any eigenvalue in R_2 has, obviously modulus greater than α, hence $|\lambda| > 1$, see figure 2.

Let now $\lambda \in R_1$, there are two different cases:

- Case 1: $\sigma < 1$

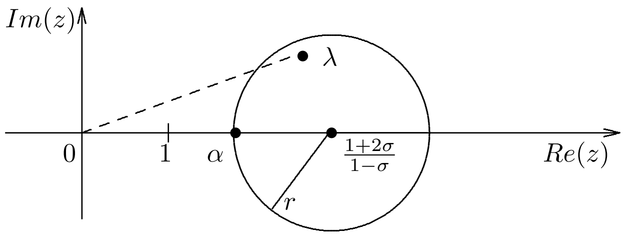

Figure 3: Gershgorin circle R_1 for $\sigma < 1$.

In this case, the center of the Gershgorin circle R_1 is $c = \frac{1+2\sigma}{1-\sigma}$, and the radius $r = \left|\frac{-\sigma}{1-\sigma}\right|$. As $\sigma < 1$, $c > 0$ and $r = \frac{\sigma}{1-\sigma}$. Thus,

$$\alpha = c - r = \frac{1+2\sigma}{1-\sigma} - \frac{\sigma}{1-\sigma} = \frac{1+\sigma}{1-\sigma} > 1. \tag{32}$$

Therefore any eigenvalue λ of M^{-1} which is in R_1 will satisfy $|\lambda| > 1$ in this case, see figure 3.

- Case 2: $\sigma > 1$

  As $\sigma > 1$, we have $c < 0$ and $r = \frac{\sigma}{\sigma-1}$. Thus,

$$\alpha = c + r = \frac{1+2\sigma}{1-\sigma} + \frac{\sigma}{\sigma-1} = \frac{1+\sigma}{1-\sigma} < -1. \tag{33}$$

  Again any eigenvalue λ of M^{-1} in R_1 will have its modulus greater than 1, see figure 4.

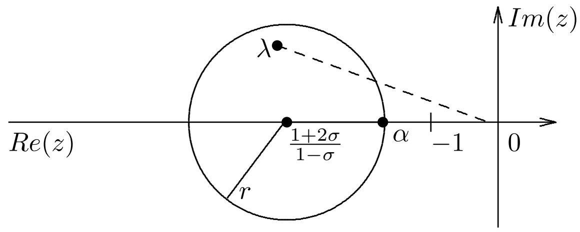

Figure 4: Gershgorin circle R_1 for $\sigma > 1$.

Euler implicit with implicit boundary conditions

For this case the matrices $\mathsf{B} = \mathsf{I}$ and A is given by

$$\mathsf{A} = \begin{bmatrix} 1+3\sigma & -\sigma & 0 & 0 & \dots \\ -\sigma & 1+2\sigma & -\sigma & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & -\sigma & 1+2\sigma & -\sigma \\ \dots & 0 & 0 & -\sigma & 1+3\sigma \end{bmatrix}_{m\times m} \tag{34}$$

The matrix A can be rewritten as

$$\mathsf{A} = \mathsf{I} - \sigma\tilde{\mathsf{A}}, \tag{35}$$

where

$$\tilde{\mathsf{A}} = \begin{bmatrix} -3 & 1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & 1 & -2 & 1 \\ \dots & 0 & 0 & 1 & -3 \end{bmatrix}_{m\times m} \tag{36}$$

Theorem 6. *The eigenvalues of* $\mathsf{M} = \mathsf{A}^{-1}\mathsf{B}$ *all have modulus smaller than 1.*

Proof: Note that the matrix $\tilde{\mathsf{A}}$ satisfies the conditions of theorem 3 case 2, hence its eigenvalues can be calculated as

$$\lambda_i^{\tilde{\mathsf{A}}} = -2 + 2\cos(\frac{i\pi}{m}) \quad \text{for} \quad i = 1, \dots, m. \tag{37}$$

From (35), the eigenvalues of A are given by:

$$\begin{aligned} \lambda_i^{\mathsf{A}} &= 1 - \sigma\left(-2 + 2\cos(\frac{i\pi}{m})\right) = 1 + 2\sigma\left(1 - \cos(\frac{i\pi}{m})\right) \\ &= 1 + 4\sigma\left(\sin^2(\frac{i\pi}{2m})\right), \quad i = 1, \dots, m. \end{aligned} \tag{38}$$

From (38) we see that

$$|\lambda_i^{\mathsf{A}}| > 1, \quad \forall\sigma > 0 \quad \text{or} \quad \left|\frac{1}{\lambda_i^{\mathsf{A}}}\right| < 1, \quad \forall\sigma > 0, \tag{39}$$

and we have proved that the eigenvalues of the iteration matrix $\mathsf{M} = \mathsf{A}^{-1}\mathsf{B}$ satisfy

$$|\lambda_i^{\mathsf{M}}| < 1, \quad \forall\sigma > 0. \tag{40}$$

Crank-Nicolson with explicit boundary conditions

For this case the matrices A and B are

$$\mathsf{A} = \begin{bmatrix} 1+\sigma & -\frac{\sigma}{2} & 0 & 0 & \dots \\ -\frac{\sigma}{2} & 1+\sigma & -\frac{\sigma}{2} & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & -\frac{\sigma}{2} & 1+\sigma & -\frac{\sigma}{2} \\ \dots & 0 & 0 & -\frac{\sigma}{2} & 1+\sigma \end{bmatrix}_{m\times m} \tag{41}$$

and

$$\mathsf{B} = \begin{bmatrix} 1-2\sigma & \frac{\sigma}{2} & 0 & 0 & \dots \\ \frac{\sigma}{2} & 1-\sigma & \frac{\sigma}{2} & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & \frac{\sigma}{2} & 1-\sigma & \frac{\sigma}{2} \\ \dots & 0 & 0 & \frac{\sigma}{2} & 1-2\sigma \end{bmatrix}_{m\times m} \quad (42)$$

The matrices A and B can be rewritten as

$$\mathsf{A} = \mathsf{I} + \sigma\tilde{\mathsf{A}}, \quad (43)$$

where

$$\tilde{\mathsf{A}} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & \dots \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \dots & 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix}_{m\times m} \quad (44)$$

and

$$\mathsf{B} = \mathsf{I} + \sigma\tilde{\mathsf{B}}, \quad (45)$$

where

$$\tilde{\mathsf{B}} = \begin{bmatrix} -2 & \frac{1}{2} & 0 & 0 & \dots \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ \dots & 0 & 0 & \frac{1}{2} & -2 \end{bmatrix}_{m\times m} \quad (46)$$

Theorem 7. *The eigenvalues of the matrix* $\mathsf{M} = \mathsf{A}^{-1}\mathsf{B}$ *satisfy:*

1. $|\lambda_i^{\mathsf{M}}| < 1$, $i = 1, \dots, m$ *if* $\sigma \in (0, 2)$*;*
2. $|\lambda_i^{\mathsf{M}}| \geq 1$, *for some* i *if* $\sigma \in [2, \infty)$.

Unfortunately the hypotheses of the theorem 3 are not satisfied for the matrix $\tilde{\mathsf{B}}$, so it cannot be used to find its eigenvalues.

We have not been able to fully prove theorem 7. All the numerical evidence obtained from finding the eigenvalues of M for several values of σ and m, indicate that it is true. We have been able to show item 1 of theorem for $\sigma \in (0, \frac{2}{3})$ only. The proof is given below.

Note that the matrix B can be written in the form:

$$\mathsf{B} = \begin{bmatrix} -\sigma & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & 0 & 0 & \\ \dots & 0 & 0 & 0 & -\sigma \end{bmatrix} + \begin{bmatrix} 1-\sigma & \frac{\sigma}{2} & 0 & 0 & \dots \\ \frac{\sigma}{2} & 1-\sigma & \frac{\sigma}{2} & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & \frac{\sigma}{2} & 1-\sigma & \frac{\sigma}{2} \\ \dots & 0 & 0 & \frac{\sigma}{2} & 1-\sigma \end{bmatrix} = \mathsf{E} + \mathsf{F} \quad (47)$$

The eigenvalue of F can be calculated from theorem 3 case 1 giving

$$\lambda_i^{\mathsf{F}} = 1 - \sigma \left(1 - \cos(\frac{i\pi}{m+1})\right), i = 1 \dots m. \tag{48}$$

The eigenvalues of the matrix E are obviously given by $\lambda_1^{\mathsf{E}} = \lambda_m^{\mathsf{E}} = -\sigma$ and all the others are zero. Therefore, using theorem 2 the eigenvalues of B satisfy

$$1 - \sigma \left(2 - \cos(\frac{i\pi}{m+1})\right) \leq \lambda_i^{\mathsf{B}} \leq 1 - \sigma \left(1 - \cos(\frac{i\pi}{m+1})\right). \tag{49}$$

In the inequality 49 the left hand side is larger in absolute value than the right hand side, so that, taking modulus we obtain

$$\left|1 - \sigma \left(2 - \cos(\frac{i\pi}{m+1})\right)\right| \geq |\lambda_i^{\mathsf{B}}| \geq \left|1 - \sigma \left(1 - \cos(\frac{i\pi}{m+1})\right)\right|. \tag{50}$$

The eigenvalue of B of maximum modulus is obtained when $i = m$, i.e,

$$|\lambda_i^{\mathsf{B}}| \leq \left|1 - \sigma \left(2 - \cos(\frac{m\pi}{m+1})\right)\right|. \tag{51}$$

Therefore

$$|\lambda_{max}^{\mathsf{B}}| \leq \left|1 - \sigma \left(2 - \cos(\frac{m\pi}{m+1})\right)\right|. \tag{52}$$

For the matrix A again theorem 3 case 1 gives

$$\lambda_i^{\mathsf{A}} = 1 + \sigma \left(1 + \cos(\frac{i\pi}{m+1})\right), i = 1 \dots m. \tag{53}$$

The eigenvalue of A of minimum modulus is obtained when $i = m$, i.e,

$$|\lambda_{min}^{\mathsf{A}}| = \left|1 + \sigma \left(1 + \cos(\frac{m\pi}{m+1})\right)\right|. \tag{54}$$

Thus as the matrices A and B are symmetric we have

$$|\lambda^{\mathsf{M}}| \leq \| \mathsf{M} \|_2 = \| \mathsf{A}^{-1}\mathsf{B} \|_2 \leq \| \mathsf{A}^{-1} \|_2 \| \mathsf{B} \|_2 = \frac{|\lambda_{max}^{\mathsf{B}}|}{|\lambda_{min}^{\mathsf{A}}|} = \left|\frac{1 - 2\sigma + \sigma\cos(\frac{m\pi}{m+1})}{1 + \sigma + \sigma\cos(\frac{m\pi}{m+1})}\right|. \tag{55}$$

Therefore

$$\left|\frac{1 - 2\sigma + \sigma\cos(\frac{m\pi}{m+1})}{1 + \sigma + \sigma\cos(\frac{m\pi}{m+1})}\right| < 1 \Rightarrow \sigma < \frac{2}{3}. \tag{56}$$

On the other hand, for item 2 we have only been able to prove it for large σ. Indeed, for large σ the eigenvalues of M tend to the eigenvalues of the matrix $\tilde{\mathsf{M}} = \tilde{\mathsf{A}}^{-1}\tilde{\mathsf{B}}$. Let λ be an eigenvalue of $\tilde{\mathsf{M}}$ with eigenvector $\mathbf{v}$, then

$$\tilde{\mathsf{M}}\mathbf{v} = \lambda\mathbf{v} \quad \text{or} \quad \tilde{\mathsf{A}}^{-1}\tilde{\mathsf{B}}\mathbf{v} = \lambda\mathbf{v} \quad \text{or} \quad \tilde{\mathsf{B}}\mathbf{v} = \lambda\tilde{\mathsf{A}}\mathbf{v}, \tag{57}$$

hence λ is a root of the characteristic polynomial $det(\tilde{\mathsf{B}} - \lambda\tilde{\mathsf{A}}) = 0$. Considering the matrices $\tilde{\mathsf{A}}$ and $\tilde{\mathsf{B}}$ and Gaussian elimination it is fairly straightforward to show that $det(\tilde{\mathsf{B}} + 3\tilde{\mathsf{A}}) = 0$ or $\lambda = -3$ is an eigenvalue of $\tilde{\mathsf{M}}$.

Crank-Nicolson with implicit boundary conditions - 1

We recall that, in this case we are considering the Crank-Nicolson method with the implicit boundary conditions (7), giving the matrices

$$\mathsf{A} = \begin{bmatrix} 1+\frac{3}{2}\sigma & -\frac{\sigma}{2} & 0 & 0 & \dots \\ -\frac{\sigma}{2} & 1+\sigma & -\frac{\sigma}{2} & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & -\frac{\sigma}{2} & 1+\sigma & -\frac{\sigma}{2} \\ \dots & 0 & 0 & -\frac{\sigma}{2} & 1+\frac{3}{2}\sigma \end{bmatrix}_{m\times m} \tag{58}$$

and

$$\mathsf{B} = \begin{bmatrix} 1-\frac{3}{2}\sigma & \frac{\sigma}{2} & 0 & 0 & \dots \\ \frac{\sigma}{2} & 1-\sigma & \frac{\sigma}{2} & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & \frac{\sigma}{2} & 1-\sigma & \frac{\sigma}{2} \\ \dots & 0 & 0 & \frac{\sigma}{2} & 1-\frac{3}{2}\sigma \end{bmatrix}_{m\times m} \tag{59}$$

The matrices A and B are rewritten again as

$$\begin{aligned} \mathsf{A} &= \mathsf{I} + \sigma\tilde{\mathsf{A}}, \\ \mathsf{B} &= \mathsf{I} - \sigma\tilde{\mathsf{B}}, \end{aligned} \tag{60}$$

where

$$\tilde{\mathsf{A}} = \tilde{\mathsf{B}} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 & 0 & \dots \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \dots & 0 & 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}_{m\times m} \tag{61}$$

Theorem 8. *The eigenvalues of* $\mathsf{M} = \mathsf{A}^{-1}\mathsf{B}$ *all have modulus smaller than 1.*

Proof: The matrices $\tilde{\mathsf{A}}$ and $\tilde{\mathsf{B}}$ satisfy the assumptions of theorem 3 case 3, hence their eigenvalues can be computed from

$$\lambda_i^{\tilde{\mathsf{A}}} = \lambda_i^{\tilde{\mathsf{B}}} = 1 + \cos(\frac{(i-1)\pi}{m}), \quad i = 1, \ldots, m. \tag{62}$$

Thus, the eigenvalues of the A and B are

$$\lambda_i^{\mathsf{A}} = 1 + \sigma \left(1 + \cos(\frac{(i-1)\pi}{m})\right), \quad i = 1, \ldots, m, \tag{63}$$

$$\lambda_i^{\mathsf{B}} = 1 - \sigma \left(1 + \cos(\frac{(i-1)\pi}{m})\right), \quad i = 1, \ldots, m, \tag{64}$$

respectively. Obviously the eigenvectors of A and B are the same, hence the eigenvalues of the iteration matrix $\mathsf{M} = \mathsf{A}^{-1}\mathsf{B}$ satisfy:

$$|\lambda_i^{\mathsf{M}}| = \left|\frac{\lambda_i^{\mathsf{B}}}{\lambda_i^{\mathsf{A}}}\right| = \left|\frac{1 - \sigma \left(1 + \cos(\frac{(i-1)\pi}{m})\right)}{1 + \sigma \left(1 + \cos(\frac{(i-1)\pi}{m})\right)}\right| < 1, \quad \forall \sigma > 0, \tag{65}$$

and this proves the theorem.

Crank-Nicolson with implicit boundary conditions - 2

This case is quite artificial, but we present it just to show how the implicit boundary conditions can improve the overall stability of the numerical method. The Crank-Nicolson method is applied with the implicit boundary condition

$$u_0^{n+1} = -u_1^{n+1}, \quad u_{m+1}^{n+1} = -u_m^{n+1}, \quad u_0^{n} = -u_1^{n+1} \quad \text{and} \quad u_{m+1}^{n} = -u_m^{n+1}. \tag{66}$$

In this case, the matrices are given by

$$\mathsf{A} = \begin{bmatrix} 1+2\sigma & -\frac{\sigma}{2} & 0 & 0 & \ldots \\ -\frac{\sigma}{2} & 1+\sigma & -\frac{\sigma}{2} & 0 & \ldots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \ldots & 0 & -\frac{\sigma}{2} & 1+\sigma & -\frac{\sigma}{2} \\ \ldots & 0 & 0 & -\frac{\sigma}{2} & 1+2\sigma \end{bmatrix}_{m\times m} \tag{67}$$

and

$$\mathsf{B} = \begin{bmatrix} 1-\sigma & \frac{\sigma}{2} & 0 & 0 & \ldots \\ \frac{\sigma}{2} & 1-\sigma & \frac{\sigma}{2} & 0 & \ldots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \ldots & 0 & \frac{\sigma}{2} & 1-\sigma & \frac{\sigma}{2} \\ \ldots & 0 & 0 & \frac{\sigma}{2} & 1-\sigma \end{bmatrix}_{m\times m} \tag{68}$$

Theorem 9. *The eigenvalues of* $\mathsf{M} = \mathsf{A}^{-1}\mathsf{B}$ *all have modulus smaller than 1.*

Proof: For this case we have only been able to prove stability for $\sigma \in (0,1)$, despite all the numerical evidence that it is unconditionally stable. Note that the matrix A can be written in the form:

$$\mathsf{A} = \begin{bmatrix} \sigma & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & 0 & 0 & \\ \dots & 0 & 0 & 0 & \sigma \end{bmatrix}_{m\times m} + \begin{bmatrix} 1+\sigma & -\frac{\sigma}{2} & 0 & 0 & \dots \\ -\frac{\sigma}{2} & 1+\sigma & -\frac{\sigma}{2} & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & -\frac{\sigma}{2} & 1+\sigma & -\frac{\sigma}{2} \\ \dots & 0 & 0 & -\frac{\sigma}{2} & 1+\sigma \end{bmatrix}_{m\times m} = \mathsf{E}+\mathsf{F} \tag{69}$$

The eigenvalues of the matrix F are given by theorem 3 case 1

$$\lambda_i^{\mathsf{F}} = 1 + \sigma\left(1 + \cos(\frac{i\pi}{m+1})\right), i = 1\dots m. \tag{70}$$

The eigenvalues of the matrix E are obviously given by $\lambda_1^{\mathsf{E}} = \lambda_m^{\mathsf{E}} = \sigma$ and all the others are zero. Therefore, using theorem 2 the eigenvalues of A satisfy

$$1 + \sigma\left(1 + \cos(\frac{i\pi}{m+1})\right) \le \lambda_i^{\mathsf{A}} \le 1 + \sigma\left(1 + \cos(\frac{i\pi}{m+1})\right) + \sigma. \tag{71}$$

From equation (71) we deduced that

$$|\lambda_i^{\mathsf{A}}| \le \left|1 + \sigma\left(1 + \cos(\frac{i\pi}{m+1})\right) + \sigma\right| \le \left|1 + \sigma\left(1 + \cos(\frac{m\pi}{m+1})\right) + \sigma\right|. \tag{72}$$

Therefore

$$|\lambda_{min}^{\mathsf{A}}| = \left|1 + \sigma\left(1 + \cos(\frac{m\pi}{m+1})\right)\right|. \tag{73}$$

For the matrix B again theorem 3 case 1 gives

$$\lambda_i^{\mathsf{B}} = 1 - \sigma\left(1 - \cos(\frac{i\pi}{m+1})\right), i = 1\dots m. \tag{74}$$

The eigenvalue of B of maximum modulus is obtained when $i = m$, i.e,

$$|\lambda_{max}^{\mathsf{B}}| = \left|1 - \sigma\left(1 - \cos(\frac{m\pi}{m+1})\right)\right|. \tag{75}$$

Thus as the matrices A and B are symmetric we have

$$\begin{gathered} |\lambda^{\mathsf{M}}| \le \| \mathsf{M} \|_2 = \| \mathsf{A}^{-1}\mathsf{B} \|_2 \le \| \mathsf{A}^{-1} \|_2 \| \mathsf{B} \|_2 = \frac{|\lambda_{max}^{\mathsf{B}}|}{|\lambda_{min}^{\mathsf{A}}|} = \\ \left|\frac{1-\sigma+\sigma\cos(\frac{m\pi}{m+1})}{1+\sigma+\sigma\cos(\frac{m\pi}{m+1})}\right| < 1 \Rightarrow \sigma < 1. \end{gathered} \tag{76}$$

4 NAVIER-STOKES TEST

We now test the significance of the stability results derived for the model problem described in the previous section for solving the two-dimensional Navier-Stokes equations for free surface flows. In dimensionless conservative form, the Navier-Stokes equations for incompressible viscous Newtonian flows can be written as

$$\mathbf{u}_t + \nabla \cdot (\mathbf{uu}) = -\nabla p + Re^{-1}\nabla^2\mathbf{u} + Fr^{-2}\mathbf{g}, \tag{77}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{78}$$

where t is time, **u** is the velocity vector field, p is pressure and **g** is the gravity field. The non-dimensional parameters $Re = LU/\nu$ and $Fr = U/\sqrt{gL}$ are the Reynolds and Froude numbers, respectively, where L and U are appropriate length and the velocity scales, and ν is the kinematic viscosity of the fluid. To solve Eqs. (77) and (78) appropriate boundary conditions need to be invoked. On solid boundaries (rigid walls), no-slip conditions apply, i.e, the normal and tangential components of the velocity are taken to be zero. On the free surface, it is necessary to impose conditions on the velocity and pressure. For two-dimensional flows, these conditions, in the absence of surface tension, can be seen in.[21] We use a test problem of the flow of a fluid between two parallel plates, separated by a distance $L = 1\text{m}$, with $Re < 1$ for which the analytical solution is given in.[25] Two types of flow will be considered namely: *Hagen-Poiseuille* flow, in which the channel is initially full and there is no free surface, i.e. confined flow; and *Fountain* flow, in which the channel is initially empty and fluid is injected at the channel's entrance with a parabolic velocity profile. In this problem there is a free surface moving along the channel. The spatial step used was $\delta x = \delta y = 0.05\text{m}$ and the velocity scale was $U = 1.0\text{m/s}$. More details of the parameters used in this simulation can be found in.[21] The following methods were considered: the original explicit method[10] with explicit boundary conditions(GENSMAC); the modified GENSMAC scheme[21] using the Euler implicit method with explicit boundary conditions(GENSMAC-EI); the modified GENSMAC scheme using the Crank-Nicolson method with explicit boundary conditions(GENSMAC-CN/EBC); the modified GENSMAC scheme using the Crank-Nicolson method with implicit boundary conditions 2(GENSMAC-CN/IBC); the semi-implicit scheme[21] using the Euler implicit method with explicit boundary conditions(SI-EI); the semi-implicit scheme using the Crank-Nicolson method with explicit boundary conditions(SI-CN/EBC) and the semi-implicit scheme using the Crank-Nicolson method with implicit boundary conditions 2(SI-CN/IBC). In[21] a modified GENSMAC method was proposed. In that paper the authors showed that the Euler implicit method, applied in the GENSMAC formulation, is unconditionally stable for confined flows. However for free surface flows, it is still subject to the parabolic-like stability condition, alike the explicit scheme. To overcome the stability problem for free surface flows, the authors developed a semi-implicit(SI) scheme using the staggered grid, with new techniques for the treatment of the free surface conditions. In this work, we implemented the Crank-Nicolson method for both the modified GENSMAC and for the SI schemes proposed by Oishi et al.[21] The stability of the Crank-Nicolson method was studied for both explicit and implicit boundary conditions for confined and free surface flows. The type of implicit boundary condition applied was case 2. The results of the GENSMAC-type and SI-type methods are displayed in Tables 1 and 2 for free surface flows, and in Tables 3 and 4 for confined flows. In theses Tables, the relative error (Er) (in the l_2 norm and computed from the analytical solution presented in[25]), the time-step (δt) and CPU time are displayed. Tables 1 and 2 show that GENSMAC-type methods are conditionally stable for free surface flows. The instability showed up in this case, is not due to the explicit or implicit treatment of the boundary conditions on rigid walls, but due to the explicit techniques used in the free surface boundary conditions. As can be noted from Tables 1 and 2, GENSMAC-type schemes present similar results in all cases. The time-step used by these methods is very small, consequently, the CPU time is high. The SI variations schemes are more stable as can be seen in Tables 1 and 2. The SI-CN/EBC method is conditionally stable because of the explicit boundary conditions, as it was proved in the previous section. Therefore, the time-step used by SI-EI and SI-CN/IBC methods are larger than that of SI-CN/EBC method. The SI schemes used total CPU times several orders of magnitude smaller than the CPU times required by GENSMAC-type methods. For confined flows, the Crank-Nicolson method with explicit boundary conditions is conditionally stable when implemented into the GENSMAC and SI formulations. This fact was proved in the previous section. As displayed in the Tables 3 and 4, the time-step used by the explicit GENSMAC is the smallest of all others methods, because of its stringent stability condition. The Euler implicit method with explicit boundary conditions and the Crank-Nicolson method with implicit boundary conditions are unconditionally stable for both GENSMAC and SI formulations. However, the numerical solutions of the GENSMAC-CN/IBC and GENSMAC-EI were influenced by the order of accuracy of the projection method used in the GENSMAC formulation,[26] which is of first-order. The SI formulations included improvements of the projection method used by GENSMAC method, and are based on second-order projection methods.[27] Therefore the SI-EI, SI-CN/EBC and SI-CN/IBC are unconditionally stable and they produced the best results as can be seen in Tables 1–4.

5 DISCUSSION

We have discussed the stability of implicit methods for solving the incompressible Navier-Stokes equations using staggered grids. A recent paper by Oishi et al.[21] addressed issues surrounding the use of implicit methods for free surface flows. However, their analysis did not consider the stability of these schemes. This has been done in this paper. This work was inspired by our numerical observations of instabilities while experimenting with Euler implicit and Crank-Nicolson methods. We showed that on staggered grids, depending on how the boundary conditions are discretized, the Crank-Nicolson method can be conditionally stable. The instability appears for large values of the time-step, when the explicit boundary conditions are used in the Crank-Nicolson method. To our knowledge, the fact that the Crank-Nicolson method with explicit boundary conditions is conditionally stable for staggered grids has never been pointed out in the literature. For the model heat equation problem, the Crank-Nicolson method with explicit boundary conditions is unstable for $\sigma \geq 2$, and the same method with implicit boundary conditions is stable for $\forall\sigma$. The study of stability of the numerical schemes were carried out through the analysis of the eigenvalues of the iteration matrix. Using known[22] and recent[24] theorems, the eigenvalues were calculated exactly or bounded permitting the study of stability. Unfortunately we have not been able to fully prove theorems 7 and 9 despite all the numerical evidence indicating that those theorem are true. We have numerically calculated the eigenvalues of the matrices in theorems 7 and 9 for many values of σ and the results show that they are valid. We are still trying to obtain a mathematically valid proof which we hope to report at the time of the conference. The Navier-Stokes test problems were used to numerically illustrated the behavior of the implicit methods and boundary conditions. The modified GENSMAC using the Euler implicit method with explicit boundary conditions and the Crank-Nicolson method with implicit boundary condition are all unconditionally stable for confined flows. The semi-implicit formulations implementing the Crank-Nicolson method with implicit boundary conditions and the Euler implicit method are unconditionally stable for confined flows and free surface flows. These results provide evidence that the stability restrictions derived for the model problem extend to the case of the Navier-Stokes equations.

Table 1: Results for the relative error (*Er*), the time-step (δt) and CPU time for *fountain* flow with $Re = 0.1$.

Method	*Er*	δt	**CPU time**-(m:s)
GENSMAC	2.2915×10^{-6}	2.5×10^{-5}	$104 : 40$
GENSMAC-EI	2.2919×10^{-6}	8.75×10^{-5}	$60 : 08$
GENSMAC-CN/EBC	2.3012×10^{-6}	5.0×10^{-5}	$89 : 26$
GENSMAC-CN/IBC	8.5422×10^{-6}	5.0×10^{-5}	$95 : 58$
SI-EI	2.2977×10^{-6}	1.25×10^{-2}	$3 : 20$
SI-CN/EBC	2.2901×10^{-6}	5.0×10^{-4}	$24 : 01$
SI-CN/IBC	2.2916×10^{-6}	1.25×10^{-2}	$3 : 05$

Table 2: Results for the relative error (*Er*), the time-step (δt) and CPU time for *fountain* flow with $Re = 0.01$.

Method	*Er*	δt	**CPU time**-(m:s)
GENSMAC	2.1915×10^{-6}	2.5×10^{-6}	$875 : 16$
GENSMAC-EI	2.2528×10^{-6}	8.5×10^{-6}	$544 : 55$
GENSMAC-CN/EBC	2.1919×10^{-6}	4.75×10^{-6}	$770 : 26$
GENSMAC-CN/IBC	8.2215×10^{-6}	4.75×10^{-6}	$822 : 15$
SI-EI	2.2958×10^{-6}	7.5×10^{-4}	$25 : 54$
SI-CN/EBC	2.2901×10^{-6}	5.0×10^{-5}	$164 : 15$
SI-CN/IBC	2.2915×10^{-6}	5.5×10^{-4}	$33 : 48$

ACKNOWLEDGEMENTS

We gratefully acknowledge the support given by *FAPESP* contracts No. 03/12612-9 and No. 00/03385-0 and *CNPq* contract No. 302152/2002-4.

Table 3: Results for the relative error (*Er*), the time-step (δt) and CPU time for *H.-Poiseuille* flow with $Re = 0.1$.

Method	*Er*	δt	**CPU time**-(m:s)
GENSMAC	2.2800×10^{-6}	2.5×10^{-5}	$104 : 31$
GENSMAC-EI	5.2951×10^{-2}	1.25×10^{-2}	$3 : 21$
GENSMAC-CN/EBC	5.0281×10^{-5}	5.0×10^{-4}	$13 : 26$
GENSMAC-CN/IBC	1.0000×10^{-3}	1.25×10^{-2}	$3 : 00$
SI-EI	2.3059×10^{-6}	1.25×10^{-2}	$3 : 19$
SI-CN/EBC	2.2916×10^{-6}	5.0×10^{-4}	$10 : 28$
SI-CN/IBC	2.3000×10^{-6}	1.25×10^{-2}	$3 : 05$

Table 4: Results for the relative error (*Er*), the time-step (δt) and CPU time for *H.-Poiseuille* flow with $Re = 0.01$.

Method	*Er*	δt	**CPU time**-(m:s)
GENSMAC	2.1955×10^{-6}	2.5×10^{-6}	$874 : 01$
GENSMAC-EI	9.3205×10^{-2}	1.25×10^{-2}	$3 : 00$
GENSMAC-CN/EBC	3.8094×10^{-5}	4.0×10^{-5}	$115 : 39$
GENSMAC-CN/IBC	8.8891×10^{-2}	1.25×10^{-2}	$3 : 09$
SI-EI	2.0115×10^{-6}	1.25×10^{-2}	$3 : 39$
SI-CN/EBC	2.2916×10^{-6}	5.0×10^{-5}	$80 : 20$
SI-CN/IBC	2.2312×10^{-6}	1.25×10^{-2}	$3 : 19$

REFERENCES

[1] F.H Harlow and J.E. Welch. Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface. *Physics of Fluids*, **8**, 2182–2189 (1965).

[2] R.K.C. Chan and R.L. Street. A computer study of finite-amplitude water waves. *Journal of Computational Physics*, **6**, 68–94 (1970).

[3] J. Viecelli. A method for including arbitrary external boundaries in the mac incompressible fluid computing technique. *Journal of Computational Physics*, **4**, 543–551 (1969).

[4] C.W. Hirt and B.D. Nichols. Volume of fluid (vof) method for the dynamics of free boundaries. *Journal of Computational Physics*, **39**, 201–225 (1981).

[5] A.A. Amsden and F. H. Harlow. A simplified mac technique for incompressible fluid flow calculations. *Journal of Computational Physics*, **6**, 332–335 (1970).

[6] C.W. Hirt, A.A Amsden, and J.L. Cook. An arbitrary lagrangian-eulerian computing method for all flow speeds. *Journal of Computational Physics*, **14**, 227–253 (1974).

[7] B.D. Nichols, C. V. Hirt, and R.S. Hotchkiss. Sola-vof: a solution algorithm for transient fluid flow with multiple free boundaries. *Los Alamos Scientific Laboratory Rep.*, (1980).

[8] H. Miyata and A. Masuko. Finite-difference simulation of nonlinear waves generated by ships of arbitrary three-dimensional configuration. *Journal of Computational Physics*, **60**, 391–436 (1985).

[9] H. Miyata. Finite-difference simulation of breaking waves. *Journal of Computational*

Physics, **65**, 179–214 (1986).
[10] M. F. Tomé and S. McKee. Gensmac: A computational marker-and-cell method for free surface flows in general domains. *J. of Comp. Phys.*, **110**, 171–186 (1994).
[11] M.F. Tomé, A. Castelo, J. A. Cuminato, N. Mangiavacchi, and S. McKee. Gensmac3d: A numerical method for solving unsteady three-dimensional free surface flows. *International Journal for Numerical Methods in Fluids*, **37**, 747–796 (2001).
[12] V. Armenio. An improved mac method (simac) for unsteady high-reynolds free surface flows. *International Journal for Numerical Methods in Fluids*, **24**, 185–214 (1997).
[13] D. E. Morton, M. J. Rudman, and J.L. Liow. A finite difference method for modelling impacting drops. In *Proceedings of FEDSM97*, pages 1–6, (1997).
[14] S. O. Kim and H. C. No. Second-order model for free surface convection and interface reconstruction. *International Journal for Numerical Methods in Fluids*, **26**, 79–100 (1998).
[15] S. Ushijima. Arbitrary lagrangian eulerian numerical prediction for local scour caused by turbulent flows. *Journal of Computational Physics*, **125**, 71–82 (1996).
[16] M.F. Tomé, A. Castelo, J.A. Cuminato, and S. McKee. A finite difference technique for simulating unsteady viscoelastic free surface flows. *Journal of Non-Newtonian Fluid Mechanics*, **106**, 61–106 (2002).
[17] M.F. Tomé, L.Grossi, A. Castelo, J.A. Cuminato, V.G. Ferreira, F. S. Sousa, and S. McKee. A numerical method for solving three-dimensional generalized newtonian free surface flows. *Journal of Non-Newtonian Fluid Mechanics*, **123**, 85–103 (2004).
[18] K. Housiadas, G. Georgiou, and J. Tsamopoulos. The steady annular extrusion of a newtoanian liquid under gravity and surface tension. *International Journal for Numerical Methods in Fluids*, **33**, 1099–1119 (2000).
[19] R.W. Lewis, S.E. Navti, and C. Taylor. A mixed lagrangian-eulerian approach to modelling fluid flow during mould filling. *International Journal for Numerical Methods in Fluids*, **25**, 931–952 (1997).
[20] M.F. Tomé, S. McKee, L. Barratt, D.A. Jarvis, and A.J. Patrick. An experimental and numerical investigations of container filling with viscous liquids. *International Journal for Numerical Methods in Fluids*, **31**, 1333–1353 (1999).
[21] C.M. Oishi, J.A. Cuminato, V.G. Ferreira, M.F. Tomé, A. Castelo, N. Mangiavacchi, and S. McKee. A stable semi-implicit method for free surface flows. *Journal of Applied Mechanics*, accepted (2005).
[22] A. Quarteroni, R. Sacco, and F. Saleri. *Numerical Mathematics*. Springer, (2000).
[23] J. H. Wilkinson. *The Algebric Eigenvalue Problem*. Oxford, England, (1965).
[24] W.C. Yueh. Eigenvalues of several tridiagonal matrices. *Applied Mathematics E-Notes*, **5**, 66–74 (2005).
[25] G. K. Batchelor. *An Introduction to Fluid Dynamics*. Cambridge, (1970).
[26] A.J. Chorin. Numerical solution of the navier-stokes. *Mathematics of Computation*, **2**, 745–762 (1968).
[27] J.B. Perot. An analysis of the fractional step method. *Journal of Computational Physics*, **108**, 51–58 (1993).