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A GLOBAL THEORY ON VARIATIONAL APPROXIMATIONS OF LINEAR PROBLEMS: SOME REMARKABLE APPLICATIONS

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Abstract. *In the early seventies Babushka [3] set up the bases of a general approximation theory of linear variational problems. Over ten years later, Dupire [6] refined this theory in his doctoral thesis defended at PUC-Rio, the Catholic University of Rio de Janeiro. This together with classical estimates of the polynomial interpolation error in Sobolev norms, has since been widely used as the basic tool to establish the convergence of finite element solutions of partial differential equations. The purpose of this work is two-fold: First we endeavour to recall the main results of Dupire[6], while pointing out some of its yet unexploited aspects; Then we show through a simple example, how both ingredients allow a straightforward convergence analysis of the finite volume method as well.*

1 INTRODUCTION

The approximation analysis of finite element methods are viewed nowadays as a well-established mathematical theory. This is mainly due to the work of several authors in the late sixties and in the early seventies. Among the outstanding studies in this direction, the contributions of Aubin [1], Aziz & Babushka [2], Bramble & Zlámal [4], Ciarlet & Raviart [5], Strang [11], should be quoted.

These early studies primarily focused elliptic problems, in which the coerciveness of the underlying continuous bilinear form, allows one to derive error estimates, in the absence of the so-called variational crimes [12], by simply considering the interpolation error measured in the natural Sobolev norms.

Before addressing the main purpose of this work, we briefly recall such mechanisms, by considering the following abstract problem setting:

Let

- Y be a real Hilbert space with the norm $\| \cdot \|_Y$;
- L be a real continuous linear functional Y (i.e. $L \in Y'$, the topological dual space of Y);
- $a : Y \times Y \longrightarrow \mathbb{R}$ be a continuous bilinear form.

For the sake of brevity, we shall use henceforth the following notation:

Definition 1.1 *Let X and Y be two normed vector spaces, wth respective norms denoted by $\| \cdot \|_X$ and $\| \cdot \|_Y$. $\mathcal{L}_{2c}(X \times Y)$ is defined to be the space of continuous bilinear forms $a : X \times Y \longrightarrow \mathbb{R}$.*

We recal that $a \in \mathcal{L}_{2c}(X \times Y)$ if and only if there exists a constant M such that

$$a(x,y) \leq M \| \ x \ \|_X \| \ y \ \|_Y, \ \forall x \in X, \ y \in Y. \tag{1}$$

In this case the norm of a, is the infimum of the constants M that verify the above inequality, that is:

$$\| \ a \ \| = \sup_{\begin{cases} (x,y) \in X \times Y \\ (x,y) \neq \vec{0} \end{cases}} \frac{a(x,y)}{\| \ x \ \|_X \| \ y \ \|_Y}. \tag{2}$$

Now we consider the linear variational problem:

$$(\mathcal{P}_Y) \quad \left\{ \begin{array}{ll} \text{Find } x \in Y & \text{such that} \\ a(x,y) = L(y) & \forall y \in Y. \end{array} \right. \tag{3}$$

The classical Lax-Milgram Theorem asserts that if a is Y-elliptic, or equivalently, if a is coercive over $Y \times Y$, that is:

$$\exists \alpha > 0 \text{ such that } a(y,y) \geq \alpha \| \; y \; \|_Y{}^2, \; \forall y \in Y. \tag{4}$$

then $(\mathcal{P}_Y)$ has a unique solution for every L. Moreover, in this case, if we approximate $(\mathcal{P}_Y)$ by $(\mathcal{P}_Z)$, a problem derived from $(\mathcal{P}_Y)$ by replacing Y with subspace Z (typically a finite-dimensional subspace of Y), and the solution $x \in Y$ with its approximation $w \in Z$, then the distance between x and w measured in the norm of Y, may be estimated using the following error bound, known as Céa's Lemma:

$$\| \; x - w \; \|_Y \leq \frac{\| \; a \; \|}{\alpha} \, d_Y(x, Z), \tag{5}$$

where $d_Y(y, Z)$ denotes the distance of an element $y \in Y$ from subspace Z, measured in the norm of Y, defined to be:

$$d_Y(y, Z) \; = \; \inf_{z \in Z} \; \| \; y - z \; \|_Y \; . \tag{6}$$

In the aim of studying some particular classes of problems not characterized by the property of Y-ellipticity, such as saddle-point problems, in the early seventies an extension of the Lax-Milgram Theorem was considered by Babushka [3]. More specifically the following problem more general than $(\mathcal{P}_Y)$ was studied, in which the solution may be searched for in another Hilbert space X equipped with the norm $\| \cdot \|_X$:

$$(\mathcal{P}_{X,Y}) \quad \left\{ \begin{array}{ll} \text{Find } x \in X & \text{such that} \\ a(x,y) = L(y) & \forall y \in Y. \end{array} \right. \tag{7}$$

Naturally enough, it is assumed here that $a : X \times Y \longrightarrow \mathbb{R}$ is bilinear and continuous, in the sense of definition (1). In so doing the following result known as the *Generalized Lax-Milgram Theorem* applies:

Theorem 1.1 *Under the assumptions that $a \in \mathcal{L}_{2c}(X \times Y)$ and $L \in Y'$, if a fulfills both conditions below:*

$$\exists \alpha_1 > 0 \textit{ such that } \forall x \in X \quad \sup_{\begin{cases} y \in Y \\ y \neq \vec{0} \end{cases}} \frac{a(x,y)}{\| \ y \ \|_Y} \geq \alpha_1 \| \ x \ \|_X; \tag{8}$$

$$\exists \alpha_2 > 0 \textit{ such that } \forall y \in Y \quad \sup_{\begin{cases} x \in X \\ x \neq \vec{0} \end{cases}} \frac{a(x,y)}{\| \ x \ \|_X} \geq \alpha_2 \| \ y \ \|_Y, \tag{9}$$

then problem $(\mathcal{P}_{X,Y})$ *has a unique solution for any* $L \in Y'$.

Definition 1.2 *A bilinear form* a *is said to be weakly coercive, if it satisfies both conditions (8) and (9).*

Notice that in the case where $X = Y$, if a is coercive it is also a weakly coercive, and one may take $\alpha_1 = \alpha_2 = \alpha$, where α is the constant of inequality (4). Moreover, according to the results due to Dupire [6], whenever a is weakly coercive, one may take $\alpha_2 = \alpha_1 = \alpha$ in condition (9), where α is the greatest constant for which both conditions hold. This fact actually suggests that the concepts of coeciveness and weak coerciveness are closely related to each other, in the sense that the latter can be viewed as a natural extension of the former. As a matter of fact, such analogy between both concepts can be seen from several other points of view. In order to illustrate this assertion, we recall hereafter an analysis given in [6], according to which an inequality entirely analogous to the one of Céa's Lemma also holds in the case where a is weakly coercive, with a constant depending on a, derived by simply replacing in (5) the constant α that characterizes coerciveness, by the (henceforth unified) constant that expresses the weak coerciveness of a, related to the subspaces of X and Y chosen to define the approximation of $(\mathcal{P}_{X,Y})$. In the next section we will examine this assertion in detail, since such analogy cannot be seen from the works more generally referred to in this connection, such as the already quoted one due to Babushka [3].

Before addressing this point, we give below some further notations and we recall some useful facts and concepts:

- The inner product of a Hilbert space Y normed by $\| \cdot \|_Y$, is denoted by $(\cdot,\cdot)_Y$,;
- Let V and Z be two closed subspaces of Y such that $Y = V \oplus Z$ (this direct sum is not necessarily orthogonal). We define the projection operator onto V parallelly to Z (resp. the projection operator onto Z parallelly to V), denoted by $P = \pi_{V//Z}$ (resp. by $Q = \pi_{Z//V} = I - P$) in the following way : Given $y \in Y$, $(Py, v)_Y = (y, v)_Y$ $\forall v \in V$ (resp. $(Qy, z)_Y = (y, z)_Y$ $\forall z \in Z$);

- For two Hilbert spaces X and Y and $a \in \mathcal{L}_{2c}(X \times Y)$, we define the operator $A \in \mathcal{L}(X,Y)$ that represents a on the left, by $(A(x),y)_Y = a(x,y)$;
- The standard norm of the above defined operator A in the space $\mathcal{L}(X,Y)$ denoted by $\| A \|$, is $\| a \|$;
- The co-norm of the operador $A \in \mathcal{L}(X,Y)$ that represents $a \in \mathcal{L}_{2c}(X \times Y)$ on the left, denoted by $conorm(A)$ is defined by:

$$conorm(A) = \inf_{\substack{x \in X \\ x \neq \vec{0}}} \frac{\| A(x) \|_Y}{\| x \|_X}.$$

Notice that $conorm(A) > 0$ if and only if condition (8) holds, and that in this case $conorm(A) \geq \alpha_1$). On the other hand, according to Dupire [6] both conditions (8) and (9) are also necessary for problem $(\mathcal{P}_{X,Y})$ to be well-posed for every $L \in Y'$. Therefore the Riesz Representation Theorem readily leads to the conclusion that they are equivalent to A being one-to-one and onto.

Finally we note that both operators P and Q defined above belong to $\mathcal{L}(X,Y)$.

2 LINEAR VARIATIONAL PROBLEMS ON SUBSPACES

Henceforth we consider that in order to solve the linear variational problem $(\mathcal{P}_{X,Y})$, we select two closed subspaces W and Z of X and Y respectively. This leads to an approximate problem $(\mathcal{P}_{W,Z})$ defined in the same way as $(\mathcal{P}_{X,Y})$, by simply replacing X with W and Y with Z, and denoting its solution by $w \in W$, in fact an approximation of $x \in X$.

Now we define the restrictions $a_{W,Z}$ of a to $W \times Z$, and L_Z of L to Z by

$$a_{W,Z} \in \mathcal{L}_{2c}(W \times Z): \; \forall (w,z) \in W \times Z, \; a_{W,Z}(w,z) = a(w,z) \tag{10}$$

$$L_Z \in Z' : \forall z \in Z \;\; L_Z(z) = L(z) \tag{11}$$

We know that there exist $A_{W,Z} \in \mathcal{L}(W,Z)$ and $f_Z \in Z$ such that

$$\forall (w,z) \in W \times Z, \; a_{W,Z}(w,z) = (A_{W,Z}(w), z)_Y \tag{12}$$

and

$$\forall z \in Z \ \ L_Z(z) = (f_Z, z)_Y. \tag{13}$$

Let us express $A_{W,Z}$ and f_Z in terms of A and f in the following way:

Let π_Z be the orthogonal projection operator onto Z. We have $\forall (w, z) \in W \times Z$ $(A(w) - A_{W,Z}(w), z)_Y = 0 \ \forall w \in W$ and $\forall z \in Z$. Thus $(A - A_{W,Z})(w) \in Z^{\perp}$, that is, $A_{W,Z}(w) = \pi_Z A(w)$ for every $w \in W$. Therefore,

$$A_{W,Z} \ = \ \pi_Z A_{|W} \tag{14}$$

which means that $A_{W,Z}$ is the combination of the orthogonal projection operator onto Z with the restriction of A to W. Furthermore, from the Generalized Lax-Milgram Theorem, problem $(\mathcal{P}_{W,Z})$ is well-posed for every L in $Y^{'}$ if and only if $a_{W,Z}$ is weakly coercive. Indeed, as one may easily check, the restrictions to Z of all the functionals $L \in Y^{'}$ sweep the whole topological dual space $Z^{'}$.

Remark 2.1 *At this point it is important to stress the fact that differently from coerciveness, the weak coerciveness does not automatically apply to subspaces. Otherwise stated, for every pair of subspaces W and Z we choose, in order to define the approximate problem $(\mathcal{P}_{W,Z})$, we must make sure that the latter is well-posed. In other terms, we must check that the bilinear form a (i.e. its restriction) is weakly coercive over $W \times Z$, or equivalently, that the operator $A_{W,Z} = \pi_Z A_{|W}$ is one-to-one and onto from W onto Z. In this case we say that $A_{W,Z}$ is weakly coercive over $W \times Z$.*

Also in the aim of simplifying expressions and notations, we further give the following definitions:

Definition 2.1 *Let $A \in \mathcal{L}(X, Y)$ and $W \subset X$ and $Z \subset Y$ be two closed subspaces. A is said to be (W, Z) w.c. if $\pi_Z A_{|W}$ is weakly coercive, that is, if it belongs to $Isom_c(W, Z)$. Moreover, in this case the co-norm of $\pi_Z \ A_{|W}$ is denoted by $\alpha_{W,Z}$*

Those definitions trivially extend to $a \in \mathcal{L}_{2c}(X \times Y)$. We immediately apply them in the context of Problem $\mathcal{P}_{X,Y}$ (cf. [6]):

Proposition 2.1 *If $a \in \mathcal{L}_{2c}(X \times Y)$ is (W, Z) w.c. then $W \oplus Ker(\pi_Z A) = X$, where $A \in \mathcal{L}(X, Y)$ represents a on the left.*

PROOF.

Let $w \in W \cap Ker(\pi_Z A)$, that is $A_{W,Z} w = \pi_Z A_{|W} w = 0$. Thus $W \cap Ker(\pi_Z A) = \{0\}$, since $A_{W,Z}$ is one-to-one and onto.

Let now $x \in X$. Recalling that A is (W, Z) *w.c.*, let $x_W \in W$ be the unique solution of the following variational problem: Find $x_W \in W$ such that, $\forall z \in Z$, $a(x_W, z) = a(w, z)$. Notice that $x - x_W \in Ker(\pi_Z A)$. Indeed, $\forall z \in Z$, $(A(x - x_W), z)_Y = a(x - x_W, z) = 0$. Hence by splitting x into the sum $x = x_W + (x - x_W)$ it is readily seen that $W \oplus Ker(\pi_Z A) = X$ ■

The following lemma due to Dupire [6] (cf. Proposition II-3.3) is a fundamental result to derive the rather fine and simple estimate of $\| x - x_W \|_X$ in terms of $d_X(x, W) = \| x - \pi_W x \|_X$, in the Proposition 2.2 given hereafter.

Lemma 2.1 *Let $A \in \mathcal{L}(X, Y)$, $W \subset X$ and $Z \subset Y$, W and Z being closed subspaces. Let also $A_1 = \pi_Z A_{|W} \in \mathcal{L}c(W, Z)$ and $A_2 = \pi_Z A_{|W^\perp}$. Then if A_1 is onto we have :*

$$[conorm(A_1)]^2 + \| A_2 \|^2 \leq \| \pi_W A \|^2 \leq \| A \|^2 . \tag{15}$$

Now we are ready to prove,

Proposition 2.2 *Let $a \in \mathcal{L}_{2c}(X \times Y)$ be such that a is (W, Z) w.c., where W and Z are closed subspaces of X and Y. Then, for every $x \in X$:*

a) $\exists! x_W \in W$ such that $\forall z \in Z$ $a(x_W, z) = a(x, z)$;

b) $\| x - x_W \|_X \leq \dfrac{\| a \|}{\alpha_{V,Z}} d_X(x, W)$,

where $\alpha_{V,Z} = conorm(\pi_Z A_{|V})$ and $d_X(x, W)$ is given by the expression (6)

PROOF.

a) This result is a direct consequence of the Generalized Lax-Milgram Theorem.

b) For every $z \in Z$, $(A(x - x_W), z)_Y = a(x - x_W, z) = 0$. Hence $\pi_Z A(x - x_W) = 0$, and if $v = \pi_W(x - x_W)$ and $w = \pi_{W^\perp}(x - x_W) = \pi_{W^\perp} x$ then $\pi_Z A(v) = -\pi_Z A(w)$. It follows that :

$$\alpha_{V,Z} \| v \|_X \leq \| \pi_Z A_{|V^\perp} \| \| w \|_X . \tag{16}$$

Moreover, from the Pythagorean Theorem we obtain:

$$\alpha_{V,Z}^2 \| \ x - x_W \ \|_X{}^2 \leq \| \ \pi_Z A_{|W^\perp} \ \|^2 \| \ w \ \|_X{}^2 + \alpha_{V,Z}^2 \| \ w \ \|_X{}^2 \tag{17}$$

On the other hand, the operator $\pi_Z A_{|W} \in \mathcal{L}(W, Z)$ is onto since it is (W, Z) $w.c.$. Thus, according to Lemma 2.1, $\| \ \pi_Z A_{|W^\perp} \ \|^2 + \alpha_{W,Z}^2 \leq \| \ \pi_Z A \ \|^2$. This implies that

$$\| \ x - x_W \ \|_X \leq \frac{\| \ \pi_Z A \ \|}{\alpha_{W,Z}} \| \ w \ \|_X \leq \frac{\| \ A \ \|}{\alpha_{W,Z}} \| \ w \ \|_X \ . \tag{18}$$

The proof is thus complete, taking into account that $w = \pi_{W^\perp} x = x - \pi_W x$, $\| \ w \ \|_X = d_X(x, W)$, and $\| \ \pi_Z A \ \| \leq \| \ A \ \| = \| \ a \ \|$. ■

3 ABSTRACT APPROXIMATION

Let us apply the results of the previous Section to the approximation of the abstract linear variational problems.

In order to do so, we first note that whenever we search for the solution x of a variational problem defined upon a pair of Hilbert spaces (X, Y), the natural steps to take are the following:

1. Check if $(\mathcal{P}_{X,Y})$ is well-posed;
2. Construct or select two finite-dimensional subspaces X_h and Y_h to approximate X and Y;
3. Check if $(\mathcal{P}_{X_h,Y_h})$ is well-posed;
4. Determine the solution $x_h \in X_h$ of $(\mathcal{P}_{X_h,Y_h})$;
5. Establish suitable upper bounds or estimates for the error $\| \ x - x_h \ \|_X$.

Steps 1. and 3. simply follow from the weak-coerciveness of a over $X \times Y$ and over $X_h \times Y_h$. As far as step 2. is concerned, a particularly handy choice is provided by the so-called Ritz method; in this way step 4. is managed by the application of suitable algorithms for solving linear systems of equations.
In this Section we address step 5.. More specifically, we establish estimates aimed at evaluating not only the approximation error with respect to the original (continuous) problem , but also the distance between two approximate solutions obtained with two different pairs of subspaces. Such estimates are referred to as abstract approximation results, since they are derived in a general hilbertian framework, expressed in terms of $d_X(x, X_h)$, without attempting to evaluate this quantity more precisely. Notice that in the

case of differential equations, for example, this is usually achieved thanks to the standard results of the theory of interpolation in Sobolev spaces [12].

Keeping this in view, we first apply Proposition 2.2 to establish:

Theorem 3.1 *Let W and Z be closed subspaces of two Hilbert spaces X and Y respectively, and $a \in \mathcal{L}_{2c}(X \times Y)$ be (X, Y) w.c. and (W, Z) f.c.. Then:*

a) For every $L \in Y'$, there exists a unique $(x, x_W) \in X \times Z$ such that

$$\forall y \in Y, \quad a(x, y) = L(y) \quad \textit{and} \quad \forall z \in Z, \quad a(x_W, z) = L(z) \tag{19}$$

b) The following error upper bound holds:

$$\| \; x - x_W \; \|_X \leq \frac{\| \; a \; \|}{\alpha_{W,Z}} \, d_X(x, W), \tag{20}$$

where x and x_W are defined in a).

Proof.

a) Is a direct consequence of Theorem 1.1.

b) First we note that $\forall \; z \in Z, \quad a(x_W, z) = a(x, z)$. Hence applying Proposition 2.2 we immediately derive (20). ■

The above result is simply a corollary of Proposition 2.2. We give it in the form of a theorem because, as already stressed in Section 1, it is a strict generalization of the classical estimate (5) for problems in which a is coercive (with $X = Y$ and $W = Z$), that is:

$$\| \; x - x_W \; \|_Y \leq \frac{\| \; a \; \|}{\alpha} d_Y(x, Z), \tag{21}$$

where α denotes the coerciveness constant (i.e., a constant that satisfies the relation establishing that a is coercive over $Y \times Y$). Indeed,

1. We do not require the coerciveness of a, but only the (X, Y) weak coerciveness and the (W, Z) weak coerciveness of a.

2. Even in the case where a is coercive, estimate (20) is finer than (21), since the constant characterizing the weak coerciveness is always greater than the coerciveness constant. Furthermore for a given subspace, the former is always greater than the one related to the whole working space, (even if this fact is not easy to exploit in practical situations).

Remark 3.1 *Notice however that the weak coerciveness does not allow any improvement in the following error estimate, applying to the case where a is both coercive and symmetric (cf. [8]):*

$$\| x - x_W \|_Y \leq (\frac{\| a \|}{\alpha})^{\frac{1}{2}} d_Y(x_W, Z). \tag{22}$$

In more practical tems, let us consider the application of the classical finite element method to solve a linear boundary value problem. Assume that $X = Y$, and that we are given a family of finite-dimensional subspaces $\{Y_h\}_h$, where h is a parameter characterizing the degree of refinement of a mesh covering the bounded domain in which the problem is posed. We associate with every value of h a finite-dimensional space Y_h to approximate Y as h goes to 0, assuming that a is weakly coercive over each one of them. Now we observe that if a is not coercive, establishing the weak coerciveness of a over every member of the family $\{Y_h\}_h$, $h > 0$ may be a delicate problem.

In this case, denoting by x_h the solution of the variational problem corresponding to a given subspace Y_h, the error estimate (20) writes:

$$\| x - x_h \|_Y \leq \frac{\| a \|}{\alpha_h} d_Y(x, Y_h), \tag{23}$$

where α_h is the co-norm of $A_h = \pi_{Y_h} A_{|Y_h}$. Hence, if α_h is independent of h, the issue of estimating the quality of the approximation of x by x_h may be confined to evaluating how good is the approximation of Y by Y_h. This is a well-established subject in the litterature on the finite element method.

4 APPLICATION TO THE ERROR ANALYSIS OF THE FINITE VOLUME METHOD

In order to illustrate the great generality of the theory treated in this paper, we show in this Section, how it may be used as efficiently in the error analysis of the finite volume method.

Consider the following problem:

Given $f \in L^2(0,1)$, find a function $u \in H^1(0,1)$ such that,

$$\left\{ \begin{array}{rcl} u' + u & = & f \\ u(0) & = & 0 \end{array} \right. \tag{24}$$

Problem (24) may be recast in the following variational form:

$$(\mathcal{P}) \quad \left\{ \begin{array}{cc} \text{Find } u \in U & \text{such that} \\ a(u,v) = L(v) & \forall v \in V. \end{array} \right. \tag{25}$$

where

$$a(u,v) \;=\; \int_0^1 u'v dx + \int_0^1 uv dx, \tag{26}$$

$$V = L^2(0,L) \text{ equipped with the norm } \| \cdot \|_{0,2}$$

and

$$U = \{v/\ v \in H^1(0,L), v(0) = 0\} \text{ is equipped with the norm } \| \cdot \|_{1,2},$$

where by definition,

$$\| \cdot \|_{0,2} = \ [\int_0^1 (\cdot)^2 dx]^{1/2}] \tag{27}$$

and

$$\| \cdot \|_{1,2} = [\| \cdot \|_{0,2}^2 + \| (\cdot)' \|_{0,2}^2]^{1/2}. \tag{28}$$

Since both V and U are Hilbert spaces, respectively for the above defined norms (cf. e.g. [9]), we may procede as follows to demonstrate that $(\mathcal{P})$ has a unique solution for every $f \in L^2(0,1)$:

First of all, given $u \in U$ we choose $v = u^{'} \in V$. Clearly $a(u,v) = \| u' \|_{0,2}^2 + \int_0^1 uu'dx$, that is:

$$a(u,v) = \| u' \|_{0,2}^2 + u^2(1)/2 \geq (1 + C_P^2)^{-1} \| u \|_{1,2}^2 \,. \tag{29}$$

where C_P is a constant for which the Friedrichs-Poincaré inequality holds for the space U (cf. e.g. [9]).

On the other hand we trivially have,

$$\| \ v \ \|_{0,2} \leq \| \ u \ \|_{1,2} \ . \tag{30}$$

The inequalities (29) and (30) imply that condition (8) is fulfilled with $\alpha = (1 + C_P^2)^{-1}$.

In order to establish that condition (9) holds, let $v \in V$, $v \neq 0$ be given. Now define $u \in U$ by $u(x) = \int_0^x v(t)dt$, $x \in [0,1]$. Since $u' = v \neq 0$ by construction, we have $a(u,v) = \| \ u' \ \|_{0,2}^2 + \int_0^1 uu'dx \geq \| \ v \ \|_{0,2}^2$ and $\| \ u \ \|_{1,2} \leq (C_P^2 + 1)^{1/2} \| \ v \ \|_{0,2}$. This yields the desired result, with a constant equal to $\alpha^{1/2}$

Let us now consider the approximation of $(\mathcal{P})$ by the vertex-centered finite volume method described as below:

First construct a partition $\mathcal{T}_h$ of the domain $(0,1)$ into control volumes I_i com $i = 1, 2, ..., N$ having equal lengths $h = L/N$, for a given $n \in \mathbb{N}^*$, $I_i = ([i-1]h, ih)$. Next we approximate $(\mathcal{P})$ by:

$$(\mathcal{P}_h) \quad \left\{ \begin{array}{ll} \text{Find } u_h \in U_h & \text{such that} \\ a(u_h, v) = L(v) & \forall v \in V_h. \end{array} \right. \tag{31}$$

where

$$U_h = \{v/ \ v \in C^0[0,L], \ v(0) = 0, \ v_{/I_i} \in P_1 \ \ \forall i, \ i = 1, 2, ..., N\} \tag{32}$$

and

$$V_h = \{v/ \ v_{/I_i} \in P_0 \ \ \forall i, \ i = 1, 2, ..., N\}. \tag{33}$$

P_k being the space of polynomials of degree less than or equal to k.

Problem $(\mathcal{P}_h)$ reduces to the integration of the differential (24) in each control volume, followed by the substitution of u through its approximation $u_h \in U_h$. Notice that the method defined in this manner may also be interpreted as a usual conforming finite element method of the piecewise linear type, since $U_h \subset U$ and $V_h \subset V$.

As one may easily demonstrate, approximate problem $(\mathcal{P}_h)$ fulfills both conditions (8) and (9), known to be necessary and sufficient for it to have a unique solution $\forall f \in L^2(0, L)$. In order to prove this assertion, we may employ the very same arguments as we did above for the continuous problem $(\mathcal{P})$. Indeed, the derivative of a function in U_h belongs to V_h, and conversely the integral from 0 to x of a function in V_h necessarily yields a function of U_h. In so doing, we readily conclude that in the case of the approximate problem too, condition (8) is fulfilled with $\alpha_h = \alpha$.

Now if in $(\mathcal{P})$ we consider the case where $f \in H^1(0, L)$, we may easily check that $u \in H^2(0, L)$ (that is $u^{''} \in L^2(0, 1)$). As a consequence, using the classical results that apply to the piecewise polynomial approximation (cf. [12]), we may assert that there exists a constant C independent of h, such that :

$$d_U(u, U_h) \leq Ch \parallel u^{''} \parallel_{0,2} . \tag{34}$$

Taking into account (20), it immediately follows that,

$$\parallel u - u_h \parallel_{1,2} \leq \frac{C\sqrt{2}}{\alpha} h \parallel u^{''} \parallel_{0,2} . \tag{35}$$

The error analysis applied in this Section, typically employed in the litterature in the study of finite element methods, extend to other finite volume schemes, even in higher dimension space, eventhough it may involve rather complex technicalities. As an illustration of this assertion we refer to another authors' work to appear shortly [10]. Actually they deeply believe that works in this direction like the one by Idelsohn and Oñate [7] can only contribute to dismystify a little more the yet widespread belief that both methods obey fundamentally different principles.

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