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COMPUTATION OF THE VIBRATION MODES OF PLATES AND SHELLS COUPLED WITH A FLUID.

Erwin Hernández*, Luis Hervella-Nieto†, and Rodolfo Rodríguez*

*GI²MA
Departamento de Ingeniería Matemática
Universidad de Concepción
Casilla 160-C, Concepción, Chile
e-mail: erwin@ing-mat.udec.cl,rodolfo@ing-mat.udec.cl
web page: http://lanin.ing-mat.udec.cl/~rodolfo/

†Departamento de Matemática
Universidade da Coruña, Spain
e-mail: luisher@udc.es

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Abstract.
We consider the approximations of the vibration modes of an elastic thin structure (shell or plate) in contact with a compressible fluid. We use the classical Naghdi model over a reference domain and its approximation using the MITC4 finite element method for the structure. The equations for the fluid are discretized with Raviart-Thomas elements, whereas a non conforming coupling is used on the fluid-solid interface. We report numerical experiments assessing the efficiency of this coupled scheme.

1 INTRODUCTION

This paper deals with the numerical computation of the vibration modes of a fluid-structure interaction problem in a 3D-domain. This is a very important engineering problem (e.g. for treatment of noise in cars or planes). It is well known that a large amount of work has been devoted to this subject (see for example [15]).

We have interested in one of problems of this kind: to compute elastoacoustic vibrations when the structure an elastic shell and the fluid is ideal and compressible, both with small displacements.

In the framework of thin solid structures, a big amount of work has been developed during the last years by different communities under different points of view. There exist two main ways of approximating shells problems: finite element methods that result from the discretization of classical shells model (namely, *two dimensional methods*) (see [8, 9]) and methods based on "degenerating" a 3D solid finite element into a shell element using some kinematical assumption in the thickness direction (see [1, 3])

For the present work, we consider one of the most important classical two-dimensional shell models: Naghdi, which is based on Reissner-Mindlin hypotheses. To discretize this we use MITC (Mixed Interpolation of Tensorial Component) finite element methods, introduced by Bathe and Dvorkin in [2], that is very likely the most used in practice. For this element, we can found some mathematical analysis when applied to Reissner-Mindlin plate equations (see, for example, [12, 10, 11]). In particular, we use the low-order MITC4 method which is the most common isoparametric quadrilateral element of this family of methods. The performance of this approach has been recently tested for vibration problems of both, plates and shells, in [13].

To determining the vibration of the fluid, usually the pressure is chosen as primary variable, however, for coupled systems, the displacement vector fields present some important properties like; for example, compatibility and equilibrium through the fluid-structure interface satisfy automatically. Though, it is well known that the displacement formulation suffers from the presence of zero-frequency spurious modes with no physical meaning. Nevertheless, an alternative approach has been introduced and analyzed in [4] to avoid the spurious modes; it consists in the use of lowest-order Raviart-Thomas element. The degrees of freedom of this element are located at the element faces and represent the normal component of the field through them.

On the fluid-solid interface a non conforming coupling is used: the kinematic constraint (i.e. equal normal displacement for fluid and shell) is imposed in a weak sense. In fact, because of this, the fluid and shell meshes do not need to be compatible on the common interface.

In this paper we consider the problem of computing the vibration modes of a shell in contact with a fluid. In Section 2 we state the vibration coupled problem. In Section 3 we treat the particular case when the shell is a plate. Finally, in Section 4, we apply the method to calculate the vibrations of a thin cylinder full of fluid.

2 STATEMENT OF THE PROBLEM.

We consider the problem of determining the free vibration modes of a three-dimensional cavity enclosing and ideal inviscid barotropic fluid. The walls of this cavity are all rigid, except for one of them which is an elastic shell structure.

Let Ω be the three-dimensional domain occupied by the fluid. We consider that $\partial\Omega$ is the union of the 2D surfaces Γ_0, Γ_1, ... , Γ_J and we assume that Γ_0 is in contact with the shell, whereas the remaining surfaces are perfectly rigid walls.

To describe the free small amplitude motions of the fluid, we consider the displacement formulation; we denote by $W = (W_1, W_2, W_3)$ the displacement fluid field.

For the shell structure, we assume that there exist a single chart ϕ that is a one-to-one mapping, which transforms a 2D domain Γ on the midsurface of the shell. Then, we consider the Naghdi shell model, which is written in terms of the rotations $\Theta = (\Theta_1, \Theta_2)$ of the fibers initially normal to the shell midsurface and the three dimensional vector field $U = (U_1, U_2, U_3)$ that correspond to the displacement of the midsurface.

The space of kinematically admissible virtual displacements is denoted by $\mathcal{U}$ and defined by

$$\mathcal{U} := \Big\{(U, \Theta, W) : \ U, \ \Theta, \ W \text{ sufficienttly smooth and } U \cdot \mathrm{n} = W \cdot \mathrm{n}\Big\} \cap \mathcal{BC},$$

where $\mathcal{BC}$ symbolically denotes the prescribed essential boundary conditions.

The governing formulation in the frequency domain for the free small amplitude motions of the coupled system is the following:

SP: *Find* $\omega > 0$ *and* $0 \neq (U, \Theta, W) \in \mathcal{U}$ *such that*

$$a\Big((U, \Theta, W), (V, \Upsilon, Z)\Big) = \omega^2 b\Big((U, \Theta, W), (V, \Upsilon, Z)\Big) \qquad \forall\, (V, \Upsilon, Z) \in \mathcal{U}.$$

The bilinear form $b(\cdot, \cdot)$ is given by

$$b\Big((U, \Theta, W), (V, \Upsilon, Z)\Big) := \int_\Gamma \rho_S \Big(t a^{\alpha\beta} U_\alpha V_\beta + t U_3 V_3 + \frac{t^3}{12} a^{\alpha\beta} \Theta_\alpha \Upsilon_\beta\Big) \sqrt{\det(a)} + \int_\Omega \rho_F W \cdot Z$$

where ρ_S and ρ_F are the density of the shell and the fluid, respectively. Here, the matrix $(a^{\alpha\beta})$ is the contravariant form of the first fundamental form of the midsurface of the shell (see [8, 9] for further details); Greek indices range over 1 and 2, the convention of the summation over the repeated indices up and down is used.

The bilinear form $a(\cdot, \cdot)$ can be written as the sum of a term of the stiffness of the fluid and other term of the stiffness of the shell; in this case, we also separate the latter in a bending term D^b, a membrane term D^m, and a shear term D^s; i.e.,

$$\begin{aligned} a\Big((U, \Theta, W), (V, \Upsilon, Z)\Big) \ &:= \ A(W, Z) + t^3 D^b\Big((U, \Theta), (V, \Upsilon)\Big) \\ &\quad + t D^m(U, V) + t k D^s\Big((U, \Theta), (V, \Upsilon)\Big), \end{aligned}$$

with k being a correction factor for the shear term and

$$\begin{aligned} A(W,Z) &:= \int_\Omega \rho_F c^2 \operatorname{div} W \operatorname{div} Z \\ D^b\Big((U,\Theta),(V,\Upsilon)\Big) &:= \int_\Gamma \frac{E^{\alpha\beta\lambda\mu}}{12}\chi_{\alpha\beta}(U,\Theta)\chi_{\lambda\mu}(V,\Upsilon)\sqrt{\det(a)}, \\ D^m(U,V) &:= \int_\Gamma E^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(U)\gamma_{\lambda\mu}(V)\sqrt{\det(a)}, \\ D^s\Big((U,\Theta),(V,\Upsilon)\Big) &:= \int_\Gamma G^{\alpha\beta}\varphi_\alpha(U,\Theta)\varphi_\beta(V,\Upsilon)\sqrt{\det(a)}. \end{aligned}$$

Here, c is the sound speed in the fluid, the tensor χ, γ, and φ are the well known bending, membrane, and shear strain operators, respectively (see, for instance [8]). The tensors present in the stiffness term, $E^{\alpha\beta\lambda\mu}$ and $G^{\alpha\beta}$ correspond to the material properties and depend on Young's modulus and Poisson's ratio for the structure, E and ν, respectively.

The spectrum of **SP** consists of the frequency $\omega = 0$ and a sequence of finite multiplicity positive frequencies converging to infinity. In the first case, the associated eingenfunctions belong to an infinite-dimensional subspace that consists of pure rotational fluid motions inducing neither variations of the pressure in the fluid nor vibrations in the shell. In fact, they do not correspond to free vibrations modes of the fluid-shell system, but arise because no irrotational constraint is imposed to fluid displacements. The rest of the spectrum are strictly positive frequencies which correspond to actual vibration modes of the fluid-solid sistem. Moreover, the corresponding fluid displacements are irootational.

According to [7, 4], to avoid typical spurious modes in the fluid-solid coupled system, the approximation of the fluid displacement vector field is made by using Raviart-Thomas element. This element discretize the whole vector field instead of each of its components separately (see [5], for further details).

On the other hand, we use the MITC4 method for the shell structure. This method is based on discretizing the bending and membrane terms using the usual isoparametric quadratic finite elements and relaxing the shear term by using reduced integration.

Let us now specify these method in our context. Let $\{\mathcal{T}_h\}$ be a family of partitions in hexahedra of Ω and $\{\mathcal{T}_h^\Gamma\}$ be a family of decomposition of Γ into convex quadrilaterals. Note that, although each $\mathcal{T}_h$ induces a decomposition on Γ, a non compatible new mesh $\{\mathcal{T}_h^\Gamma\}$ could also used. Here h stands for the maximum diameter of the elements in $K \in \mathcal{T}_h^\Gamma$ or $\mathbf{K} \in \mathcal{T}_h$, respectively.

Let $\widehat{K}$ be the unit square reference element. We denote by $Q_{i,j}(\widehat{K})$ the space of polynomials of degree less than or equal to i in the first variable and to j in the second one. We set $Q_k(\widehat{K}) = Q_{k,k}(\widehat{K})$. We denote by F_K the bilinear mapping of $\widehat{K}$ onto K, and we set $Q(K) := \{p : p \circ F_K \in Q_1(\widehat{K})\}$

The space of admissible discrete displacement is

$$\mathcal{U}_h := \Big\{(U_h, \Theta_h, W_h) : \quad U_h|_K \in Q_1(K)^3,\ \Theta_h|_K \in Q_1(K)^2,\ \forall K \in \mathcal{T}_h$$
$$W_h|_{\mathbf{K}} \in RT(\mathbf{K}),\ and \int_{\mathcal{F}} U_h \cdot \mathrm{n} = \int_{\mathcal{F}} W_h \cdot \mathrm{n}\Big\} \cap \mathcal{BC},$$

where $RT(\mathbf{K})$ denote the lowest-order Raviart-Thomas hexahedron (whose degrees of freedom are the fluxes through each of the six faces of $\mathbf{K}$). Recall that $\mathcal{BC}$ denotes the essential boundary conditions prescribed.

We have included weakly the kinematic interface constraint, because to do it strongly is too stringent (see [7]). The integrals to do this are imposed on the fluid mesh faces in contact with the shell.

Then, the discrete variational problem reads:

$\mathbf{SP_h}$: *Find* $\omega_h > 0$ *and* $0 \neq (U_h, \Theta_h, W_h) \in \mathcal{U}_h$ *such that*

$$a_h\Big((U_h, \Theta_h, W_h), (V_h, \Upsilon_h, Z_h)\Big) = \omega_h^2 b\Big((U_h, \Theta_h, W_h), (V_h, \Upsilon_h, Z_h)\Big) \qquad \forall\, (V_h, \Upsilon_h, Z_h) \in \mathcal{U}_h.$$

The bilinear form $a_h(\cdot,\cdot)$ is a perturbed form of $a(\cdot,\cdot)$; more precisely, it comes from introducing in the shear term D^s a reduction operator $\varphi \longmapsto \mathbf{R}\varphi$ with $\mathbf{R}\varphi|_K \in Q_{0,1}(K) \times Q_{1,0}(K)$, $\forall K \in \mathcal{T}\Gamma_h$ (see, for example, [11]). Thus we obtain

$$\begin{aligned} a_h\Big((U_h, \Theta_h, W_h), (V_h, \Upsilon_h, Z_h)\Big) \;&:=\; A(W_h, Z_h) + t^3 D^b\Big((U_h, \Theta_h), (V_h, \Upsilon_h)\Big) \\ &\quad + tD^m(U_h, V_h) + tkD_h^s\Big((U_h, \Theta_h), (V_h, \Upsilon_h)\Big), \end{aligned}$$

with

$$D_h^s\Big((U_h, \Theta_h), (V_h, \Upsilon_h)\Big) := \int_\Omega G^{\alpha\beta}\Big(\mathbf{R}\varphi(U_h, \Theta_h)\Big)_\alpha \Big(\mathbf{R}\varphi(V_h, \Upsilon_h)\Big)_\beta \sqrt{\det(a)}.$$

Let us emphasize that, for the shell structure, this MITC4 finite element procedure is based on meshes that are constructed in a reference 2D domain, and the numerical computations require an extensive use of the chart ϕ.

3 PLATE STRUCTURES

In this section, we consider that the shellis plane (i.e. a plate). The Reissner-Mindlin formulation for plates can be seen as a special case of the Naghdi shell model, where the plate transversal displacements terms appear separately from the in-plane terms. Moreover, the in-plane motions do not interact with the fluid. Therefore they can be solved separately. Thus the fluid-solid interaction can be expressed in terms only of the plate transversal displacement, the fiber rotations and the fluid displacements. If we choose coordinate sistem which that the plate lies in the x, y plane, then the finite element space is

$$\mathcal{U}_h^P := \Big\{(U_{3h}, \Theta_h, W_h) : \qquad U_{3h}|_K \in Q_1(K),\ \Theta_h|_K \in Q_1(K)^2,\ \forall K \in \mathcal{T}_h$$
$$W_h|_{\mathbf{K}} \in RT(\mathbf{K}),\ \text{and} \int_{\mathcal{F}} U_{3h} = \int_{\mathcal{F}} W_h \cdot \mathrm{n}\Big\} \cap \mathcal{BC},$$

and the computed free vibration modes are the solutions of the following problem (see [10, 14]):

$\mathbf{PP_h}$: *Find* $\omega_h > 0$ *and* $0 \neq (U_{3h}, \Theta_h, W_h) \in \mathcal{U}_h^P$ *such that:*

$$t^3 \tilde{a}(\Theta_h, \Upsilon_h) + \kappa t \int_\Omega \mathbf{R}(\nabla U_{3h} - \Theta_h) \cdot \mathbf{R}(\nabla V_{3h} - \Upsilon_h) + \int_\Omega \rho_F c^2 \operatorname{div} W_h \operatorname{div} Z_h$$
$$= \omega_h^2 \rho \left(t \int_\Omega U_{3h} V_{3h} + \frac{t^3}{12} \int_\Omega \Theta_h \cdot \Upsilon_h \right) + \int_\Omega \rho_F W_h \cdot Z_h \qquad \forall (V_{3h}, \Upsilon_h, Z_h) \in \mathcal{U}_h^P.$$

Here $\kappa := Ek/2(1+\nu)$ is the shear modulus (with k a correction factor), meanwhile the bilinear form a is defined by

$$\tilde{a}(\Theta_h, \Upsilon_h) := \frac{E}{12(1-\nu^2)} \int_\Omega \left[\sum_{i,j=1}^{2} (1-\nu) \varepsilon_{ij}(\Theta_h) \varepsilon_{ij}(\Upsilon_h) + \nu \operatorname{div} \Theta_h \operatorname{div} \Upsilon_h \right],$$

where ε_{ij} denote the components of the linear strain tensor.

This coupled problem have been analyzed mathematically in [10], where tetrahedral Raviart-Thomas elements for the fluid and MITC3 elements for the plate have been used. These results have been extended to MITC4 and Raviart-Thomas hexahedra in [14]. In both papers, optimal order error estimates have been obtained for the solution of $\mathbf{PP_h}$, which are valid uniformly on the thickness parameter t.

As a test of the performance of this method, we have considered a steel 3D cavity completely filled with water with all of its walls being perfectly rigid, except for one of them which is a plate. The geometric parameter are given in Figure 1. The physical parameters of plate and fluid are the following ones:

- density of the plate: $\rho_{\mathrm{P}} = 7700\,\mathrm{kg/m^3}$,
- Young modulus: $E = 1.44 \times 10^{11}\,\mathrm{Pa}$,
- Poisson coefficient: $\nu = 0.35$,
- density of the fluid: $\rho_{\mathrm{F}} = 1000\,\mathrm{kg/m^3}$,
- sound speed: $c = 1430\,\mathrm{m/s}$,

Table 1 shows the frequencies of the three lowest-frequency vibration modes computed on different meshes. Here, N stand for the number of layer of element for the fluid domain in the vertical direction. The number of layers in the other two direction being $2N$ and

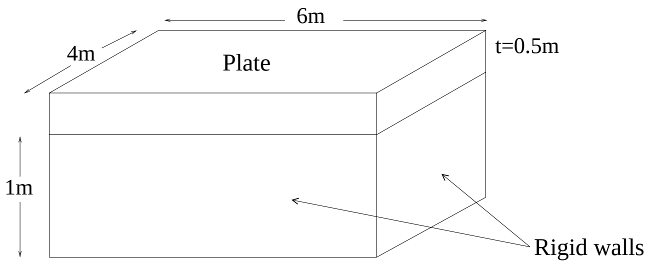

Figure 1: 3D cavity filled with fluid.

$3N$. We also include more accurate values computed by extrapolating those obtained with most refined meshes. The obtained results compare perfectly well with those in [5, 10, 14].

Table 2 shows the result obtained for the first frequency vibration mode, for plates with different thicknesses. To allow for comparison we scale the frequencies. Note that the convergence behaivors do not depend on the thicknesses.

Table 1: Vibration frequencies of a steel plate in contact with water.

Mode	N=4	N= 5	N=6	N=7	"exact"
ω_1	745.5411	744.6309	744.1355	743.8364	743.002848
ω_2	1126.6920	1123.8563	1122.3137	1121.3828	1118.786303
ω_3	1354.1576	1351.3917	1349.8881	1348.9811	1346.471972

Table 2: Scaled lowest vibration frequency for plates of different thicknesses coupled with water.

Thickness	N=4	N= 5	N=6	N=7	"exact"
0.5	745.54	744.63	744.13	743.83	743.0028
0.05	747.51	746.63	746.15	745.87	745.0697
0.005	747.53	746.65	746.18	745.89	745.0913
0.0005	747.53	746.65	746.18	745.89	745.0915

Figures 2 and 3 show the deformed plate and the fluid pressure for the two first modes in Table 1.

4 SHELL STRUCTURES

In this section we present numerical results corresponding to the solution of problem $\mathbf{SP_h}$. We consider a thin cylinder clamped by both ends and full of fluid. We use the

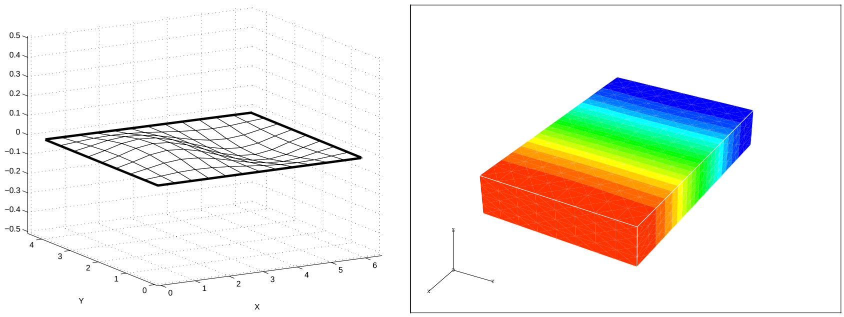

Figure 2: First modes of the fluid.

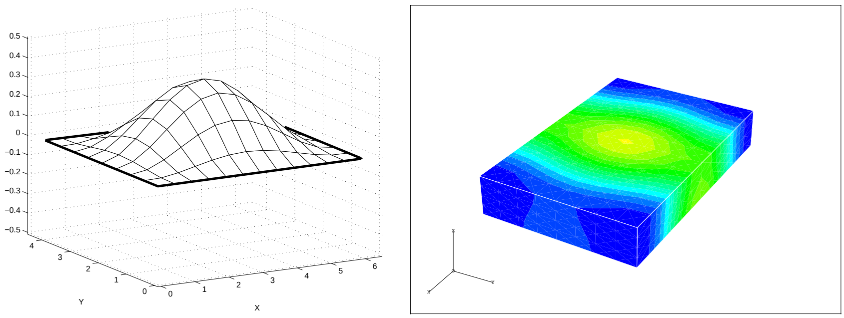

Figure 3: First modes of the plate.

same material properties as in the previous case; i.e., an steel structure full of water. The height of the cylinder is 3.5 m, inner diameter length 2.0 m, and thickness 0.1 m.

Table 3 shows the computed lowest vibration frequencies for the cylinder coupled with the water. We denote by ω_s and ω_f the shell and fluid mode, respectively. This is the same test chosen in [6, 5].

Figures 4 and 5 show the deformed cylinder and the fluid pressure for the vibration modes in Table 3.

Table 3: First vibration mode for shell of different thickness coupled with water.

mode	N=1	N= 2	N=3	"exact"
ω_s	2091.0483	1193.7631	1014.4395	856.0393927
ω_f	1199.2930	1169.1406	1162.4796	1155.7457583

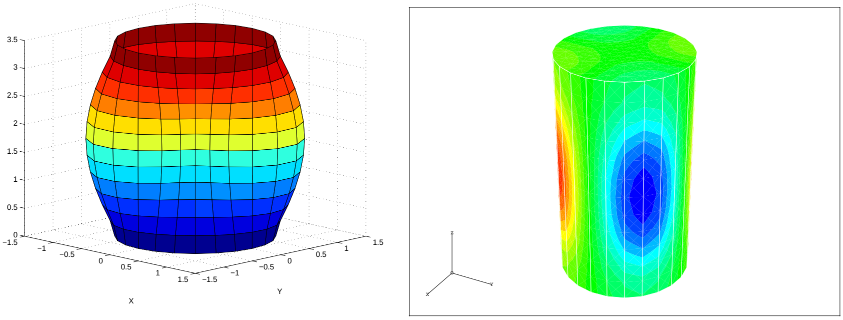

Figure 4: First vibration mode of the shell.

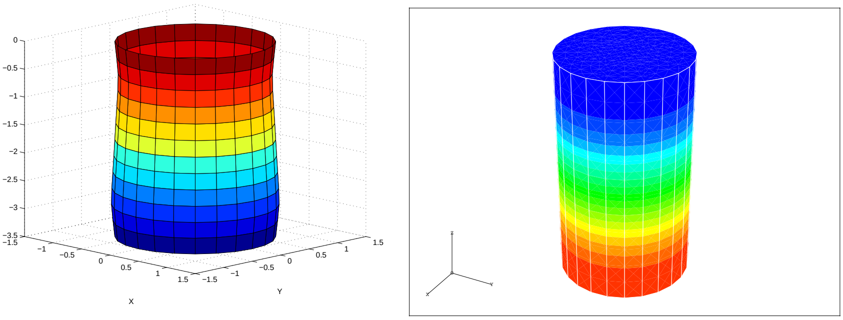

Figure 5: First mode of the fluid.

REFERENCES

[1] K.J. Bathe. Finite Element Procedures. Prentice Hall,Englewood Cliffs, NJ, 1996.

[2] K.J. Bathe and E.N. Dvorkin. A four-node plate bending element based on Mindlin/Reissner plate theory and a mixed interpolation. *Internat. J. Numer. Methods Eng.*, **21**, 367–383 (1985).

[3] K.J. Bathe, A. Iosilevich and D. Chapelle. An evaluation of the MITC shell elements. *Computers & Structures*, **75**, 1–30 (2000).

[4] A. Bermúdez, R. Durán, M.A. Muschietti, R. Rodríguez and J. Solomin. Finite element vibration analysis of fluid-solid systems without spurious modes. *SIAM J. Numer. Anal.* **32**, 1280–1295 (1995).

[5] A. Bermúdez, P. Gamallo, and R. Rodríguez. An hexahedral face element for elastoacoustic vibration problems. *Journal of Computational Acoustics.* **119**, 355–370 (1994).

[6] A. Bermúdez, L. Hervella-Nieto, R. Rodríguez. newblock Finite element computation of three dimensional elastoacoustic vibrations. *Journal of Sound and Vibration.***219**, 277–304 (1999).

[7] A. Bermúdez and R. Rodríguez. Finite element computation of the vibration modes of a fluid-solid system. *Comput. Methods Appl. Mech. and Eng.*, **119**, 355–370 (1994).

[8] M. Bernadou. Finite element methods for thin shells problems. John Wiley and Sons, New York, 1996.

[9] D. Chapelle and K.J. Bathe. Fundamental considerations for the finite element analysis of shell structures. *Computers & Structures*, **66**, 19–36 (1998).

[10] R. Durán, L. Hervella-Nieto, E. Liberman, R. Rodríguez and J. Solomin. Finite element analysis of the vibration problem of a plate coupled with a fluid. *Numerische Mathematik*, **86**, 591–616 (2000).

[11] R. Durán, E. Hernández, L. Hervella-Nieto, E. Liberman and R.Rodríguez. Error estimates for low-order isoparametric quadrilateral finite element for plates. *Submitted*, (2002).

[12] R. Durán and E. Liberman. On mixed finite element methods for the Reissner-Mindlin plate model. *Math. Comp.*, **58**, 561–573 (1992) .

[13] E. Hernández, L. Hervella-Nieto, and R. Rodríguez. Computation of the vibration modes of plates and shells by low-order MITC quadrilateral finite elements. *submitted*, (2002).

[14] E. Hernández. Approximation of the vibration modes of a plate coupled with a fluid by low-order isoparametric finite elements. *In preparation*, (2002).

[15] H.J-P. Morand and R. Ohayon. Fluid-structure interactions. John Wiley & Sons, New York, 1995.