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AN INTERFACE STRIP PRECONDITIONER FOR DOMAIN DECOMPOSITION METHODS

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Abstract. *A preconditioner for iterative solution of the interface problem in Schur Complement Domain Decomposition Methods is presented. This preconditioner is based on solving a problem in a narrow strip around the interface. It requires much less memory and computing time than classical Neumann-Neumann preconditioner and its variants, and handles correctly the flux splitting among subdomains that share the interface. Performance of this preconditioner is assessed with an analytical study of Schur complement matrix eigenvalues and several numerical experiments conducted in a sequential computational environment. Even if the crucial practical test will be the implementation in a production parallel code, the results shown here are promising.*

1 INTRODUCTION

Linear systems obtained from discretization of PDE's by means of Finite Difference or Finite Element Methods are normally solved in parallel by iterative methods[1] because they are much less coupled than direct solvers.

Schur complement domain decomposition method leads to a reduced system better suited for iterative solution than the global system, since its condition number is lower ($\propto 1/h$ in place of $\propto 1/h^2$ for the global system, h being the mesh size) and the computational cost per iteration is not so high once the subdomain matrices have been factorized. In addition, it has other advantages in comparison to global iteration. It resolves bad "inter-equation" conditioning, it can handle Lagrange multipliers and in some sense it can be thought as a mixture between a global direct solver and a global iterative one.

The efficiency of iterative methods can be further improved by using preconditioners.[2] For mechanical problems, Neumann-Neumann is the most classical one. From a mathematical point of view, the preconditioner is defined by approximating the inverse of global Schur complement matrix by the weighted sum of local Schur complement matrices. From a physical point of view, Neumann-Neumann preconditioner is based on splitting the flux applied to the interface in the preconditioning step and solving local Neumann problems in each subdomain. This strategy is good only for symmetric operators.

We propose here a preconditioner based on solving a problem in a *"strip"* of nodes around the interface. When the width of the strip is narrow, the computational cost and memory requirements are low and the iteration count is high, when the strip is wide the converse is true. This preconditioner performs better for non-symmetric operators and doesn't suffer from the *"rigid body modes"* for *"internal floating subdomains"* as is the case for the Neumann-Neumann preconditioner. A detailed computation of the eigenvalue spectra for a simple case is shown, and several numerical examples are presented.

2 SCHUR COMPLEMENT DOMAIN DECOMPOSITION METHOD

It is clear that knowing the eigenvalue spectrum of the Schur complement matrix is one of the most important issues in order to develop suitable preconditioners. To obtain analytical expressions for Schur complement matrix eigenvalues and also the influence of several preconditioners, we consider a simplified problem, namely the solution of the Poisson problem in a unit square:

$$\Delta\phi = g, \quad \text{in } \Omega = \{0 \leq x, y \leq 1\} \tag{1}$$

with boundary conditions

$$\phi = \bar{\phi}, \quad \text{at } \Gamma \tag{2}$$

where ϕ is the unknown, $g(x, y)$ is a given source term and Γ is the boundary. Consider now the division of Ω in N_s non-overlapping subdomains $\Omega_1, \Omega_2, \ldots, \Omega_{N_\text{s}}$, such that $\Omega = \Omega_1 \bigcup \Omega_2 \bigcup \ldots \bigcup \Omega_{N_\text{s}}$. For simplicity we assume that the subdomains are rectangles of unit height and width L_j. In practice this is not the best partition, but it will allow us to compute the eigenvalues of the interface problem in closed form. Afterwards, we will show through numerical examples that the conclusions drawn from this simple case can be applied to more general partitions and operators. Let $\Gamma_{\rm int} = \Gamma_1 \bigcup \Gamma_2 \bigcup \ldots \bigcup \Gamma_{N_{\rm s}-1}$ be the interior interfaces among adjacent subdomains. Given a guess for the trace of ϕ at the interior subdomains $\psi_j = \phi|_{\Gamma_j}$, we can solve each interior problem independently as

$$\begin{aligned} \Delta\phi &= g, \quad \text{in } \Omega_j, \\ \phi &= \begin{cases} \psi_{j-1}, & \text{at } \Gamma_{j-1} \\ \psi_j, & \text{at } \Gamma_j \\ \psi, & \Gamma_{\text{up},j} + \Gamma_{\text{down},j} \end{cases} . \end{aligned} \tag{3}$$

where $\psi_0 = \bar{\phi}\big|_{x=0}$ and $\psi_{N_{\rm s}} = \bar{\phi}\big|_{x=1}$ are given.

2.1 The Stekhlov operator

Not any combination of trace values $\{\psi_j\}$ gives the solution to the original problem (1). Indeed the solution to (1) is obtained when the trace values are chosen in such a way that the flux balance condition at the internal interfaces is satisfied,

$$f_j = \left.\frac{\partial\phi}{\partial x}\right|_{\Gamma_j}^{-} - \left.\frac{\partial\phi}{\partial x}\right|_{\Gamma_j}^{+} = 0 \tag{4}$$

where the $\pm$ superscripts stand for the derivative taken from the left and right sides of the interface. We can think at the correspondence between the ensemble of interface values $\boldsymbol{\psi} = \{\psi_1, \ldots, \psi_{N_{\rm s}-1}\}$ and the ensemble of flux imbalances $\mathbf{f} = \{f_1, \ldots, f_{N_{\rm s}-1}\}$ as an interface operator $\mathcal{S}$ such that

$$\mathbf{f} = \mathcal{S}\boldsymbol{\psi} + \mathbf{f}_0 \tag{5}$$

where all inhomogeneities coming from the source term g and Dirichlet boundary conditions $\bar{\phi}$ are concentrated in the constant term $\mathbf{f}_0$, and the homogeneous operator $\mathcal{S}$ is equivalent to solving the equation set (3) with $g = 0$ and homogeneous Dirichlet boundary conditions at the external boundary $x, y = 0, 1$.

Here, $\mathcal{S}$ is the *"Stekhlov operator"*. In a more general setting, it relates the unknown values and fluxes at boundaries when the internal domain is in equilibrium. In the case of internal boundaries, it can be generalized by replacing the fluxes by the flux imbalances. The Schur complement matrix is a discrete version of the Stekhlov operator, and we will show that in this simplified case we can compute the Stekhlov operator eigenvalues in closed form, and then a good estimate for the corresponding Schur complement matrix ones.

2.2 Eigenvalues of Stekhlov operator

We will further assume that only two subdomains (one internal interface) are present, so that a given interface value function ψ (we drop the interface subindex) is an eigenfunction of the Stekhlov operator if the corresponding flux imbalance $f = \mathcal{S}\psi$ is proportional to ψ, i.e. $f = \omega\psi$, being ω the corresponding eigenvalue. The flux imbalance is computed by solving the Poisson problem in each subdomain with homogeneous Dirichlet boundary condition at the external boundary and ψ at the interface. If the domain were an infinite strip in the y direction, then by translation invariance of the problem along that direction we could guess that the eigenfunctions are sinusoids, and then they could be easily computed. An infinite problem equivalent to the original one can be posed by replacing the homogeneous Dirichlet boundary conditions by symmetry reflection conditions. This amounts to assume the eigenfunctions are proportional to sinusoids, such that an odd number of half wavelengths fit in the unit height square, i.e.

$$\psi(y) = \sin k_n y, \quad k_n = 2\pi/\lambda_n, \quad L = (2n+1)\lambda_n, \quad n = 0, ..., \infty \tag{6}$$

where $L = 1$ is the side length, k_n is the wave number, and λ_n is the wavelength. The solution of Laplace problem in each subdomain is

$$\phi_n(x, y) = \begin{cases} [\sinh(k_n x)/\sinh(k_n L_1)] \sin k_n y; & 0 \leq x \leq L_1, \\ [\sinh(k_n(L-x))/\sinh(k_n L_2)] \sin k_n y; & L_1 \leq x \leq L_2. \end{cases} \tag{7}$$

The flux imbalance can be computed as

$$\begin{aligned} f &= \mathcal{S}\psi_n, \\ &= \mathcal{S}^-\psi_n + \mathcal{S}^+\psi_n \\ &= \left.\frac{\partial \phi_n}{\partial x}\right|_{x=L_1^-} - \left.\frac{\partial \phi_n}{\partial x}\right|_{x=L_1^+}, \\ &= k_n \left(\frac{\cosh(k_n L_1)}{\sinh(k_n L_1)} + \frac{\cosh(k_n L_2)}{\sinh(k_n L_2)} \right) \sin(k_n y), \\ &= k_n \left[\coth(k_n L_1) + \coth(k_n L_2)\right] \sin(k_n y), \end{aligned} \tag{8}$$

so that this demonstrates that (6) is an eigenfunction of the Stekhlov operator with eigenvalues

$$\omega_n = \mathrm{eig}(\mathcal{S})_n = \mathrm{eig}(\mathcal{S}^-)_n + \mathrm{eig}(\mathcal{S}^+)_n = k_n \left[\coth(k_n L_1) + \coth(k_n L_2)\right]. \tag{9}$$

$\mathcal{S}^\pm$ are the Stekhlov operators for each of the left and right subdomains,

$$\begin{aligned} \left.\frac{\partial v_1}{\partial x}\right|_{L_1^-} &= \mathcal{S}^- v_1 \\ \left.\frac{\partial v_2}{\partial x}\right|_{L_1^+} &= -\mathcal{S}^+ v_2 \end{aligned} \tag{10}$$

and their eigenvalues are

$$\mathrm{eig}(\mathcal{S}^\mp)_n = k_n \, \coth(k_n L_{1,2}) \tag{11}$$

For large n the hyperbolic cotangents tend both to unity. This shows that the eigenvalues of the Stekhlov operator grow proportionally to n for large n, and then it condition number is infinity. However, when considering the discrete case the wave number k_n is limited by the largest frequency that can be represented by the mesh, which is $k_{\max} = \pi/h$ where h is the mesh spacing. The maximum eigenvalue is

$$\omega_{\max} = 2k_{\max} = \frac{2\pi}{h}, \tag{12}$$

which grows proportionally to $1/h$. As the lowest eigenvalue is independent of h, this means that the condition number of the Schur complement matrix grows as $1/h$. This is to be compared with the condition number of discrete Laplace operator, which typically grows as $1/h^2$. Of course, this reduction in the condition number is not directly translated to total computation time, since we have to take account of factorization of subdomain matrices and forward and backward substitutions involved in each iteration to solve internal problems. However, the overall balance is positive and reduction in the condition number, beside its inherent parallelism, turns out to be one of the main strengths of domain decomposition methods.

In figure 1 we can see the first and tenth eigenfunctions computed directly from the Schur complement matrix for a 2 subdomain partition, whereas in figure 2 we see the first and twenty-fourth eigenfunction for a 9 subdomain partition. It is verified that eigenvalue magnitude is related to eigenfunction frequency along the inter-subdomain interface, and that the penetration of the eigenfunctions towards subdomains interiors decays strongly for higher modes.

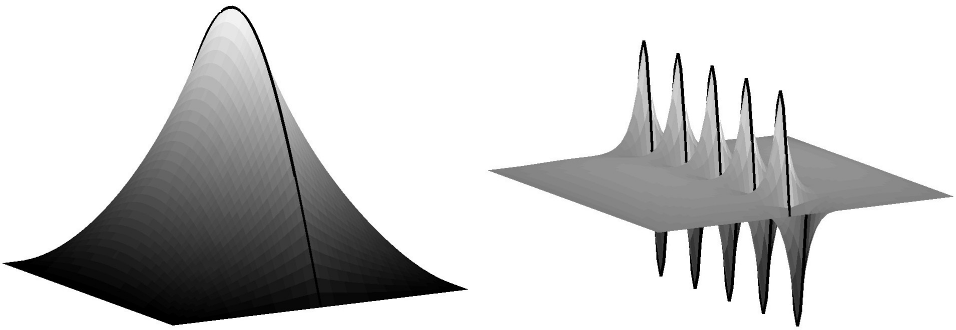

Figure 1: Eigenfunctions of Schur complement matrix with 2 subdomains. Left: 1-st eigenfunction. Right: 10-th eigenfunction.

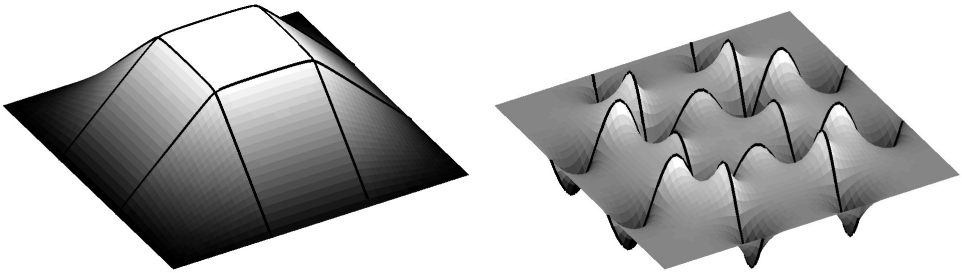

Figure 2: Eigenfunctions of Schur complement matrix with 9 subdomains. Left: 1-st eigenfunction. Right: 24-th eigenfunction.

3 PRECONDITIONERS FOR THE SCHUR COMPLEMENT MATRIX

In order to further improve the efficiency of iterative methods, a preconditioner has to be added so that the condition number of the Schur complement matrix is lowered. The most known preconditioners for mechanical problems are Neumann-Neumann and its variants[3,4] for Schur complements methods, and Dirichlet for FETI methods and its variants.[5–8] It can be proved that they reduce the condition number of the preconditioned operator to $O(1)$ (i.e. independent of h)in some especial cases.

3.1 The Neumann-Neumann preconditioner

Consider the Neumann-Neumann preconditioner

$$\mathcal{P}_{\rm NN} v = f \tag{13}$$

where

$$v(y) = {}^1\!/\!_2[v_1(L_1^-, y) + v_2(L_1^+, y)], \tag{14}$$

and v_1, v_2 are defined in $\Omega_{1,2}$ through the following problems

$$\begin{aligned} &\Delta v_1 = 0, \quad \text{in} \ \ \Omega_1, \\ &v_1 = 0, \quad \text{at} \quad \Gamma_0 + \Gamma_{\text{down},1} + \Gamma_{\text{up},1}, \\ &\frac{\partial v}{\partial x} = {}^1\!/\!_2 f, \quad \text{at} \quad \Gamma_1, \end{aligned} \tag{15}$$

and

$$\begin{aligned} &\Delta v_2 = 0, \quad \text{in} \ \ \Omega_2, \\ &v_2 = 0, \quad \text{at} \quad \Gamma_2 + \Gamma_{\text{down},2} + \Gamma_{\text{up},2}, \\ &\frac{\partial v}{\partial x} = -{}^1\!/\!_2 f, \quad \text{at} \quad \Gamma_1. \end{aligned} \tag{16}$$

The preconditioner consists in assuming that the flux imbalance f is applied on the interface and that, since the operator is symmetric, this *"heat load"* is equally split among the two subdomains. Then, we have a problem in each subdomain with the same boundary conditions in the exterior boundaries, and a non-homogeneous Neumann boundary condition at the inter-subdomain interface.

Again, we will show that the eigenfunctions of the Neumann-Neumann preconditioner are (6). Effectively, we can propose for v_1 the following form

$$v_1 = a \sinh(k_n x) \sin(k_n y) \tag{17}$$

where a is determined from the boundary condition at the interface in (15) and results in

$$a = \frac{1}{2k_n \cosh(k_n L_1)} \tag{18}$$

and similarly for v_2, so that

$$\begin{aligned} v_1(x,y) &= \frac{1}{2k_n} \frac{\sinh(k_n x)}{\cosh(k_n L_1)} \sin(k_n y), \\ v_2(x,y) &= \frac{1}{2k_n} \frac{\sinh(k_n (L_1 - x))}{\cosh(k_n L_2)} \sin(k_n y). \end{aligned} \tag{19}$$

And then, the value of $\mathcal{P}_{\mathrm{NN}}^{-1} f = v$ is

$$\mathcal{P}_{\mathrm{NN}}^{-1} f = v(y) = \frac{1}{4k_n} \left[\tanh(k_n L_1) + \tanh(k_n L_2)\right] \sin(k_n y), \tag{20}$$

so that the eigenvalues of $\mathcal{P}_{\mathrm{NN}}$ are

$$\mathrm{eig}(\mathcal{P}_{\mathrm{NN}})_n = 4k_n \left[\tanh(k_n L_1) + \tanh(k_n L_2)\right]^{-1}. \tag{21}$$

As its definition suggests, it can be verified that

$$\mathrm{eig}(\mathcal{P}_{\mathrm{NN}})_n = 4[\mathrm{eig}(\mathcal{S}^-)_n^{-1} + \mathrm{eig}(\mathcal{S}^+)_n^{-1}]^{-1} \tag{22}$$

As the Neumann-Neumann preconditioner (13) and the Stekhlov operator (8) diagonalize in the same basis (6) (i.e., they *"commute"*), the eigenvalues of the preconditioned operator are simply the quotients of respective eigenvalues, i.e.

$$\mathrm{eig}(\mathcal{P}_{\mathrm{NN}}^{-1} \mathcal{S})_n = {}^1\!/\!_4 [\tanh(k_n L_1) + \tanh(k_n L_2)]\,[\coth(k_n L_1) + \coth(k_n L_2)]. \tag{23}$$

We see that all $\tanh(k_n L_j)$ and $\coth(k_n L_j)$ factors tend to unity for $n \to \infty$, then we have

$$\mathrm{eig}(\mathcal{P}_{\mathrm{NN}}^{-1} \mathcal{S})_n \to 1, \quad \text{for} \quad n \to \infty, \tag{24}$$

so that this means that the preconditioned operator $\mathcal{P}_{\mathrm{NN}}^{-1}\mathcal{S}$ has a condition number $O(1)$, i.e. it doesn't degrade with mesh refinement. This is optimal, and is a well known feature of the Neumann-Neumann preconditioner. In fact, for a symmetric decomposition of the domain (i.e. $L_1 = L_2 = /half$), we have

$$\mathrm{eig}(\mathcal{P}_{\mathrm{NN}}^{-1}\mathcal{S})_n = \frac{1}{4} 2\tanh(k_n/2)\, 2\coth(k_n/2) = 1, \tag{25}$$

so that the preconditioner is equal to the operator and convergence is achieved in one iteration. However, the computational cost of the preconditioner is relatively high, since it amounts to solve a problem in each subdomain with Neumann boundary conditions at the interfaces, i.e. a problem with more unknowns than the internal subdomain problem. At first sight, one could argue that the number of additional unknowns is relatively small because it represents the unknowns on a surface, in contrast with the internal unknowns that correspond to a volume, so that it is asymptotically negligible. However, in practical problems it can be relatively high. For instance, for a regular cube mesh of $10 \times 10 \times 10$ hexahedral elements (equivalent to $5,000$ tetrahedral elements) the number of unknowns in the interior are $9^3 = 729$, while the number of unknowns for the Neumann-Neumann preconditioner is $11^3 = 1331$ nodes, near twice.

Note that comparing (9) and (22) we can see that the preconditioning is good as long as

$$\mathrm{eig}(\mathcal{S}^-)_n \approx \mathrm{eig}(\mathcal{S}^+)_n. \tag{26}$$

This is true for symmetric operators and symmetric domain partitions (i.e. $L_1 \approx L_2$). Even for $L_1 \neq L_2$, if the operator is symmetric, then (26) is valid for large eigenvalues. However, this fails for non-symmetric operators as in the advection-diffusion case, and also for irregular interfaces. Another disadvantage of the Neumann-Neumann preconditioner is the occurrence of indefinite internal Neumann problems, which leads to the need of solve a coarse problem[3,4] in order to resolve the *"rigid body modes"* for internal floating subdomains.

3.2 The Interface Strip (IS) preconditioner

A key point about the Stekhlov operator is that its high frequency eigenfunctions decay very strongly far from the interface, so that a preconditioning that represents correctly the high frequency modes can be constructed if we solve a problem on a narrow strip around the interface. More precisely, the n-th eigenfunction with wave number k_n given by (6) decays far from the interface as $\exp(-k_n|s|)$ where s is the distance to the interface (the hyperbolic sine factors appearing in (8)). Then, this high frequency modes can be correctly represented if we solve a problem on a strip of width b around the interface, provided that the interface width is very large with respect to the mode wave length λ_n. The Interface Strip preconditioner is defined as

$$\mathcal{P}_{\mathrm{IS}} v = f \tag{27}$$

where

$$\begin{aligned} \Delta v &= 0, \quad \text{in} \quad |x - L_1| < b, \\ v &= 0, \quad \text{at} \quad |x - L_1| = b \ \text{ and } \ y = 0, 1. \end{aligned} \tag{28}$$

Note that, for high frequencies (i.e. $k_n b$ large) the eigenfuctions of the Stekhlov operator will be negligible at the border of the strip, so that the boundary condition at $|x - L_1| = b$ is justified. The eigenfuctions for this preconditioner are again given by (6) and the eigenvalues can be taken from (9), replacing $L_{1,2}$ by b, i.e.

$$\text{eig}(\mathcal{P}_{\text{IS}})_n = 2\,\text{eig}(\mathcal{S}_b)_n = 2k_n \, \coth(k_n b), \tag{29}$$

where $\mathcal{S}_b$ is the Stekhlov operator corresponding to a strip of width b. For the preconditioned Stekhlov operator

$$\text{eig}(\mathcal{P}_{\text{IS}}^{-1}\mathcal{S})_n = {}^1\!/_2 \tanh(k_n b) \, [\coth(k_n L_1) + \coth(k_n L_2)] \,. \tag{30}$$

We note that $\text{eig}(\mathcal{P}_{\text{IS}}^{-1}\mathcal{S})_n \to 1$ for $n \to \infty$, so that the preconditioner is optimal, independently of b. Also, for b long enough we recover the original problem so that the preconditioner is exact (convergence is achieved in one iteration). However, in this case the use of this preconditioner is impractical, since it amounts to solve the whole problem. Note that, in order to solve the problem for v, we need information from both sides of the interface, while the Neumann-Neumann preconditioner only needs in each subdomain information from itself. This is a disadvantage from the perspective of efficiency, since we have to waste some communication time in sending the matrix coefficients in the strip from one side to the other or either compute them in both processors. However, we will see that efficient preconditioning can be achieved with few node layers and negligible communication, or either we can solve the preconditioner problem by iteration itself, so that no migration of coefficients is needed.

4 THE ADVECTIVE-DIFFUSIVE CASE

Consider now the advective diffusive case,

$$\kappa \Delta \phi - u \phi_{,x} = g \tag{31}$$

where κ is the thermal conductivity of the medium and u the advection velocity. The problem can be treated in a similar way, and the Stekhlov operators are defined as

$$\mathcal{S}^- \bar{v} = v_{,x}|_{L_1^-} \,, \tag{32}$$

where

$$\begin{aligned} &\kappa \Delta v - u v_{,x} = 0 \\ &v = 0, \quad \text{at } x = 0, y = 0, L, \\ &v = \bar{v}, \quad \text{at } x = L_1. \end{aligned} \tag{33}$$

and similarly for the other subdomain. The eigenfunctions are always given by (6). Looking for solutions of the form $v \propto \exp(\mu x)\,\sin(k_n y)$ it results in a constant coefficient second order differential equation whose characteristic polynomial is

$$\kappa\mu^2 - u\mu - \kappa k_n^2 = 0 \tag{34}$$

whose solutions are

$$\mu^{\pm} = \frac{u \pm \sqrt{u^2 + 4\kappa^2 k_n^2}}{2\kappa} \tag{35}$$

and, after some algebra

$$v = \begin{cases} \mathrm{e}^{u(x-L_1)/2\kappa}\,[\sinh(\delta x)/\sinh(\delta L_1)]; & \text{for } 0 \leq x \leq L_1 \\ \mathrm{e}^{u(x-L_1)/2\kappa}\,[\sinh(\delta(L-x))/\sinh(\delta L_2)]; & \text{for } L_1 \leq x \leq L_2, \end{cases} \tag{36}$$

and the eigenvalues are

$$\begin{aligned} \mathrm{eig}(\mathcal{S}^-)_n &= \frac{u}{2\kappa} + \coth(\delta_n L_1)\delta_n \\ \mathrm{eig}(\mathcal{S}^+)_n &= -\frac{u}{2\kappa} + \coth(\delta_n L_2)\delta_n \end{aligned} \tag{37}$$

In figure 3 we see the first and tenth eigenfuctions for a problem with an advection term at a global Pèclet number of $uL/2\kappa = 2.5$. For low frequency modes, advective effects are more pronounced and the first eigenvalue (on the left) is notably biased to the right. In contrast, for high frequency modes (like the tenth mode shown at the right) the diffusive term prevails and the eigenfunction is more symmetric about the interface, and (as in the pure diffusive case) concentrated around it. Note that now the eigenvalues for the right and left part of the Stekhlov operator may be very different due to the asymmetry introduced by the advective term. This difference in splitting is more important for the lowest mode.

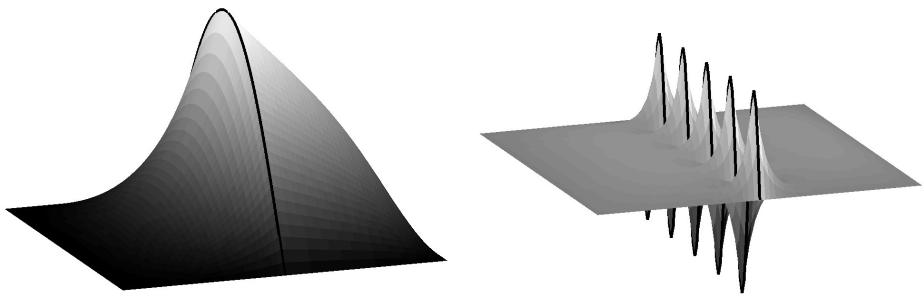

Figure 3: Eigenfunctions of Schur complement matrix with 2 subdomains and advection (global Pèclet 5). Left: 1-st eigenfunction. Right: 10-th eigenfunction.

In figures 4, 5, 6, 7 we see the eigenvalues as a function of the wave number k_n. Note that, for a given side length L only a certain sequence of wave numbers, given by (6) should be considered. However, it is perhaps easier to consider the continuous dependence of the different eigenvalues upon the wave number k.

For a symmetric partition ($L_1 = L_2 = L/2$) and a symmetric operator ($u = 0$, see 4), the symmetric flux splitting is exact and the Neumann-Neumann preconditioner is optimal. The major discrepancy between the IS preconditioner and the Stekhlov operator occurs at low frequencies and yields a condition number less than two.

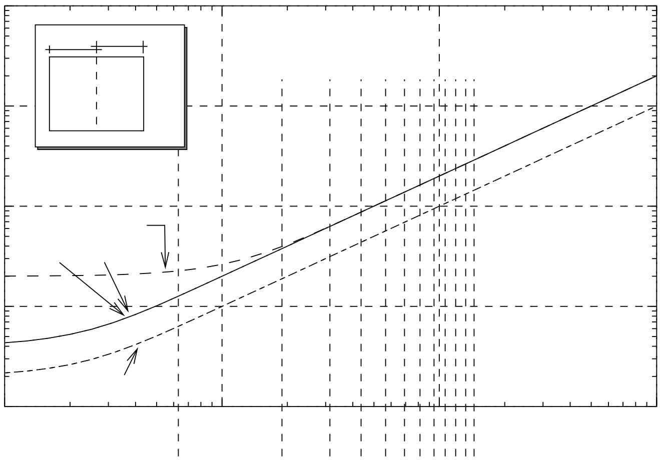

Figure 4: Eigenvalues of Stekhlov operators and preconditioners for the Laplace operator (Pe=0) and symmetric partitions ($L_1 = L_2 = L/2$).

If the partition is non-symmetric (see figure 5) then the Neumann-Neumann preconditioner is no longer exact, because $\mathcal{S}^+ \neq \mathcal{S}^-$. However, its condition number is very low whereas the IS preconditioner condition number is still under two.

For a relatively important advection term, given by a global Pèclet number of 5 (see figure 6), the asymmetry in the flux splitting is much more evident, mainly for small wave numbers, and this results in a large discrepancy between the Neumann-Neumann preconditioner and the Stekhlov operator. On the other hand, the IS preconditioner is still very close to the Stekhlov operator.

The difference between the Neumann-Neumann preconditioner and the Stekhlov operator increases for larger Pe, as can be seen in figure 7 for Pe=50.

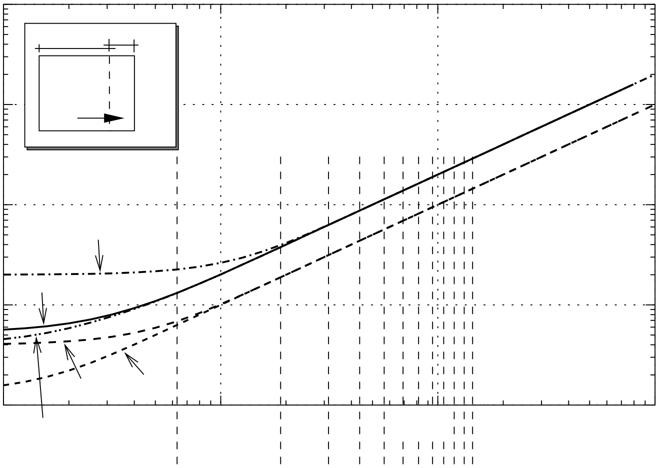

Figure 5: Eigenvalues of Stekhlov operators and preconditioners for the Laplace operator (Pe=0) and non-symmetric partitions ($L_1 = 0.75L, L_2 = 0.25L$).

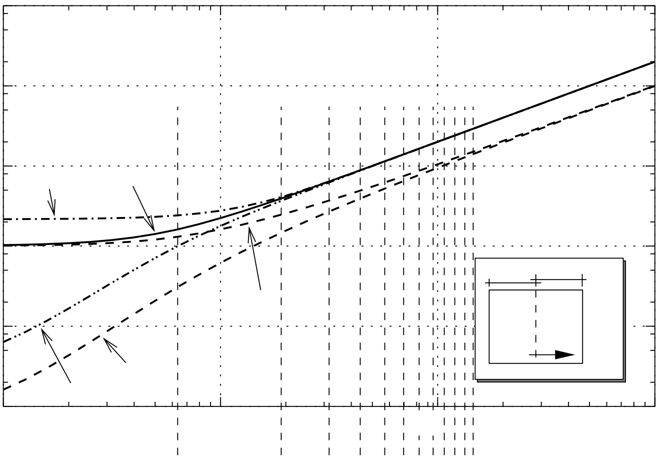

Figure 6: Eigenvalues of Stekhlov operators and preconditioners for the advection-diffusion operator (Pe=5) and symmetric partitions.

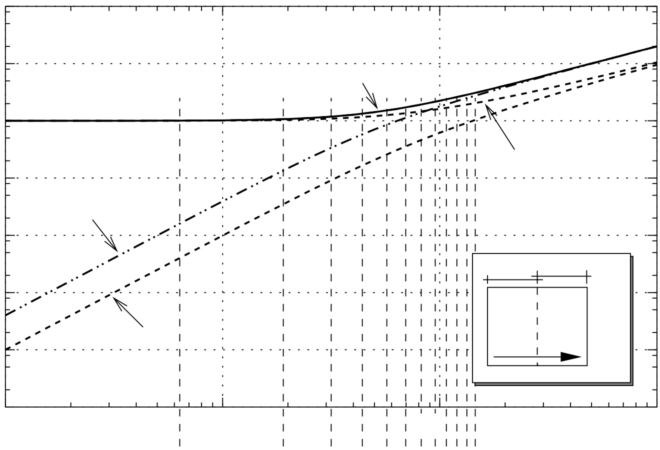

Figure 7: Eigenvalues of Stekhlov operators and preconditioners for the advection-diffusion operator (Pe=50) and symmetric partitions.

This behavior can be directly verified by computing the condition number of Schur complement matrix and preconditioned Schur complement matrix for the different preconditioners, see tables 1 and 2. We can see that both the Neumann-Neumann and IS preconditioners give a preconditioned condition number independent of mesh refinement (it almost doesn't change from a mesh of 50×50 to a mesh of 100×100), whereas the Schur complement matrix exhibits a condition number roughly proportional to $1/h$. However, the Neumann-Neumann preconditioner exhibits a large condition number for high Pèclet numbers whereas the IS preconditioner seems to perform better for advection dominated problems.

u	cond($\mathcal{S}$)	cond($\mathcal{P}_{\rm NN}^{-1}\mathcal{S}$)	cond($\mathcal{P}_{\rm IS}^{-1}\mathcal{S}$)
0	41.00	1.00	4.92
1	40.86	1.02	4.88
10	23.81	3.44	2.92
50	5.62	64.20	1.08

Table 1: Condition number for the Stekhlov operator and several preconditioners for a mesh of 50×50 elements.

u	cond($\mathcal{S}$)	cond($\mathcal{P}_{\rm NN}^{-1}\mathcal{S}$)	cond($\mathcal{P}_{\rm IS}^{-1}\mathcal{S}$)
0	88.50	1.00	4.92
1	81.80	1.02	4.88
10	47.63	3.44	2.92
50	11.23	64.20	1.08

Table 2: Condition number for the Stekhlov operator and several preconditioners for a mesh of 100×100 elements.

5 SOLUTION OF THE STRIP PROBLEM

Efficient implementation of the IS preconditioner in a parallel environment will be the subject of future research. However, we will give some hints here.

A first possibility is a fully coupled, direct solution of the interface problem. This approach involves transferring all the interface matrix to a single processor and solving the problem there. This is not a significant amount of work, but doing it in only one processor would largely imbalance the distribution of load among processors.

A second possibility is partitioning the strip problem among processors, much in the same way as the global problem is. Then, the preconditioning problem may be solved by an iterative method. Care must be taken in not to nest a non-stationary method like CG or GMRES inside another outer non-stationary method. The problem here is that a non stationary method executed a finite number of times *is not* a linear operator, unless the inner iterative method is iterated enough and then approaches the inverse of the preconditioner. In this respect, relaxed Richardson iteration is a candidate. The idea of an iterative method is also suggested by the fact that the preconditioning matrix (i.e. the matrix obtained by assembling on the strip domain with Dirichlet boundary conditions at the strip boundary) is highly diagonal dominant for narrow strips. A subsequent possibility is preconditioning the Interface Strip preconditioner problem itself with block Jacobi.

6 NUMERICAL RESULTS

Performance of the proposed preconditioner is compared in a sequential environment. For this purpose, we consider two different problems. The domain Ω in both cases is the unit square discretized on an unstructured mesh of 120×120 nodes, and decomposed in 6 rectangular subdomains. We compare the residual norm versus iteration count by using no preconditioner, Neumann-Neumann preconditioner, and the IS preconditioner (with several node layers at each interface side).

The first example is the Poisson's problem $\Delta\phi = g$, were $g = 1$ and $\phi = 0$ on all de boundary Γ. The iteration counts and the problem solution (obtained in a coarse mesh for visualization purposes) are plotted in figure 8. As it can be seen, the Neumann-Neumann preconditioner has a very low iteration count, as is expected for a symmetric operator. The IS preconditioner has a larger iteration count for thin strip widths, but it decreases as well as the strip is thickened. For a strip of five-layers width, we reach an iteration count comparable to the Neumann-Neumann preconditioner with a significantly less computational effort. Regarding memory use, the required core memory for thin strip is much less than for the Neumann-Neumann preconditioner. The strip width acts in fact as a parameter that balances the required amount of memory and the preconditioner efficiency.

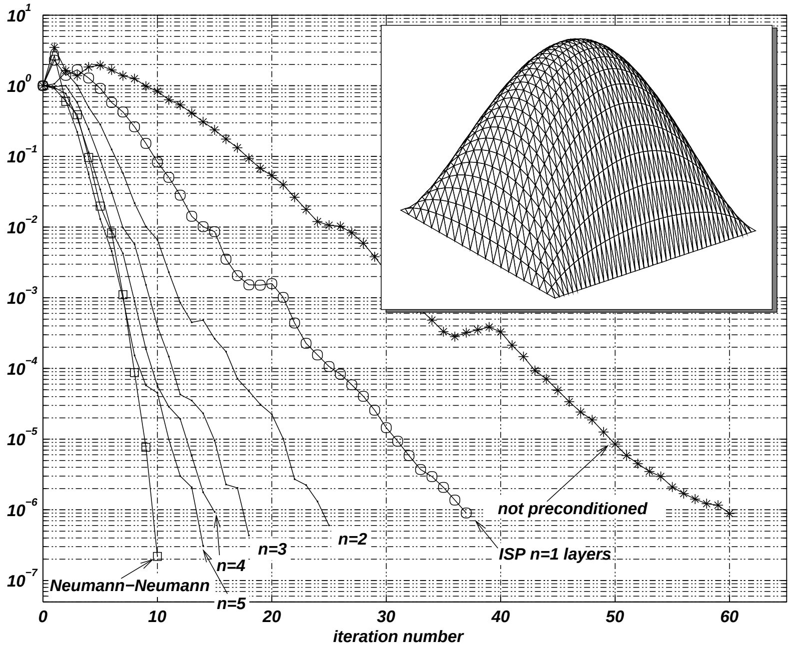

Figure 8: Solution of Poisson's problem.

The second example is an advective-diffusive problem at a global Péclet number of Pe $= 25$, $g = \delta(1/4, 7/8) + \delta(3/4, 1/8)$, and $\phi(0, y) = 0$. Therefore, the problem is strongly advective. The iteration counts and the problem solution (obtained in a coarse mesh for visualization purposes) are plotted in figure 9. In this example, the advective term introduces a strong asymmetry. The Neumann-Neumann preconditioner is far to be optimal. It is outperformed by IS preconditioner in iteration count (consequently, in computing time) and memory demands, even for thin strips.

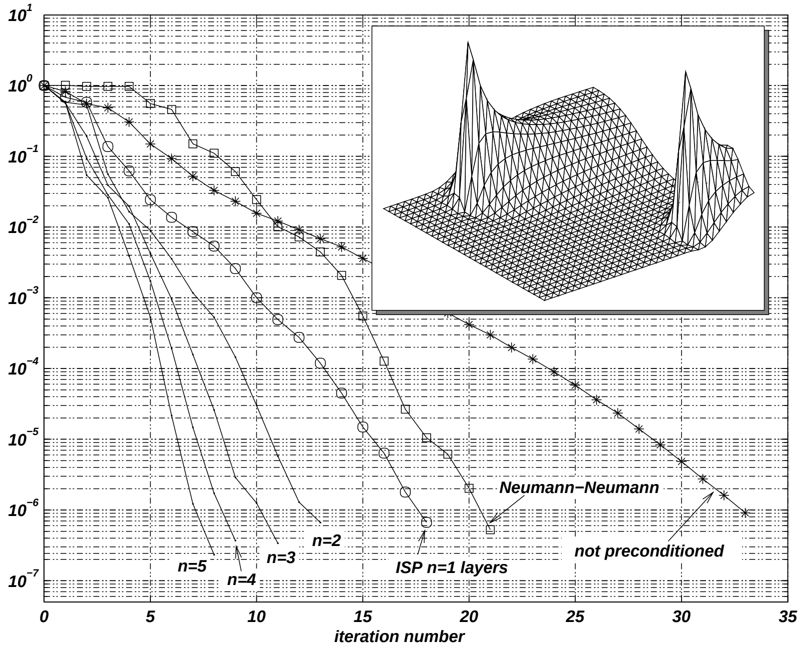

Figure 9: Solution of advective-diffusive problem.

7 CONCLUSIONS

In this paper, we have presented a new preconditioner for the Schur complement matrix, which is the heart of domain decomposition methods. This preconditioner is based on solving a problem posed in a narrow strip around the inter-subdomain interfaces. Some analytical results have been derived to present its mathematical basis. Numerical experiments have been carried out to show its convergence properties.

The IS preconditioner is easy to construct as it does not require any special calculation (it is assembled with a subset of the coefficients of subdomain matrices). It is much less memory-consuming than classical optimal preconditioners such as Neumann-Neumann (or Dirichlet in FETI methods). Moreover, it permits to decide how many memory to assign for preconditioning purposes. In addition, it does not suffer from the "rigid body modes" for internal floating subdomains.

In advective-diffusive real-life problems, were the Pèclet number can vary on the domain between low and high values, the proposed preconditioner outperforms classical ones in advection-dominated regions while it is capable to handle reasonably well diffusion-dominated regions.

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