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NON LINEAR MODES AS A TOOL TO ANALYSE NONLINEAR DYNAMICAL SYSTEMS

S. Bellizzi and R. Bouc

Laboratoire de Mécanique et d'Aoustique, Centre National de la Recherche Scientifique, 31 chemin Joseph Aiguier, 13402 MArseille, France, {bellizzi,bouc}@lma.cnrs-mr.fr,
http://www.lma.cnrs-mrs.fr

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Abstract. In structural analysis, the concept of normal modes is classically related to the linear vibration theory. Extending the concept of normal modes to the case where the restoring forces contain non-linear terms has been a challenge to many authors mainly because the principle of linear superposition does not hold for non-linear systems. The aim of this paper is to show how the concept of the Noninear Modes (NNMs) can be used to better understand the response of the nonlinear mechanical systems. The concept of NNMs is introduced here in the framework of invariant manifold theory for dynamical systems. A NNM is defined in terms of amplitude, phase, frequency, damping coefficient and mode shape, where the last three quantities are amplitude and phase dependent. An amplitude-phase transformation is performed on the nonlinear dynamical system, giving the time evolution of the nonlinear mode motion via the two first-order differential equations governing the amplitude and phase variables, as well as the geometry of the invariant manifold. The formulation adopted here is suitable for use with a Galerkin-based computational procedure. It will be shown how the NNMs give access to the existence and stability of periodic orbits such as limit cycle.

1 INTRODUCTION

The modal analysis is the natural tool to characterize the linear mechanical systems (in particular for numerical modelling, prediction and experimental characterization). It is based on the modal parameters (frequencies, modal shape and ratio damping), which constitute intrinsic characteristics of the system. The modal analysis uses intensively the principle of superposition, which allows to uncouple the equations from the movement and to express the free or forced responses. For the non-linear systems, these techniques have to be reformulated in particular because the principle of superposition does not apply but also to take account the dependencies of the resonant frequencies (and more generally modal shapes) with respect to the amplitude of the oscillations.

The extension of the modal theory to nonlinear mechanical systems appears the first time under the name Nonlinear Normal Modes linear (NNM) in the work of Rosenberg (1962) for a system of n-masses interconnected by nonlinear springs. The term NNM describes a vibration in unison, which means that "all masses execute equiperiodic motions, all pass through equilibrium at the same instant, all attain maximum displacement at the same instant and the position of any one mass at any given instant of time defines uniquely that of every other mass at the same instant". More precisely, a NNM is defined as a family of periodic solutions of the equations of motion corresponding to simple curves in the configuration space.

More recently, Shaw and Pierre (1991) extended the concept of NNM in the context of phase space . The proposed approach is geometric in nature and utilizes the theory of invariant manifolds for dynamical systems. A NNM of autonomous system is defined as (see Shaw and Pierre (1993)) "a two-dimensional invariant manifold in the phase space. This manifold passes through a stable equilibrium point of the system and, at that point, it is tangent to a plan, which is an eigenspace of the system linearized about that equilibrium". In the manifold, the modal equation of motion is a one-degree of freedom nonlinear oscillator. This definition is valid for dissipative mechanical systems. A nonlinear superposition technique is also proposed and its validity is discussed in Pellicano and Mastroddi (1997). The construction of the NNMs for piecewise linear systems is considered in Jiang et al. (2004), the systems with internal resonance is treated in Jiang et al. (2005).

The invariant manifold approach is very close to the methods based on the theory of normal forms (see Jezequel and Lamarque (1991), Nayfeh (1993) and Touzé et al. (2004)) where the invariant manifold and the modal equations of motion are extracted from the minimal representation.

The review paper Vakakis (1997) and the book Vakakis et al. (1996) contain an almost complete account of the history of the subject. Moreover, the main tool to analyse dynamical nonlinear systems can be found in Guckenheimer and Holmes (1983), Nayfeh and Mook (1984) and Szemplinska-Stupnicka (1990).

In this paper, the NNMs are introduced using an amplitude-phase formulation recently proposed in Bellizzi and Bouc (2005) and Bellizzi and Bouc (2007) in the line with the approach developed in Pesheck et al. (2002). For nonlinear conservative systems, as in the linear case, a nonlinear mode is described (see Bellizzi and Bouc (2005)) in terms of mode shape and frequency, where the distinctive feature that these two quantities are amplitude and phase dependent. For a given modal motion the amplitude is constant and the time evolution of the modal motion is defined by a first-order differential equation governing the total phase motion, from which the period of the oscillation can easily be deduced. It was established that the frequency and mode shape functions solve a 2π-periodic (with respect to the phase variable) nonlinear differential eigenvalue problem. This formulation also gives a parametric description of the associated invariant manifold. For autonomous mechanical systems including displacement and velocity nonlinear terms, a NNM is described (see Bellizzi and Bouc (2007)) in terms of mode shape, frequency and damping where these three functions are amplitude and phase dependent. For a given modal motion the time evolution of the modal motion is defined by two first-order differential equations governing the amplitude and the total phase motions. This formulation gives access to the parametric equation of the invariant manifold associated to the NNM in the phase space. In terms of signal processing the frequency function and the damping function give access to the amptitude modulation and frequency moduation of the modal motions.

In addition to introducing the concept of NNMs, the objective of this paper is to show that the NNM can provide a valuable theoretical tool for understanding some nonlinear phenomena. We will focus on the limit cycle analysis of auto-oscillation systems. We will show that the amplitude phase formulation gives, without explicit integration of the equations of motion, access to the transient, the limit cycle, its period and its stability. The process will be illustrated for a two-coupled van der Pol oscillators.

2 NONLINEAR MODES FORMULATION

Let's consider the equations of motion

$$\mathbf{M}\ddot{\mathbf{Q}}(t) + \mathbf{F}(\dot{\mathbf{Q}}(t), \mathbf{Q}(t)) = \mathbf{0} \tag{1}$$

where the n-dimensional vectors $\mathbf{Q}$ and $\dot{\mathbf{Q}}$ represent displacement and velocity, respectively, $\mathbf{M}$ denotes the mass matrix and $\mathbf{F}$ is a n-vector function defined from the forces and moment acting on the system. We assume that $\mathbf{M}$ is symmetric positive definite matrix and $\mathbf{F}$ satisfies $\mathbf{F}(\mathbf{0},\mathbf{0}) = \mathbf{0}$ (i.e. $\mathbf{0}$ is an equilibrium point).

2.1 Definition of a NNM

Let's consider a family of motions of (1) who can express themselves in the form

$$\left\{ \begin{array}{lcl} \mathbf{Q}(t) & = & v(t)\mathbf{X}(v(t),\phi(t)) \\ \dot{\mathbf{Q}}(t) & = & v(t)\mathbf{Y}(v(t),\phi(t)) \end{array} \right. \tag{2}$$

where the scalar functions v and ϕ, called amplitude and phase respectively, satisfy the two first-order differential equations

$$\left\{ \begin{array}{lcl} \dot{v}(t) & = & v(t)\xi(v(t),\phi(t)) \\ \dot{\phi}(t) & = & \Omega(v(t),\phi(t)) \end{array} \right. \quad \text{with} \quad \left\{ \begin{array}{lcl} v(0) & = & a \\ \phi(0) & = & \varphi \end{array} \right. \tag{3}$$

where a and φ are constant parameters defining the initial conditions of the motion.

We assume that

- $\mathbf{X}$ and $\mathbf{Y}$ are n-vector functions, 2π-periodic with respect to the variable ϕ;
- ξ is a odd scalar function π-periodic with respect to the variable ϕ;
- Ω is a positive scalar function π-periodic with respect to the variable ϕ.

If such a family of motions generated by equations (2-3) and parameterized by (a, φ) for $(a, \varphi) \in \mathbb{R}^+ \times [0, 2\pi]$ exists, it defines a NNM. Hence, each NNM is characterized by the four functions $\mathbf{X}$, $\mathbf{Y}$, ξ and Ω. The two vector functions, $\mathbf{X}$ and $\mathbf{Y}$, characterize the geometrical properties of the NNM. For a given NNM, all the modal motions take place in an invariant set of the phase space defined by

$$\left\{ \begin{array}{lcl} \mathbf{Q} & = & a\mathbf{X}(a,\varphi) \\ \dot{\mathbf{Q}} & = & a\mathbf{Y}(a,\varphi) \end{array} \right. , \ (a,\varphi) \in \mathbb{R}^+ \times [0, 2\pi]. \tag{4}$$

The scalar functions Ω and ξ capture the dynamical properties of the NNM. The fast and slow motions of the modal motions are characterized by the frequency function Ω and the damping function ξ, respectively.

2.2 Linear case

If $\mathbf{F}(\mathbf{Q}, \dot{\mathbf{Q}}) = \mathbf{KQ} + \mathbf{C}\dot{\mathbf{Q}}$, the linear modes of (1) are classical defined by the eigenproblem

$$\left(\begin{array}{cc} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{array} \right) \mathbf{\Psi}\lambda + \left(\begin{array}{cc} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{array} \right) \mathbf{\Psi} = 0. \tag{5}$$

A solution $(\lambda, \mathbf{\Psi})$ of equation (5) (with $\mathbf{\Psi} = \left(\boldsymbol{\psi}^T, \lambda\boldsymbol{\psi}^T\right)^T$) defines a Linear Normal Mode (LNM) if $\mathbf{C} = 0$ and a complex mode if not.

The modal motions associated to $(\lambda, \mathbf{\Psi})$ take the form

$$\left\{ \begin{array}{lcl} \mathbf{Q}(t) & = & ae^{\eta t}\left(\boldsymbol{\psi}^c \cos(\omega t + \varphi) - \boldsymbol{\psi}^s \sin(\omega t + \varphi)\right) \\ \dot{\mathbf{Q}}(t) & = & ae^{\eta t}\left((\eta\boldsymbol{\psi}^c - \omega\boldsymbol{\psi}^s)\cos(\omega t + \varphi) - (\eta\boldsymbol{\psi}^s + \omega\boldsymbol{\psi}^c)\sin(\omega t + \varphi)\right) \end{array} \right. \tag{6}$$

where $\lambda = \eta + i\omega$ (it is assume that $\omega > 0$) and $\boldsymbol{\psi} = \boldsymbol{\psi}^c + i\boldsymbol{\psi}^s$ with $i^2 = -1$. As previously, the parameters a and φ fix the initial conditions of the motion.

It is easy to verify that the expressions (6) can be re-written in form of equations (2-3) with

$$\left\{ \begin{array}{lcl} \mathbf{X}(v,\phi) & = & \boldsymbol{\psi}^c \cos\phi - \boldsymbol{\psi}^s \sin\phi \\ \mathbf{Y}(v,\phi) & = & (\xi\boldsymbol{\psi}^c - \Omega\boldsymbol{\psi}^s)\cos\phi - (\xi\boldsymbol{\psi}^s + \Omega\boldsymbol{\psi}^c)\sin\phi \\ \xi(v,\phi) & = & \eta \\ \Omega(v,\phi) & = & \omega \end{array} \right. . \tag{7}$$

Hence the formulation (2-3) appears as an extension of the linear modal analysis where Ω and ξ are constant functions.

As it is well known, the invariant manifold associated to a linear mode is a vector subspace of dimension 2 and the modal motions can be, depending on the damping matrix, periodic or exponential decreasing.

2.3 Nonlinear conservative case

If $\mathbf{F}(\mathbf{Q}, \dot{\mathbf{Q}}) = \mathbf{F}(\mathbf{Q})$ with $\mathbf{F}(\mathbf{Q}) = -\mathbf{F}(-\mathbf{Q})$, without loss of generality, the NNM can be sought under the form

$$\mathbf{Q}(t) = a\mathbf{\Psi}(a, \phi(t))\cos\phi(t) \text{ with } \dot{\phi}(t) = \Omega(a, \phi(t)) \text{ and } \phi(0) = \varphi \tag{8}$$

where

- $\mathbf{\Psi}$ is n-vector functions, π-periodic with respect to the variable ϕ;
- Ω is a positive scalar function π-periodic with respect to the variable ϕ.

The formulation (8) is evidently related to the formulation (2-3) with

$$\begin{cases} \mathbf{X}(v,\phi) & = & \mathbf{\Psi}(v,\phi)\cos\phi \\ \xi(v,\phi) & = & 0 \end{cases} \tag{9}$$

and if such a NNM exists, all the modal motions are periodic and the period is given by

$$T(a) = \int_{\varphi}^{\varphi+2\pi} \frac{1}{\Omega(a,\phi)} d\phi = 2\int_{0}^{\pi} \frac{1}{\Omega(a,\phi)} d\phi \tag{10}$$

showing that the period is only amplitude-dependent.

Substituting equation (8) into equation (1), equation (1) reduces to the following differential equations in the variable ϕ

$$\mathbf{M}\mathbf{\Psi}(\Omega^2\cos\phi + \frac{1}{2}(\Omega^2)_\phi \sin\phi) = \mathbf{L}(\Omega^2,\mathbf{\Psi};\phi) + \frac{1}{a}\mathbf{F}(\mathbf{\Psi} a\cos\phi) \tag{11}$$

where

$$\mathbf{L}(\Omega^2,\mathbf{\Psi};\phi) = \Omega^2\cos\phi\mathbf{M}\mathbf{\Psi}_{\phi\phi} - 2\Omega^2\sin\phi\mathbf{M}\mathbf{\Psi}_\phi + \frac{1}{2}(\Omega^2)_\phi\cos\phi\mathbf{M}\mathbf{\Psi}_\phi \tag{12}$$

and $(.)_\phi = \frac{\partial}{\partial\phi}(.)$. The differential rule $(\Omega^2)_\phi = 2\Omega\Omega_\phi$ has been used to work with the unknown function Ω^2 in place of Ω.

For fix a, (11) is a system of 2π periodic partial differential equations with respect to the two unknown functions $\mathbf{\Psi}$ and Ω. A well-posed problem can be obtained imposing, as usual, a normalization condition. As proposed in Bellizzi and Bouc (2005), the scalar equation can be used

$$\mathbf{\Psi}^T\mathbf{M}\mathbf{\Psi} = 1, \tag{13}$$

and it can be shown that for a well-defined periodic solution of equations (11) and (13), we always have

$$\Omega^2(a,\phi) > 0,\ \forall(a,\phi) \tag{14}$$

and Ω^2 can be expressed in terms of $\mathbf{\Psi}$ as

$$\Omega^2(a,\phi) = \frac{2}{a\sin^2\phi}\left(\int_0^\phi \exp(-\int_\sigma^\phi \beta(a,\nu)d\nu) I(a,\sigma)d\sigma\right) \tag{15}$$

where

$$I(a,\phi) = \sin\phi\mathbf{\Psi}^T(a,\phi)\mathbf{F}(\mathbf{\Psi}(a,\phi)a\cos\phi)$$

and

$$\beta(a,\phi) = 2\gamma^2(a,\phi)\frac{\cos\phi}{\sin\phi} \text{ with } \gamma^2(a,\phi) = \frac{\partial\mathbf{\Psi}^T(a,\phi)}{\partial\phi}\mathbf{M}\frac{\partial\mathbf{\Psi}(a,\phi)}{\partial\phi}.$$

Inequality (14) ensures the existence of a real positive resonance frequency function Ω and equation (15) reduces to $\Omega^2 = \mathbf{\Psi}^T\mathbf{K}\mathbf{\Psi}$ in the linear case.

Moreover, if we assume that the eigenvalues associated with the pair of matrices $(\mathbf{M}, \partial_X\mathbf{F}(\mathbf{0}))$ are distinct (and are all positive) then we can prove that for each a in some neighbourhood of $a = 0$, there exist n well-defined solutions to equations (11) and (13). Each solution is unique in some neighbourhood of $a = 0$. Consequently there exist n NNMs which can be viewed as a continuation of the n LNMs of the underlying linear system (1). For each NNM, the invariant set in the phase space is tangent to the vector subspace of dimension 2 characterizing to the associated linear mode of the underlying linear system.

2.4 Nonlinear autonomous case

In this case, the general formulation (2-3) has to be considered to define the NNM. Substituting equations (2) and (3) into equation (1), equation (1) reduces to the following partial differential equations in the variable ϕ and v

$$(\mathbf{X} + v\mathbf{X}_v)\,\xi + \mathbf{X}_\phi \Omega = \mathbf{Y} \tag{16}$$

$$\mathbf{M}\,(\mathbf{Y} + v\mathbf{Y}_v)\,\xi + \mathbf{M}\mathbf{Y}_\phi \Omega + \frac{1}{v}\mathbf{F}(v\mathbf{Y}, v\mathbf{X}) = \mathbf{0} \tag{17}$$

where $(.)_v = \frac{\partial}{\partial v}(.)$ and $(.)_\phi = \frac{\partial}{\partial \phi}(.)$.

As in the conservative case, a well-posed problem can be obtained imposing, as usual, normalization conditions. As proposed in Bellizzi and Bouc (2007), the following two scalar equations

$$\sin^2\phi {\mathbf{X}^{\text{oc}}}^T \mathbf{M}\mathbf{X}^{\text{oc}} + \cos^2\phi {\mathbf{X}^{\text{os}}}^T \mathbf{M}\mathbf{X}^{\text{os}} = \sin^2\phi\cos^2\phi, \tag{18}$$

$${\mathbf{X}^{\text{oc}}}^T \mathbf{M} {\mathbf{X}^{\text{os}}}^T \sin\phi\cos\phi = 0 \tag{19}$$

can be used where $\mathbf{X}^{\text{oc}}$ (respectively $\mathbf{X}^{\text{es}}$, $\mathbf{X}^{\text{os}}$, $\mathbf{X}^{\text{es}}$) denote the odd cosine (respectively even cosine, odd sinus, even sinus) terms in the Fourier series (with respect to the variable ϕ) of $\mathbf{X}$ (i.e. $\mathbf{X} = \mathbf{X}^{\text{oc}} + \mathbf{X}^{\text{ec}} + \mathbf{X}^{\text{os}} + \mathbf{X}^{\text{es}}$). The normalization condition (18) reduces to (13) whereas (19) is trivially satisfied in case of conservative systems. Moreover, noting that the functions $\mathbf{X}^{\text{oc}}$ and $\mathbf{X}^{\text{os}}$ can be factored as

$$\mathbf{X}^{\text{oc}}(v,\phi) = \mathbf{\Psi}^{\text{c}}(v,\phi)\cos\phi, \quad \mathbf{X}^{\text{os}}(v,\phi) = \mathbf{\Psi}^{\text{s}}(v,\phi)\sin\phi \tag{20}$$

where $\mathbf{\Psi}^{\text{c}}$ and $\mathbf{\Psi}^{\text{s}}$ are even, π-periodic functions with respect to ϕ, equations (18-19) reduce to

$${\mathbf{\Psi}^{\text{c}}}^T \mathbf{M}\mathbf{\Psi}^{\text{c}} + {\mathbf{\Psi}^{\text{s}}}^T \mathbf{M}\mathbf{\Psi}^{\text{s}} = 1$$

$${\mathbf{\Psi}^{\text{c}}}^T \mathbf{M}\mathbf{\Psi}^{\text{s}} = 0,$$

in the linear case (see section 2.2).

Finally, a NNM is obtained solving the equations (16-19) with respect to $\mathbf{X}$, $\mathbf{Y}$, Ω and ξ in a domain $[0, v_{max}] \times [0, 2\pi]$. A numerical approach based on Galerkin method is proposed in Bellizzi and Bouc (2007).

3 SOME APPILCATIONS

3.1 A simple example

We consider the nonlinear conservative system also treated in Pesheck et al. (2002)

$$\left\{ \begin{array}{lcl} \ddot{x}_1 + \omega_1^2 x_1 + 0.405x_1^3 + 1.34x_1^2x_2 + 1.51x_1x_2^2 + 0.349x_2^3 & = & 0 \\ \ddot{x}_2 + \omega_2^2 x_2 + 0.448x_1^3 + 1.51x_1^2x_2 + 1.05x_1x_2^2 + 4.580x_2^3 & = & 0 \end{array} \right. \tag{21}$$

where $\omega_1 = 0.689$ and $\omega_2 = 3.244$ denote the natural frequencies of the underlying linear system.

The NNM have been computed under the form (8) solving the 2π-periodic algebro-differential equations (11)(13) using the balance harmonic principle with the truncated expansions

$$\mathbf{\Psi}(a,\phi) \approx \sum_{k=0}^{N_\phi} \mathbf{\Psi}_k(a)\cos(2k)\phi \text{ and } \Omega(a,\phi) \approx \sum_{k=0}^{N_\phi} \Omega_k(a)\cos 2k\phi. \tag{22}$$

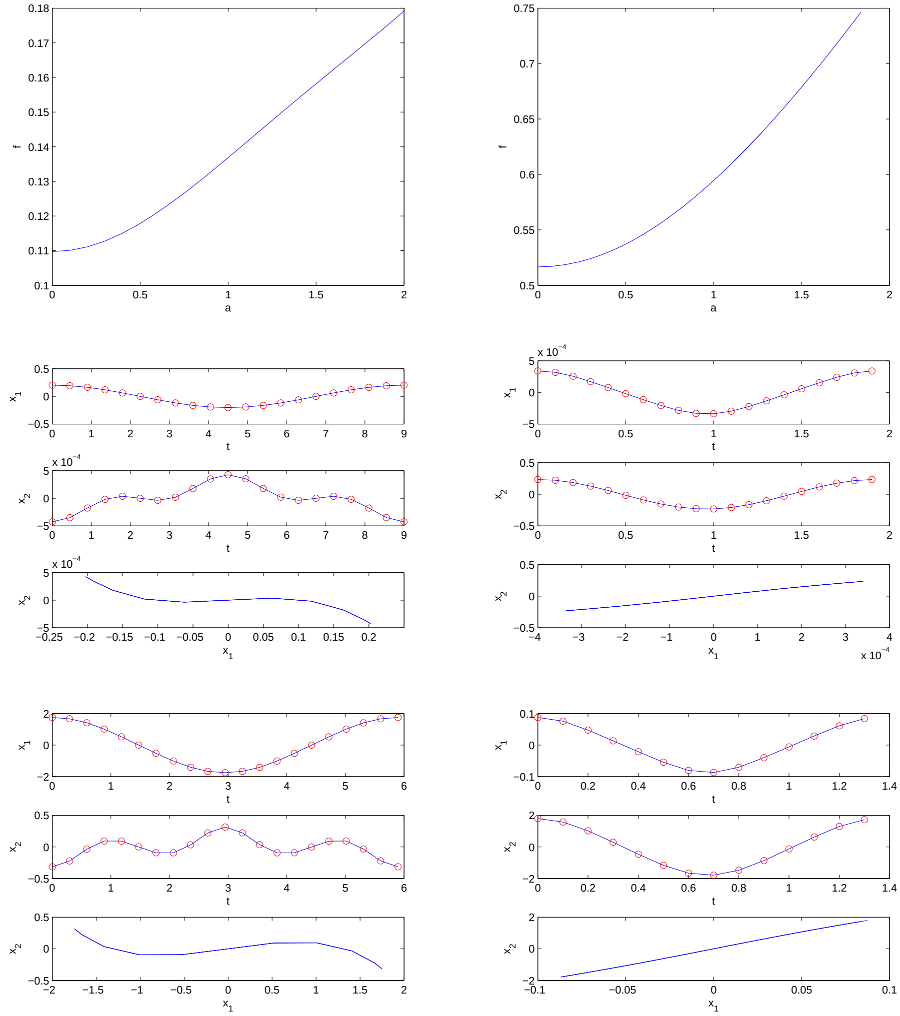
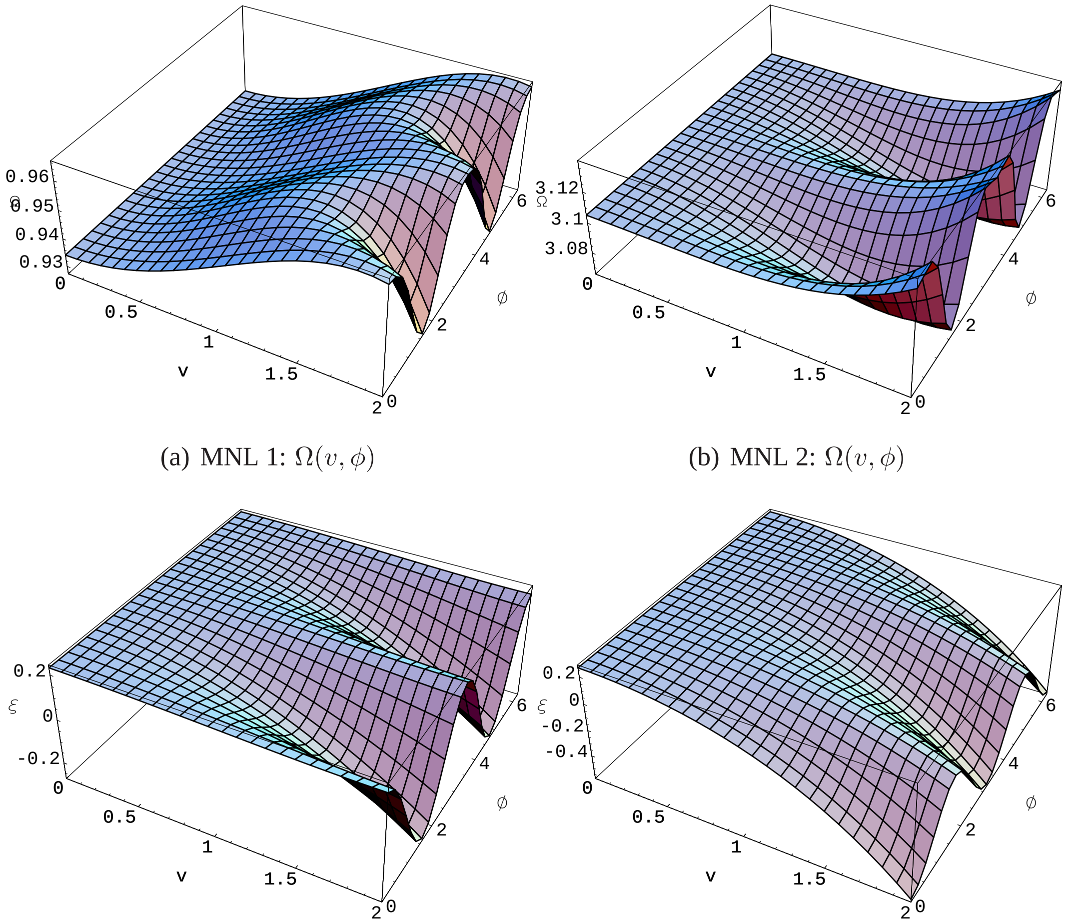


Figure 1: First (left colomn) and second (right colomn) NNM of system (21). Top row: Resonance frequency functions versus a. Middle row: modal motions with the initial condition $a = 0.25$ and $\varphi = 0$, NNM approach (continuous lines), direct simulation (dotted lines). Bottom row: modal motions with the initial condition $a = 1.8$ and $\varphi = 0$, NNM approach (continuous lines), direct simulation (dotted lines).

where $\mathbf{\Psi}_k(a)$ and $\Omega_k(a)$ denote the unknown coefficients. Finally, the resulting algebraic equations have been solved using the continuation method named Asymptotic Numerical Method Cochelin et al. (2007).

The behaviour of the two NNMs is illustrated in figure 1 where the approximations (22) have been obtained with $N_\phi = 4$. The backbone curves (top row in figure 1) show a hardening behaviour of the two NNMs. For each NNM, the backbone curve is defined as the evolution of the frequency $f = \frac{1}{T}$ versus the amplitude a where T denotes the period of the modal motions (see equation (13)). Modal motions are also plotted (middle and bottom rows) in figure 1. Modal motions have been obtained for each NNM solving equations (8) over one period given by equation (13) with two different initial values $a = 0.25, \varphi = 0$ (middle row) and $a = 1.8, \varphi = 0$ (bottom row). In order to check the validiy of the approximations, the reference solutions (with initial conditions given by equation (8) at $t = 0$) were computed by direct solving of equation (21) using a Runge-Kutta method (plotted in dotted lines). The results indicate good agreement. The nonlinearity does not affect the two NNM in the same way



(a) MNL 1: $\Omega(v,\phi)$ (b) MNL 2: $\Omega(v,\phi)$

(c) MNL 1: $\xi(v,\phi)$ (d) MNL 2: $\xi(v,\phi)$

Figure 2: Frequency ((a), (b)) damping ((c, (d)) functions characterizing the two MNLs of system (23).

in the configuration space. For the second NNM, the modal curves in the configuration space (phase subspace (X_1, X_2)) are straight whereas, for the first NNM, the modal curves are curved.

3.2 A self-oscillation system

We consider the two-coupled van der Pol oscillators

$$\left\{ \begin{array}{lcl} \ddot{q}_1 + c_{11}\dot{q}_1 + c_{12}\dot{q}_2 - \epsilon_1(1 - q_1^2 - \delta_1 q_2^2)\dot{q}_1 + q_1 & = & 0 \\ \ddot{q}_2 + c_{21}\dot{q}_1 + c_{22}\dot{q}_2 - \epsilon_2(1 - \delta_2 q_1^2 - q_2^2)\dot{q}_2 + 9q_2 & = & 0 \end{array} \right. \tag{23}$$

with $\epsilon_1 = \epsilon_2 = 0.5$, $\delta_1 = \delta_2 = 2.5$, $c_{11} = c_{22} = 0$ and $c_{12} = -c_{21} = 0.8$.

The NNM have been computed under the form (2-3) solving the 2π-periodic algebro-partial differential equations (16-19) using the balance principle over the domain $[0, v_{max}] \times [0, 2\pi]$ with the truncated expansions:

$$\left\{ \begin{array}{lcl} \mathbf{X}^{\text{ce}}(v,\phi) & \approx & \mathbf{X}^{\text{c}}_{\text{lin}}(\phi) + \sum_{p=1}^{N_v}\sum_{k=0}^{N_\phi} \mathbf{X}^{\text{c}}_{p,2k+1} v^p \cos(2k+1)\phi \\ \mathbf{X}^{co}(v,\phi) & \approx & \sum_{p=1}^{N_v}\sum_{k=0}^{N_\phi} \mathbf{X}^{\text{c}}_{p,2k} v^p \cos 2k\phi \\ \mathbf{X}^{\text{se}}(v,\phi) & \approx & \mathbf{X}^{\text{s}}_{\text{lin}}(\phi) + \sum_{p=1}^{N_v}\sum_{k=0}^{N_\phi} \mathbf{X}^{\text{s}}_{p,2k+1} v^p \sin(2k+1)\phi \\ \mathbf{X}^{so}(v,\phi) & \approx & \sum_{p=1}^{N_v}\sum_{k=1}^{N_\phi} \mathbf{X}^{\text{s}}_{p,2k} v^p \sin 2k\phi \end{array} \right. \tag{24}$$

$$\begin{cases} \mathbf{Y}^{\mathrm{ce}}(\upsilon,\phi) & \approx & \mathbf{Y}^{\mathrm{c}}_{\mathrm{lin}}(\phi) & + & \sum_{p=1}^{N_\upsilon}\sum_{k=0}^{N_\phi}\mathbf{Y}^{\mathrm{c}}_{p,2k+1}\upsilon^p\cos(2k+1)\phi \\ \mathbf{Y}^{co}(\upsilon,\phi) & \approx & & & \sum_{p=1}^{N_\upsilon}\sum_{k=0}^{N_\phi}\mathbf{Y}^{\mathrm{c}}_{p,2k}\upsilon^p\cos 2k\phi \\ \mathbf{Y}^{\mathrm{se}}(\upsilon,\phi) & \approx & \mathbf{Y}^{\mathrm{s}}_{\mathrm{lin}}(\phi) & + & \sum_{p=1}^{N_\upsilon}\sum_{k=0}^{N_\phi}\mathbf{Y}^{\mathrm{s}}_{p,2k+1}\upsilon^p\sin(2k+1)\phi \\ \mathbf{Y}^{so}(\upsilon,\phi) & \approx & & & \sum_{p=1}^{N_\upsilon}\sum_{k=1}^{N_\phi}\mathbf{Y}^{\mathrm{s}}_{p,2k}\upsilon^p\sin 2k\phi \end{cases} \tag{25}$$

$$\begin{cases} \Omega(\upsilon,\phi) & \approx & \Omega_{\mathrm{lin}} + \sum_{p=1}^{N_\upsilon}\sum_{k=0}^{N_\phi}\Omega_{p,2k}\upsilon^p\cos 2k\phi \\ \xi(\upsilon,\phi) & \approx & \xi_{\mathrm{lin}} + \sum_{p=1}^{N_\upsilon}\sum_{k=0}^{N_\phi}\xi_{p,2k}\upsilon^p\cos 2k\phi \end{cases} \tag{26}$$

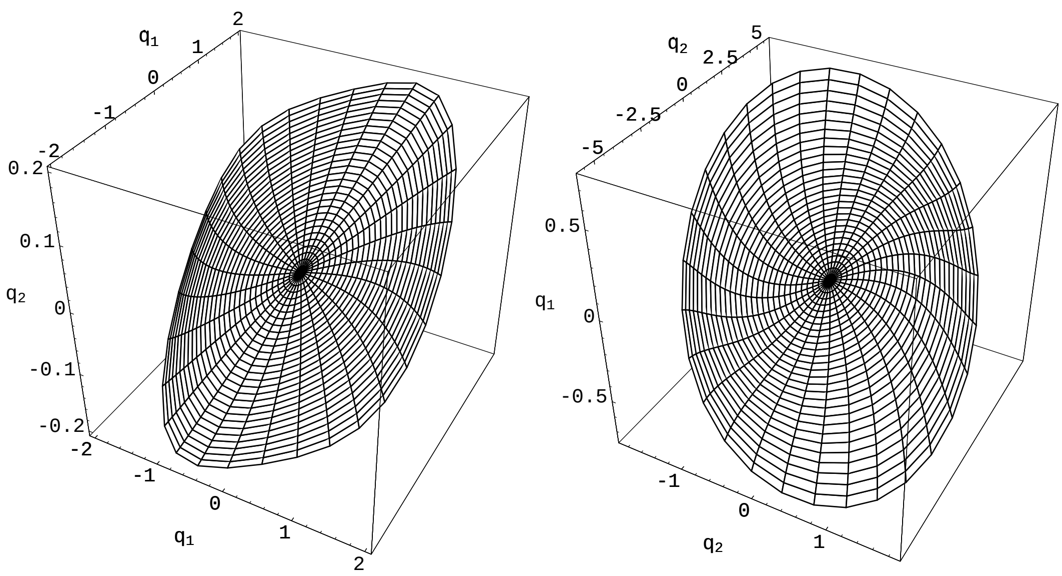
where $\mathbf{X}^{\mathrm{c}}_{p,2k+1}$, $\mathbf{X}^{\mathrm{c}}_{p,2k}, \cdots, \xi_{p,2k}$ denote the unknown coefficients and the terms $\mathbf{X}^{\mathrm{c}}_{\mathrm{lin}}$, $\mathbf{X}^{\mathrm{s}}_{\mathrm{lin}}$, $\mathbf{Y}^{\mathrm{c}}_{\mathrm{lin}}$, $\mathbf{Y}^{\mathrm{s}}_{\mathrm{lin}}$, Ω_{lin} and ξ_{lin} are selected from the modes of the associated linear system (see equations (7)). The complete procedure is described in Bellizzi and Bouc (2007)).

Figure 2 shows the frequency Ω and damping ξ functions characterizing the two NNMs obtained with $\upsilon_{max} = 2$, $N_\upsilon = 2$ and $N_\phi = 2$ and from respectively

- Linear mode 1: $\eta = 0.23$, $\omega = 0.93$, $\psi^c = (0.002, -0.092)^T$, $\psi^s = (0.995, 0.023)^T$
- Linear mode 2: $\eta = 0.27$, $\omega = 3.10$, $\psi^c = (0.022, 0.961)^T$, $\psi^s = (0.274, -0.006)^T$.

For each mode, the damping function ξ decreases with υ from positive to negative values.

Figure 3 shows the invariant manifold associated to each NNM.



(a) MNL 1 in the phase subspace$(Q_1, \dot{Q}_1, Q_2)$ (b) MNL 2 in the phase subspace $(Q_2, \dot{Q}_2, Q_1)$

Figure 3: Invariant manifold of the two MNLs of system (23).

The dynamic behaviour of the modal motions in the invariant manifold are governed by the two first-order differential equations (3). Hence, a periodic motion may occur on the invariant manifold if there exits a periodic solution to (3) or equivalently to

$$\frac{d\upsilon}{d\phi} = \upsilon\tau(\upsilon,\phi) \text{ where } \tau(\upsilon,\phi) = \frac{\xi(\upsilon,\phi)}{\Omega(\upsilon,\phi)}. \tag{27}$$

The existence of a periodic solution of equation (27) can be deduced from the existence of an equilibrium point in the associeated averaged equation (using the average principle in the context of perturbation theory, see Hale (1969))

$$\frac{dv}{d\phi} = v < \tau > (v) \text{ where } < \tau > (.) = \frac{1}{2\pi} \int_0^{2\pi} (\frac{\xi(.,\phi))}{\Omega(.,\phi))} d\phi. \tag{28}$$

The functions $< \tau >$ versus v are plotted in figure 4 for the two nonlinear modes. Equation (28) takes the equilibrium point $v_1^* = 1.89$ in the first nonlinear mode and $v_2^* = 1.45$ in the second nonlinear mode. Each equilibrium point characterizes an asymptotically stable ($\frac{d}{dv} < \tau > (v_i^*) < 0$) limit cycle on the associated invariant manifold. The approximations can be improved by solving equation (27). The harmonic balance method yields the following truncated expansion

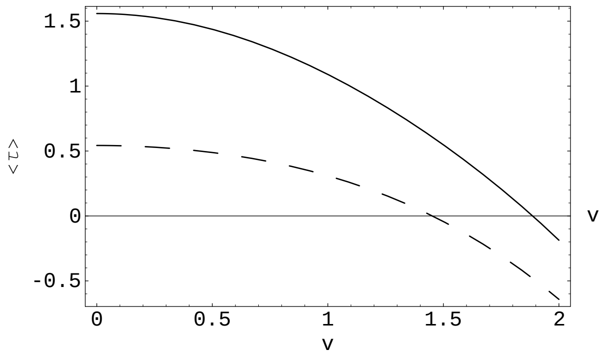


Figure 4: Evolution of $< \tau >$ versus v for the nonlinear modes 1 (continuous line) and 2 (dashed line).

$$\begin{aligned} v_1^*(\phi) &= 1.906 + 0.046 \cos 2\phi - 0.017 \cos 4\phi + 0.199 \sin 2\phi + 0.006 \sin 4\phi, \\ v_2^*(\phi) &= 1.453 + 0.003 \cos 2\phi - 0.0004 \cos 4\phi - 0.026 \sin 2\phi - 0.0001 \sin 4\phi. \end{aligned} \tag{29}$$

giving the limit cycle approximations (with period $T_1 = 6.58$ and $T_2 = 2.03$) for $i = 1, 2$

$$\begin{pmatrix} \mathbf{Q}(t) \\ \dot{\mathbf{Q}}(t) \end{pmatrix}_i = v_i^*(\phi(t)) \begin{pmatrix} \mathbf{X}_i(v_i^*(\phi(t)), \phi(t)) \\ \mathbf{Y}_i(v_i^*(\phi(t), \phi(t)) \end{pmatrix} \text{ with } \dot{\phi} = \Omega_i(v_i^*(\phi), \phi), \ \phi(0) = \varphi. \tag{30}$$

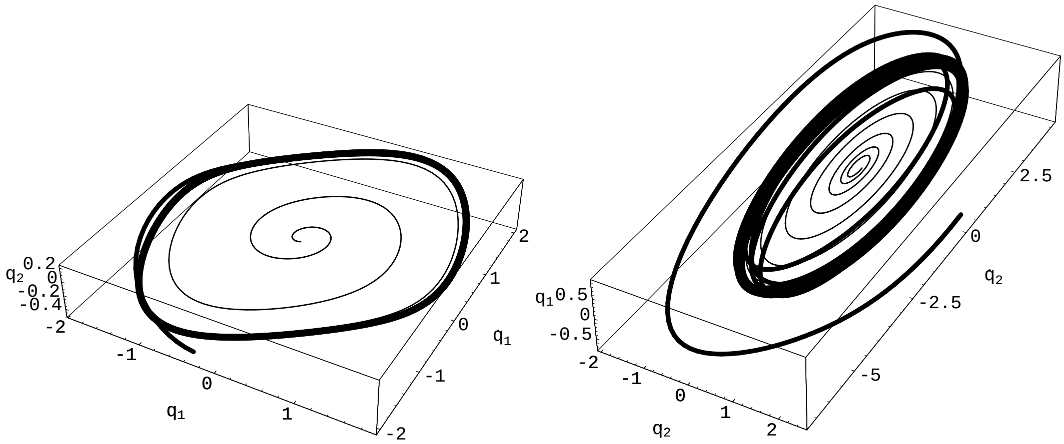
The last question, which now arises, is whether or not the limit cycles are stable in the phase space. To answer this question Floquet's theory (see Hale (1969)) is applied. Rewriting equation (23) in the first order autonomous differential system

$$\frac{d}{dt}\mathbf{Z}(t) = \mathbf{G}(\mathbf{Z}(t)) \text{ with } \mathbf{Z} = \left(\mathbf{Q}^T, \dot{\mathbf{Q}}^T\right)^T$$

the stability of the periodic solutions (30) can be deduced from the eigenvalues of the monodromy matrix associated with the fundamental matrix solution of the 2π-periodic variational linear differential system (see Hale (1969) page 119)

$$\frac{d}{d\phi}\mathbf{\Delta Z}(\phi) = \frac{1}{\Omega_i(v_i^*(\phi), \phi)} [\partial \mathbf{G}_Z(\mathbf{Z}_i(\phi))] \mathbf{\Delta Z}(\phi). \tag{31}$$

The monodromy matrix is computed over one period, using the four canonical basis vectors as initial conditions successively. The computations show that the periodic orbit (v_1^*) associated with the first nonlinear mode is stable on its invariant manifold and unstable in the phase space (two complex conjugate multipliers are outside the unit circle), whereas the periodic orbit (v_2^*) associated with the second nonlinear mode is stable on its invariant manifold, as well as in the phase space (one multiplier lies on the unit circle and all the others are located inside the unit circle). The local and global stability are illustrated in figure 5 by solving numerically equation (23). Motions are plotted with initial conditions (near the periodic orbit) on the invariant manifold and outside the invariant manifold for the first NNM , figure 5(a), and the second NNM, figure 5(b).



(a) MNL 1 in the phase subspace $(Q_1, \dot{Q}_1, Q_2)$ (b) MNL 2 in the phase subspace $(Q_2, \dot{Q}_2, Q_1)$

Figure 5: Illustration of the local and global stability of the limit cycle associated with the first (a) and the second (b) nonlinear mode. Motions obtained by solving (23) with initial conditions inside the invariant manifold (continuous line) and outside the invariant manifold (bold continuous line).

4 CONCLUSION

An amplitude-phase transformation procedure was described here for characterizing the NNMs of the nonlinear mechanical systems in the framework of invariant manifold theory. This formulation is a natural extension of the linear tools and gives an interpretation of the NNM in terms of frequency, damping and mode shapes. A NNM can also be viewed as a familly of motions (damped motions, periodic orbits,...).

In addition to introducing the concept of NNMs, we have shown that the NNM can provide a valuable theoretical tool for understanding some nonlinear phenomena. For example, bifurcation analysis can be performed, and the existence and stability of periodic orbits on the associated invariant manifold can be studied from the two first-order coupled differential equations governing the amplitude and phase variables, which describe the dynamics.

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