Asociación Argentina de Mecánica Computacional

Mecánica Computacional Vol XXVII, págs. 2221-2239 (artículo completo)
Alberto Cardona, Mario Storti, Carlos Zuppa. (Eds.)
San Luis, Argentina, 10-13 Noviembre 2008

NUMERICAL APPROXIMATION OF EIGENVALUE PROBLEMS BY ADAPTIVE FINITE ELEMENT METHODS

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Keywords: Eigenvalue problems, adaptive finite elements, convergence, optimality, regularity.

Abstract. In this article we present an algorithm for the approximation through adaptive finite element methods of solutions to second order elliptic eigenvalue problems, considering Lagrange finite elements of any degree. We show the convergence of the algorithm for simple as well as multiple eigenvalues under a minimal refinement of marked elements, for all *reasonable* marking strategies, and starting from any initial triangulation. Finally, we discuss briefly the quasi-optimality of the adaptive method and conclude with some numerical experiments that illustrate the advantages of adaptivity and the relationship between order of convergence and regularity.

1 INTRODUCTION

In many practical applications it is of interest to find or approximate the eigenvalues and eigenfunctions of elliptic problems. Finite element approximations for these problems have been widely used and analyzed under a general framework. Optimal a priori error estimates for the eigenvalues and eigenfunctions have been obtained (see Babuška and Osborn (1991, 1989); Raviart and Thomas (1983); Strang and Fix (1973) and the references therein).

Adaptive finite element methods are an effective tool for making an efficient use of the computational resources; for certain problems, it is even indispensable to their numerical resolvability. A quite popular, natural adaptive version of classical finite element methods consists of the loop

$$\textsc{Solve} \to \textsc{Estimate} \to \textsc{Mark} \to \textsc{Refine},$$

that is: solve for the finite element solution on the current grid, compute the a posteriori error estimator, mark with its help elements to be subdivided, and refine the current grid into a new, finer one. The ultimate goal of adaptive methods is to equidistribute the error and the computational effort obtaining a sequence of meshes with optimal complexity. A general result of convergence for linear problems has been obtained in Morin et al. (2008), where very general conditions on the linear problems and the adaptive methods that guarantee convergence are stated. Optimality for adaptive methods using Dörfler's marking strategy (Dörfler, 1996) has been proved in Cascon et al. (2008); Stevenson (2007) for linear problems.

In this article we propose a convergent algorithm using adaptive finite element methods for approximating the eigenvalue problem consisting in finding $\lambda \in \mathbb{R}$, and $u \not\equiv 0$ such that

$$-\nabla \cdot (\mathcal{A}\nabla u) = \lambda \mathcal{B} u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

under general assumptions on $\mathcal{A}$, $\mathcal{B}$ and Ω that we state precisely in Section 2.1.

As we mentioned before, adaptive methods are based on a posteriori error estimators, that are computable quantities depending on the discrete solution and data, and indicate a distribution of the error. A posteriori error estimators for eigenvalue problems have been constructed by using different approaches in Verfürth (1996); Verfürth (1994); Durán et al. (2003); Larson (2000), they have been developed for $\mathcal{A} \equiv I$ and $\mathcal{B} \equiv 1$, but the same proofs can be carried over to the general case considered here; see Giani and Graham (2007). An important aspect to be mentioned here is that the a posteriori error estimators are reliable only if the underlying mesh is sufficiently fine.

Before proceeding with the details of the statement we note some properties of our adaptive algorithm:

- It does not require that the initial mesh $\mathcal{T}_0$ is fine enough. Any initial mesh that captures the discontinuities of $\mathcal{A}$ will guarantee convergence.
- It is possible to use any of the popular marking strategies, not only Dörfler's (Dörfler, 1996). The only assumption is that non-marked elements have error estimators smaller than marked ones, see condition (3.2) in Section 3 below. If the marking is done according to Dörfler's strategy, then the resulting meshes are quasi-optimal.
- The marking is done according to the residual type a posteriori error estimators presented in Section 2.3. Even though there are some *oscillation terms* in the efficiency of the estimators, we do not require any marking due to these terms. We only need to mark according to the error estimators, which is what is usually done in practice.

- The result holds with a minimal refinement of marked elements, one bisection suffices. We do not require the enforcement of the so-called *interior node property*.

The rest of the article is organized as follows. In Section 2 we state precisely the problem that we study, describe the approximants and mention some already known results about a priori and a posteriori estimation. In Section 3 we state the adaptive loop. In Section 4 we state briefly the convergence of the adaptive algorithm. In Section 5 we state the quasi-optimality of the adaptive process and finally, in Section 6 we explore the performance of the adaptive methods through the computation of the first and second eigenpair of the Laplacian on a domain consisting of three quarters of a circle.

2 PROBLEM STATEMENT AND NUMERICAL APPROXIMATION

In this section we state precisely the continuous problem that we study and the discrete problems that we consider as approximants to the continuous one. Furthermore, we will define a posteriori error estimators and mention their reliability and efficiency.

2.1 General setting

We consider the general eigenvalue problem consisting in finding $\lambda \in \mathbb{R}$ and $u \not\equiv 0$ such that

$$-\nabla \cdot (\mathcal{A}\nabla u) = \lambda \mathcal{B} u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^d$ is a bounded open set with a Lipschitz boundary. In particular, we suppose that Ω is a polygonal domain if $d = 2$ and a polyhedral domain if $d = 3$. Here, $\mathcal{A}$ is a piecewise Lipschitz symmetric-matrix-valued function which is uniformly positive definite, i.e., there exist constants $a_1, a_2 > 0$ such that

$$a_1|\xi|^2 \leq \mathcal{A}(x)\xi \cdot \xi \leq a_2|\xi|^2, \qquad \forall\, \xi \in \mathbb{R}^d, \qquad \forall\, x \in \Omega,$$

and $\mathcal{B}$ is a scalar function such that

$$b_1 \leq \mathcal{B}(x) \leq b_2, \qquad \forall\, x \in \Omega,$$

for some constants $b_1, b_2 > 0$.

In order to state the variational formulation of this problem we introduce the following functional spaces. If $A \subset \Omega$, we denote by $L^2(A)$ the space of the square integrable functions on A with the norm

$$\|v\|_A := \left(\int_A |v|^2 \right)^{1/2},$$

and by $H^1(A)$ the Sobolev space consisting in functions in $L^2(A)$ whose first order weak derivatives are also in $L^2(A)$, with the norm

$$\|v\|_{H^1(A)} := \left(\|v\|_A^2 + \|\nabla v\|_A^2 \right)^{1/2}.$$

Finally, $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ of functions which vanish on $\partial\Omega$.

We consider the bilinear forms $a, b : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ given by

$$a(u,v) := \int_\Omega \mathcal{A}\nabla u \cdot \nabla v, \qquad \text{and} \qquad b(u,v) := \int_\Omega \mathcal{B} uv,$$

and their induced norms

$$\|v\|_a := a(v,v)^{1/2}, \quad v \in H_0^1(\Omega), \qquad \text{and} \qquad \|v\|_b := b(v,v)^{1/2}, \quad v \in L^2(\Omega).$$

By the assumptions on $\mathcal{A}$ and $\mathcal{B}$, there exist positive constants c_1, c_2, c_3, c_4 such that

$$c_1\|v\|_{H_0^1(\Omega)} \leq \|v\|_a \leq c_2\|v\|_{H_0^1(\Omega)}, \qquad \forall\, v \in H_0^1(\Omega),$$

and

$$c_3\|v\|_\Omega \leq \|v\|_b \leq c_4\|v\|_\Omega, \qquad \forall\, v \in L^2(\Omega).$$

Now, the weak formulation of the problem is

Continuous eigenvalue problem. Find $\lambda \in \mathbb{R}$ and $u \in H_0^1(\Omega)$ satisfying

$$\begin{cases} a(u,v) = \lambda\, b(u,v), & \forall\, v \in H_0^1(\Omega), \\ \|u\|_b = 1. \end{cases} \tag{2.1}$$

It is well known (Babuška and Osborn, 1991) that under our assumptions on $\mathcal{A}$ and $\mathcal{B}$ problem (2.1) has a countable sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \nearrow \infty$$

and corresponding eigenfunctions

$$u_1, u_2, u_3, \ldots$$

which can be assumed to satisfy

$$b(u_i,u_j) = \delta_{ij} := \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$$

where in the sequence $\{\lambda_j\}_{j\in\mathbb{N}}$, the λ_j are repeated according to geometric multiplicity.

For each fixed eigenvalue λ of (2.1) we define

$$M(\lambda) := \{u \in H_0^1(\Omega) \mid u \text{ satisfies (2.1)}\},$$

and notice that if λ is simple, then $M(\lambda)$ contains two functions, whereas if λ is not simple, it consists of a sphere in the subspace generated by the eigenfunctions. Furthermore, we have that the eigenfunctions u of the problem (2.1) belong to $H^{1+r}(\Omega)$, for some $r \in (0,1]$ depending only on Ω and $\mathcal{A}$.

2.2 Discrete problem

In order to define the numerical approximations to the continuous problem (2.1) we will consider finite element spaces defined over *triangulations* of the domain Ω. Let $\mathcal{T}_0$ be an initial conforming triangulation of Ω, that is, a partition of Ω into d-simplices such that if two elements intersect, they do so at a vertex or a full edge/face of both elements. Let $\mathbb{T}$ denote the set of all conforming triangulations of Ω obtained from $\mathcal{T}_0$ by refinement using the *newest vertex* bisection procedure in two dimensions and the bisection procedure of Kossaczký in three dimensions (Schmidt and Siebert, 2005).

For any triangulation $\mathcal{T} \in \mathbb{T}$, $\mathcal{S}$ will denote the set of interior sides, where by side we mean an edge if $d = 2$ and a face if $d = 3$. And $\kappa_{\mathcal{T}}$ will denote the regularity of $\mathcal{T}$, defined as

$$\kappa_{\mathcal{T}} := \max_{T \in \mathcal{T}} \frac{\operatorname{diam}(T)}{\rho_T},$$

where $\operatorname{diam}(T)$ is the length of the longest edge of T, and ρ_T is the radius of the largest ball contained in it. It is also useful to define the meshsize $h_{\mathcal{T}} := \max_{T \in \mathcal{T}} h_T$, where $h_T := |T|^{1/d}$.

Due to the refinement procedures considered here, the family of triangulations $\mathbb{T}$ is shape regular, i.e.,

$$\sup_{\mathcal{T} \in \mathbb{T}} \kappa_{\mathcal{T}} < \infty,$$

where this uniform constant only depends on the initial triangulation $\mathcal{T}_0$.

For each interior side $S \in \mathcal{S}$ we define $\omega_{\mathcal{T}}(S)$ as the union of the two elements in $\mathcal{T}$ sharing S. For $T \in \mathcal{T}$, $\mathcal{N}_{\mathcal{T}}(T)$ denotes the set of neighbors of T in $\mathcal{T}$, i.e., the subset of $\mathcal{T}$ consisting of the elements which share at least a vertex with T, and $\omega_{\mathcal{T}}(T)$ denotes the neighborhood of T, i.e., the union of the neighbors of T.

Let $\ell \in \mathbb{N}$ be fixed, and let $\mathbb{V}_{\mathcal{T}}$ be the finite element space consisting of continuous functions vanishing on $\partial\Omega$ which are polynomials of degree $\leq \ell$ in each element of $\mathcal{T}$, i.e,

$$\mathbb{V}_{\mathcal{T}} := \{v \in H_0^1(\Omega) \mid v_{|_T} \in \mathcal{P}_\ell(T), \quad \forall\, T \in \mathcal{T}\}.$$

Thus, we use a conforming and nested approximation because $\mathbb{V}_{\mathcal{T}} \subset \mathbb{V}_{\mathcal{T}_*} \subset H_0^1(\Omega)$ whenever $\mathcal{T}_*$ is a refinement of $\mathcal{T}$.

We are now in a position to define the

Discrete eigenvalue problem. Find $\lambda_{\mathcal{T}} \in \mathbb{R}$ and $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$ such that

$$\begin{cases} a(u_{\mathcal{T}}, v) = \lambda_{\mathcal{T}}\, b(u_{\mathcal{T}}, v), & \forall\, v \in \mathbb{V}_{\mathcal{T}}, \\ \|u_{\mathcal{T}}\|_b = 1. \end{cases} \tag{2.2}$$

If $\{\phi_1, \phi_2, \ldots, \phi_{N_{\mathcal{T}}}\}$ is a basis for $\mathbb{V}_{\mathcal{T}}$ and $K := (a(\phi_j, \phi_i))$, and $M := (b(\phi_j, \phi_i))$ denote the stiffness and mass matrices, respectively, the corresponding linear system is

$$K\mathbf{U} = \lambda_{\mathcal{T}} M\mathbf{U},$$

where $\mathbf{U} := (\mathbf{U_i})_{i=1}^{N_{\mathcal{T}}}$ is the coefficient vector defining $u_{\mathcal{T}}$, i.e. $u_{\mathcal{T}} = \sum_{i=1}^{N_{\mathcal{T}}} \mathbf{U_i}\phi_i$. Since K and M are symmetric and positive definite, the problem (2.2) has a finite sequence of eigenvalues

$$0 < \lambda_{1,\mathcal{T}} \leq \lambda_{2,\mathcal{T}} \leq \lambda_{3,\mathcal{T}} \leq \ldots \leq \lambda_{N_{\mathcal{T}},\mathcal{T}},$$

and corresponding eigenfunctions

$$u_{1,\mathcal{T}}, u_{2,\mathcal{T}}, u_{3,\mathcal{T}}, \ldots, u_{N_{\mathcal{T}},\mathcal{T}},$$

which can be assumed to satisfy

$$b(u_{i,\mathcal{T}}, u_{j,\mathcal{T}}) = \delta_{ij}.$$

For $j = 1, 2, \ldots, N_{\mathcal{T}}$, it follows from the minimum-maximum principles that $\lambda_j \leq \lambda_{j,\mathcal{T}}$, and it also follows that if $\mathcal{T}_*$ is any refinement of $\mathcal{T}$ then $\lambda_{j,\mathcal{T}_*} \leq \lambda_{j,\mathcal{T}}$. Furthermore, for $j \in \mathbb{N}$ we have that

$$\lambda_{j,\mathcal{T}} \longrightarrow \lambda_j, \qquad \text{as} \quad h_{\mathcal{T}} \longrightarrow 0. \tag{2.3}$$

2.3 A posteriori error estimators

A posteriori estimates for eigenvalue problems have been developed by Larson (2000), Durán et al. (2003), Giani and Graham (2007). Next, we present the residual type a posteriori estimates for eigenvalue problems that they have developed and state some of their properties. In order to define the estimators we assume that the triangulation $\mathcal{T}$ matches the discontinuities of $\mathcal{A}$. More precisely, we assume that the discontinuities of $\mathcal{A}$ are aligned with the sides of $\mathcal{T}$. Observe that in particular, $\mathcal{A}_{|_T}$ is Lipschitz continuous for all $T \in \mathcal{T}$.

For $\mu \in \mathbb{R}$ and $v \in \mathbb{V}_\mathcal{T}$ we define the *element residual* $R(\mu, v)$ by

$$R(\mu, v)_{|_T} := -\nabla \cdot (\mathcal{A}\nabla v) - \mu\mathcal{B}v, \tag{2.4}$$

for all $T \in \mathcal{T}$, and the *jump residual* $J(v)$ by

$$J(v)_{|_S} := (\mathcal{A}\nabla v)_{|_{T_1}} \cdot \overrightarrow{n_1} + (\mathcal{A}\nabla v)_{|_{T_2}} \cdot \overrightarrow{n_2}, \tag{2.5}$$

for every interior side $S \in \mathcal{S}$, where T_1 and T_2 are the elements in $\mathcal{T}$ which share S and $\overrightarrow{n_i}$ is the outward normal unit vector of T_i on S, for $i = 1, 2$. We define $J(v)_{|_{\partial\Omega}} := 0$.

The *local error estimator* $\eta_\mathcal{T}(\mu, v; T)$ is given by

$$\eta_\mathcal{T}(\mu, v; T)^2 := h_T^2 \left\| R(\mu, v) \right\|_T^2 + h_T \left\| J(v) \right\|_{\partial T}^2,$$

for all $T \in \mathcal{T}$, and the *global error estimator* $\eta_\mathcal{T}(\mu, v)$ by

$$\eta_\mathcal{T}(\mu, v)^2 := \sum_{T \in \mathcal{T}} \eta_\mathcal{T}(\mu, v; T)^2.$$

The *local oscillation term* $\operatorname{osc}_\mathcal{T}(\mu, v; T)$ is given by

$$\operatorname{osc}_\mathcal{T}(\mu, v; T)^2 := h_T^2 \left\| R - \overline{R} \right\|_{\omega_\mathcal{T}(T)}^2 + h_T \left\| J - \overline{J} \right\|_{\partial T}^2,$$

for all $T \in \mathcal{T}$, where, for every $T' \in \mathcal{N}_\mathcal{T}(T)$, $\overline{R}_{|_{T'}}$ is the $L^2(T')$-projection of $R := R(\mu, v)$ onto $\mathcal{P}_\ell$, and for every side $S \subset \partial T$, $\overline{J}_{|S}$ is the $L^2(S)$-projection of $J := J(v)$ onto $\mathcal{P}_\ell$. The *global oscillation term* $\operatorname{osc}_\mathcal{T}(\mu, v)$ is given by

$$\operatorname{osc}_\mathcal{T}(\mu, v)^2 := \sum_{T \in \mathcal{T}} \operatorname{osc}_\mathcal{T}(\mu, v; T)^2.$$

We have that the global a posteriori error estimator defined above is reliable if the meshsize is small enough.

Reliability of the global error estimator. Let $j \in \mathbb{N}$, and let $u_\mathcal{T}$ be an eigenfunction corresponding to the j-th eigenvalue $\lambda_\mathcal{T}$ of the discrete problem (2.2), then, if $h_\mathcal{T}$ is small enough, there exists an eigenfunction u corresponding to the j-th eigenvalue λ of the continuous problem (2.1) such that

$$\left\| u - u_\mathcal{T} \right\|_a \leq C_U \eta_\mathcal{T}(\lambda_\mathcal{T}, u_\mathcal{T}),$$

where C_U is a constant depending on the data, but not on u or the meshsize $h_\mathcal{T}$.

When the global oscillation term is small, the global a posteriori error estimator is an efficient indicator of the error in the sense that a big estimator implies a big error.

Efficiency of the global error estimator. Let $j \in \mathbb{N}$. Let $u_{\mathcal{T}}$ be an eigenfunction corresponding to the j-th eigenvalue $\lambda_{\mathcal{T}}$ of the discrete problem (2.2) and let u be an eigenfunction corresponding to the j-th eigenvalue λ of the continuous problem (2.1). Then, there holds

$$C_L \eta_{\mathcal{T}}(\lambda_{\mathcal{T}}, u_{\mathcal{T}}) \leq \|u - u_{\mathcal{T}}\|_a + \operatorname{osc}_{\mathcal{T}}(\lambda_{\mathcal{T}}, u_{\mathcal{T}}),$$

where C_L is a constant depending on the data, but not on u or the meshsize $h_{\mathcal{T}}$.

We define $n_d := 3$ if $d = 2$ and $n_d := 6$ if $d = 3$. This guarantees that after n_d bisections to an element, new nodes appear on each side and in the interior.

Discrete local efficiency. Let $T \in \mathcal{T}$ and let $\mathcal{T}_*$ be the triangulation of Ω which is obtained from $\mathcal{T}$ by bisecting n_d times each element of $\mathcal{N}_{\mathcal{T}}(T)$. Let $\lambda_{\mathcal{T}}$ and $u_{\mathcal{T}}$ be a solution to the discrete problem (2.2). Let $\mathbb{W}$ be a subspace of $H_0^1(\Omega)$ such that $\mathbb{V}_{\mathcal{T}_*} \subset \mathbb{W}$. If $\mu \in \mathbb{R}$ and $w \in \mathbb{W}$ satisfy

$$\begin{cases} a(w, v) = \mu\, b(w, v), & \forall\, v \in \mathbb{W}, \\ \|w\|_b = 1, \end{cases}$$

then[1]

$$\eta_{\mathcal{T}}(\lambda_{\mathcal{T}}, u_{\mathcal{T}}; T) \lesssim \|\nabla(w - u_{\mathcal{T}})\|_{\omega_{\mathcal{T}}(T)} + h_T \|\mu w - \lambda_{\mathcal{T}} u_{\mathcal{T}}\|_{\omega_{\mathcal{T}}(T)} + \operatorname{osc}_{\mathcal{T}}(\lambda_{\mathcal{T}}, u_{\mathcal{T}}; T).$$

Considering that for the oscillation term we have that

$$\operatorname{osc}_{\mathcal{T}}(\lambda_{\mathcal{T}}, u_{\mathcal{T}}; T) \lesssim h_T(2 + \lambda_{\mathcal{T}})\|u_{\mathcal{T}}\|_{H^1(\omega_{\mathcal{T}}(T))},$$

the discrete local efficiency implies

$$\eta_{\mathcal{T}}(\lambda_{\mathcal{T}}, u_{\mathcal{T}}; T) \lesssim \|\nabla(w - u_{\mathcal{T}})\|_{\omega_{\mathcal{T}}(T)} + h_T \|\mu w\|_{\omega_{\mathcal{T}}(T)} + h_T(1 + \lambda_{\mathcal{T}})\|u_{\mathcal{T}}\|_{H^1(\omega_{\mathcal{T}}(T))}. \quad (2.6)$$

3 ADAPTIVE LOOP

Now, we describe the adaptive method to approximate the j-th eigenvalue and one of its eigenfunctions. From now on, we keep $j \in \mathbb{N}$ fixed, and let λ denote the j-th eigenvalue of (2.1) and u an eigenfunction in $M(\lambda)$.

The algorithm for approximating λ and $M(\lambda)$ is an iteration of the following main steps:

(1) $(\lambda_k, u_k) := \mathsf{SOLVE}(\mathbb{V}_k)$.

(2) $\{\eta_k(T)\}_{T \in \mathcal{T}_k} := \mathsf{ESTIMATE}(\lambda_k, u_k, \mathcal{T}_k)$.

(3) $\mathcal{M}_k := \mathsf{MARK}(\{\eta_k(T)\}_{T \in \mathcal{T}_k}, \mathcal{T}_k)$.

(4) $\mathcal{T}_{k+1} := \mathsf{REFINE}(\mathcal{T}_k, \mathcal{M}_k)$, increment k.

This is the same loop considered in Morin et al. (2008), the difference lies in the building blocks which we now describe in detail.

If $\mathcal{T}_k$ is a conforming triangulation of Ω, the module SOLVE takes the space $\mathbb{V}_k := \mathbb{V}_{\mathcal{T}_k}$ as input argument and outputs the j-th eigenvalue of the discrete problem (2.2) with $\mathcal{T} = \mathcal{T}_k$, i.e., $\lambda_k := \lambda_{j,\mathcal{T}_k}$, and a corresponding eigenfunction $u_k \in \mathbb{V}_k$. Therefore, λ_k and u_k satisfy

$$\begin{cases} a(u_k, v_k) = \lambda_k\, b(u_k, v_k), & \forall\, v_k \in \mathbb{V}_k, \\ \|u_k\|_b = 1. \end{cases} \quad (3.1)$$

[1] From now on, whenever we write $A \lesssim B$ we mean that $A \leq CB$ with a constant C that may depend on $\mathcal{A}$, $\mathcal{B}$, the domain Ω and the regularity $\kappa_{\mathcal{T}}$ of $\mathcal{T}$, but not on other properties of $\mathcal{T}$ such as element size or uniformity.

It is worth mentioning at this point that *any* algorithm for computing eigenvalues of discrete problems is able to produce such an output (λ_k, u_k), and thus our assumptions are very practical. In contrast, the a priori error estimates on eigenvalue problems state that, given an exact eigenfunction u, there exists a discrete eigenfunction $u_{\mathcal{T}}$ which satisfies certain estimates. This kind of statement is far from the philosophy of a posteriori estimation and adaptivity, where an estimation or convergence result about the *computed* discrete solutions is sought.

Given $\mathcal{T}_k$ and the corresponding outputs λ_k and u_k of SOLVE, the module ESTIMATE computes and outputs the a posteriori error estimators $\{\eta_k(T)\}_{T\in\mathcal{T}_k}$, where

$$\eta_k(T) := \eta_{\mathcal{T}_k}(\lambda_k, u_k; T).$$

Based upon the a posteriori error indicators $\{\eta_k(T)\}_{T\in\mathcal{T}_k}$, the module MARK collects elements of $\mathcal{T}_k$ in $\mathcal{M}_k$. The only requirement that we make on the module MARK is that the set of marked elements $\mathcal{M}_k$ contains at least one element of $\mathcal{T}_k$ holding the largest value of estimator. That is, there exists one element $T_k^{\max} \in \mathcal{M}_k$ such that

$$\eta_k(T_k^{\max}) = \max_{T\in\mathcal{T}_k} \eta_k(T).$$

Whenever a marking strategy satisfies this assumption, we call it *reasonable*, since this is what practitioners do in order to maximize the error reduction with a minimum effort. The most commonly used marking strategies, e.g., *Maximum strategy* and *Equidistribution strategy*, fulfill this condition, which is sufficient to guarantee that

$$T \in \mathcal{T}_k \setminus \mathcal{M}_k \qquad \Longrightarrow \qquad \eta_k(T) \lesssim \eta_k(\mathcal{M}_k) := \left(\sum_{T\in\mathcal{M}_k} \eta_k(T)^2 \right)^{1/2}. \tag{3.2}$$

This condition is slightly weaker, and will be sufficient to guarantee the convergence of this algorithm. The original *Dörfler's strategy* (Dörfler, 1996) also satisfies (3.2).

The refinement procedure REFINE takes the triangulation $\mathcal{T}_k$ and the subset $\mathcal{M}_k \subset \mathcal{T}_k$ as input arguments. We require that all elements of $\mathcal{M}_k$ are refined (at least once), and that a new conforming triangulation $\mathcal{T}_{k+1}$ of Ω, which is a refinement of $\mathcal{T}_k$, is returned as output.

In this way, starting with an initial conforming triangulation $\mathcal{T}_0$ of Ω and iterating the steps (1)–(4) of this algorithm, we obtain a sequence of successive conforming refinements of $\mathcal{T}_0$ called $\mathcal{T}_1, \mathcal{T}_2, \ldots$ and the corresponding outputs (λ_k, u_k), $\{\eta_k(T)\}_{T\in\mathcal{T}_k}$, $\mathcal{M}_k$ of the modules SOLVE, ESTIMATE and MARK, respectively.

For simplicity, we consider for the module REFINE, the concrete choice of the *newest vertex* bisection procedure in two dimensions and the bisection procedure of Kossaczký in three dimensions (Schmidt and Siebert, 2005). As we have mentioned before, both these procedures refine the marked elements and some additional ones in order to keep conformity, and they also guarantee that

$$\kappa := \sup_{k\in\mathbb{N}_0} \kappa_{\mathcal{T}_k} < \infty,$$

i.e., $\{\mathcal{T}_k\}_{k\in\mathbb{N}_0}$ is a sequence shape regular of triangulations of Ω.

We stress that the marking in the module MARK, is done only according to the error estimators; no marking due to oscillation is necessary. It is also worth mentioning that we do not assume REFINE to enforce the so-called *interior node property*, and convergence is guaranteed nevertheless.

4 CONVERGENCE OF THE ADAPTIVE LOOP

In this section we analyze the convergence of the adaptive loop described in the last section. Following similar ideas to those of Morin et al. (2008), with some modifications due to the different nature of the problem, it is possible to prove the convergence; see Garau et al. (2008a) for details. It consists in proving the following steps:

- The full sequence of discrete eigenvalues converges to a number λ_∞ and a subsequence of the discrete eigenfunctions converges to some function u_∞.
- The global a posteriori error estimator converges to zero (for the subsequence).
- The pair $(\lambda_\infty, u_\infty)$ is an eigenpair of the continuous problem.
- The full sequence of the discrete eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}_0}$ converges to a eigenvalue λ and the full sequence of the discrete eigenfunctions $\{u_k\}_{k\in\mathbb{N}_0}$ converges to the set of associated eigenfunctions $M(\lambda)$.

4.1 Convergence to a limiting pair

Now, we prove that the sequence of discrete eigenpairs $\{(\lambda_k, u_k)\}_{k\in\mathbb{N}_0}$ obtained by SOLVE throughout the adaptive loop of Section 3 has the following property: λ_k converges to some $\lambda_\infty \in \mathbb{R}$ and there exists a subsequence $\{u_{k_m}\}_{m\in\mathbb{N}_0}$ of $\{u_k\}_{k\in\mathbb{N}_0}$ converging in $H^1(\Omega)$ to a function u_∞.

Let us define the limiting space as $\mathbb{V}_\infty := \overline{\cup\mathbb{V}_k}^{H_0^1(\Omega)}$, and note that $\mathbb{V}_\infty$ is a closed subspace of $H_0^1(\Omega)$, and therefore it is itself a Hilbert space with the inner product inherited from $H_0^1(\Omega)$.

Since $\mathcal{T}_{k+1}$ is always a refinement of $\mathcal{T}_k$, by the minimum-maximum principle $\{\lambda_k\}_{k\in\mathbb{N}_0}$ is a decreasing sequence bounded below by λ. Therefore, there exists $\lambda_\infty > 0$ such that $\lambda_k \searrow \lambda_\infty$. From (3.1) it follows that

$$\|u_k\|_a^2 = a(u_k, u_k) = \lambda_k b(u_k, u_k) = \lambda_k \|u_k\|_b^2 = \lambda_k \to \lambda_\infty, \tag{4.1}$$

and therefore, that $\{u_k\}_{k\in\mathbb{N}_0}$ is a bounded sequence in $\mathbb{V}_\infty$. Then, there exists a subsequence $\{u_{k_m}\}_{m\in\mathbb{N}_0}$ weakly convergent in $\mathbb{V}_\infty$ to a function $u_\infty \in \mathbb{V}_\infty$, so

$$u_{k_m} \rightharpoonup u_\infty \quad \text{in} \quad H_0^1(\Omega). \tag{4.2}$$

Using Rellich's theorem we can extract a subsequence of the last one, which we still denote $\{u_{k_m}\}_{m\in\mathbb{N}_0}$, such that

$$u_{k_m} \longrightarrow u_\infty \quad \text{in} \quad L^2(\Omega). \tag{4.3}$$

If $k_0 \in \mathbb{N}_0$ and $k_m \geq k_0$, for all $v_{k_0} \in \mathbb{V}_{k_0}$ we have that $a(u_{k_m}, v_{k_0}) = \lambda_{k_m} b(u_{k_m}, v_{k_0})$, and when m tends to infinity, we obtain that $a(u_\infty, v_{k_0}) = \lambda_\infty b(u_\infty, v_{k_0})$. Since $k_0 \in \mathbb{N}_0$ and $v_{k_0} \in \mathbb{V}_{k_0}$ are arbitrary we have that

$$a(u_\infty, v) = \lambda_\infty b(u_\infty, v), \qquad \forall\, v \in \mathbb{V}_\infty. \tag{4.4}$$

On the other hand, since that $\|u_{k_m}\|_b = 1$, considering (4.3) we conclude that $\|u_\infty\|_b = 1$. Now, taking into account (4.4) we have that

$$\|u_\infty\|_a^2 = \lambda_\infty \|u_\infty\|_b^2 = \lambda_\infty.$$

From (4.1) it follows that $\|u_{k_m}\|_a^2 = \lambda_{k_m} \longrightarrow \lambda_\infty$, and therefore, $\|u_{k_m}\|_a \to \|u_\infty\|_a$. This, together with (4.2) yields

$$u_{k_m} \longrightarrow u_\infty \quad \text{in} \quad H_0^1(\Omega).$$

Summarizing, we have proved the following

Theorem 4.1. *Let $\{(\lambda_k, u_k)\}_{k\in\mathbb{N}_0}$ be the sequence obtained by the module* SOLVE *in the adaptive loop described in Section 3. Then, $\lambda_k \to \lambda_\infty \in \mathbb{R}$ and a subsequence $u_{k_m} \to u_\infty \in \mathbb{V}_\infty$ in $H_0^1(\Omega)$, where the limiting pair $(\lambda_\infty, u_\infty)$ satisfies*

$$\begin{cases} a(u_\infty, v) = \lambda_\infty\, b(u_\infty, v), & \forall\, v \in \mathbb{V}_\infty, \\ \|u_\infty\|_b = 1. \end{cases}$$

It is important to notice that carrying over the steps stated above, from any subsequence $\{(\lambda_{k_m}, u_{k_m})\}_{m\in\mathbb{N}_0}$ of $\{(\lambda_k, u_k)\}_{k\in\mathbb{N}_0}$, we can extract another subsequence $\{(\lambda_{k_{m_n}}, u_{k_{m_n}})\}_{n\in\mathbb{N}_0}$, such that $u_{k_{m_n}}$ converges in $H^1(\Omega)$ to some function $\tilde{u}_\infty \in \mathbb{V}_\infty$ that satisfies

$$\begin{cases} a(\tilde{u}_\infty, v) = \lambda_\infty\, b(\tilde{u}_\infty, v), & \forall\, v \in \mathbb{V}_\infty, \\ \|\tilde{u}_\infty\|_b = 1. \end{cases}$$

4.2 Convergence of the global error estimator

We now show that the global a posteriori estimator defined in Section 2.3 tends to zero.

In order not to clutter the notation, we will still denote by $\{u_k\}_{k\in\mathbb{N}_0}$ the subsequence $\{u_{k_m}\}_{m\in\mathbb{N}_0}$, and by $\{\mathcal{T}_k\}_{k\in\mathbb{N}_0}$ the subsequence $\{\mathcal{T}_{k_m}\}_{m\in\mathbb{N}_0}$. Also, we will replace the subscript $\mathcal{T}_k$ by k (e.g. $\mathcal{N}_k(T) := \mathcal{N}_{\mathcal{T}_k}(T)$ and $\omega_k(T) := \omega_{\mathcal{T}_k}(T)$), and whenever Ξ is a subset of $\mathcal{T}_k$, $\eta_k(\Xi)^2$ will denote the sum $\sum_{T\in\Xi} \eta_k(T)^2$.

Theorem 4.2 (Convergence of the global error estimator)**.** *If $\{\mathcal{T}_k\}_{k\in\mathbb{N}_0}$ denote the triangulations corresponding to the convergent subsequence of discrete eigenpairs from Theorem 4.1 and $\{\eta_k(\mathcal{T}_k)\}_{k\in\mathbb{N}_0}$ the a posteriori error estimators given by the module* ESTIMATE *in the adaptive loop, then*

$$\lim_{k\to\infty} \eta_k(\mathcal{T}_k) = 0.$$

Sketch of the proof. We can classify the elements in $\mathcal{T}_k$ in the following three disjoint groups:

- Elements which themselves and their neighbors are refined at least n_d times are in $\mathcal{T}_k^0$;
- elements which neither themselves nor their neighbors are ever refined are in $\mathcal{T}_k^+$;
- and the other ones are in $\mathcal{T}_k^*$.

Therefore, the global error estimator can be decomposed as

$$\eta_k(\mathcal{T}_k)^2 = \eta_k(\mathcal{T}_k^0)^2 + \eta_k(\mathcal{T}_k^+)^2 + \eta_k(\mathcal{T}_k^*)^2,$$

and it will be sufficient show that $\eta_k(\mathcal{T}_k^0)$, $\eta_k(\mathcal{T}_k^*)$ and $\eta_k(\mathcal{T}_k^+)$ tend to zero as k tends to infinity.

$\boxed{1}$ In order to prove that $\eta_k(\mathcal{T}_k^0) \to 0$, we notice that the elements in $\mathcal{T}_k^0$ are sufficiently refined, and we can thus use the the local efficiency of the estimator (2.6) with $w = u_\infty$, the convergence stated in Theorem 4.1 and the fact that the meshsize function $h_{k|_T} := |T|^{1/d}$ converges uniformly to zero[2] over Ω_k^0, where Ω_k^0 is the union of the neighborhoods of the elements in $\mathcal{T}_k^0$.

[2] This claim is a consequence of the fact that the sequence of triangulations is obtained by refinement only, and that every time an element $T \in \mathcal{T}_k$ is refined into $\mathcal{T}_{k+1}$, $h_{k+1}(x) \le \left(\frac{1}{2}\right)^{1/d} h_k(x)$ for almost every $x \in T$.

[2] To prove that $\eta_k(\mathcal{T}_k^*) \to 0$ we can use the the local efficiency of the estimator (2.6) with $w = u \in H_0^1(\Omega)$ an eigenfunction of the continuous problem, the convergence stated in Theorem 4.1 and the fact that the measure of the interface Ω_k^* tends to zero, where Ω_k^* is the union of the neighborhoods of the elements in $\mathcal{T}_k^*$.
[3] Finally, to prove that $\eta_k(\mathcal{T}_k^+) \to 0$ we notice that the elements in $\mathcal{T}_k^+$ whose elements are never refined. At this point we will use the assumption (3.2) on the marking strategy. This is technically the most difficult part, but the idea is to use the fact that since the elements in $\mathcal{T}_k^+$ are not marked to be refined, $\eta_k(T) \lesssim \eta_k(\mathcal{T}_k^0) + \eta_k(\mathcal{T}_k^*)$ for all $T \in \mathcal{T}_k^+$, and thus $\eta_k(T) \to 0$ for all $T \in \mathcal{T}_k^+$. Resorting to a generalized Lebesgue dominated convergence theorem we obtain the claim. □

The details of this proof are contained in Garau et al. (2008a).

4.3 The limiting pair is an eigenpair

Next, we prove that $(\lambda_\infty, u_\infty)$ is an eigenpair of the continuous problem (2.1). The idea in Morin et al. (2008) to prove that u_∞ is the exact solution to the continuous problem, consisted in using the *reliability* of the a posteriori error estimators. Such a bound does not hold in this case unless the underlying triangulation is sufficiently fine. We do not enforce such a condition on the initial triangulation $\mathcal{T}_0$, since the term *sufficiently fine* is not easily quantifiable. Instead we resort to another idea, we will bound $a(u_\infty, v) - \lambda_\infty b(u_\infty, v)$ by the residuals of the discrete problems, which are in turn bounded by the global error estimator, and was already proved to converge to zero.

Theorem 4.3. *The limiting pair $(\lambda_\infty, u_\infty)$ of Theorem 4.1 is an eigenpair of the continuous problem* (2.1)*. That is,*

$$\begin{cases} a(u_\infty, v) = \lambda_\infty\, b(u_\infty, v), & \forall\, v \in H_0^1(\Omega), \\ \|u_\infty\|_b = 1. \end{cases}$$

Proof. We know that $\|u_\infty\|_b = 1$ due to Theorem 4.1. It remains to prove that

$$a(u_\infty, v) = \lambda_\infty b(u_\infty, v), \qquad \forall\, v \in H_0^1(\Omega).$$

Let $v \in H_0^1(\Omega)$, and let $v_k \in \mathbb{V}_k$ be the Scott-Zhang interpolant (Scott and Zhang, 1990, 1992) of v, which satisfies

$$\|v - v_k\|_T \lesssim h_T \|\nabla v\|_{\omega_k(T)} \qquad \text{and} \qquad \|v - v_k\|_{\partial T} \lesssim h_T^{1/2} \|\nabla v\|_{\omega_k(T)}.$$

From (3.1) we have

$$a(u_k, v_k) = \lambda_k b(u_k, v_k),$$

for all k, and then

$$\begin{aligned} |a(u_\infty, v) - \lambda_\infty b(u_\infty, v)| &= |a(u_\infty, v) - \lambda_\infty b(u_\infty, v) - a(u_k, v_k) + \lambda_k b(u_k, v_k)| \\ &\le |a(u_k, v - v_k) - \lambda_k b(u_k, v - v_k)| + |b(\lambda_k u_k - \lambda_\infty u_\infty, v)| + |a(u_\infty - u_k, v)|. \end{aligned} \tag{4.5}$$

The second term in (4.5) can be bounded as

$$\begin{aligned} |b(\lambda_k u_k - \lambda_\infty u_\infty, v)| &\le |\lambda_k| |b(u_k - u_\infty, v)| + |\lambda_k - \lambda_\infty| |b(u_\infty, v)| \\ &\lesssim \left(\lambda_0 \|u_k - u_\infty\|_\Omega + |\lambda_k - \lambda_\infty| \|u_\infty\|_\Omega\right) \|v\|_\Omega . \end{aligned}$$

And the third term in (4.5) is bounded by

$$|a(u_\infty - u_k, v)| \lesssim \|\nabla(u_\infty - u_k)\|_\Omega \|\nabla v\|_\Omega .$$

Finally, the first term in (4.5) can be bounded using integration by parts on each element and following the steps of the proof of the a posteriori upper bound, as follows:

$$\begin{aligned}|a(u_k, v - v_k) - \lambda_k b(u_k, v - v_k)| &= \left|\sum_{T\in\mathcal{T}_k} \int_T \mathcal{A}\nabla u_k \cdot \nabla(v - v_k) - \lambda_k \int_T \mathcal{B} u_k (v - v_k)\right| \\ &= \left|\sum_{T\in\mathcal{T}_k} \int_T R(\lambda_k, u_k)(v - v_k) + \frac{1}{2}\int_{\partial T} (v - v_k) J(u_k)\right|,\end{aligned}$$

with $R(\lambda_k, u_k)$ and $J(u_k)$ as defined in (2.4) and (2.5). Now, by Hölder and Cauchy-Schwarz inequalities we obtain

$$\begin{aligned}|a(u_k, v - v_k) - \lambda_k b(u_k, v - v_k)| &\le \sum_{T\in\mathcal{T}_k} \|R(\lambda_k, u_k)\|_T \|v - v_k\|_T + \|J(u_k)\|_{\partial T} \|v - v_k\|_{\partial T} \\ &\lesssim \sum_{T\in\mathcal{T}_k} \|R(\lambda_k, u_k)\|_T h_T \|\nabla v\|_{\omega_k(T)} + \|J(u_k)\|_{\partial T} h_T^{1/2} \|\nabla v\|_{\omega_k(T)} \\ &\lesssim \left(\sum_{T\in\mathcal{T}_k} h_T^2 \|R(\lambda_k, u_k)\|_T^2 + h_T \|J(u_k)\|_{\partial T}^2\right)^{1/2} \|\nabla v\|_\Omega \\ &= \eta_k(\mathcal{T}_k) \|\nabla v\|_\Omega .\end{aligned}$$

Summarizing, we have that

$$|a(u_\infty, v) - \lambda_\infty b(u_\infty, v)| \lesssim \Big((1 + \lambda_0) \|u_k - u_\infty\|_{H^1(\Omega)} + |\lambda_k - \lambda_\infty| \|u_\infty\|_\Omega + \eta_k(\mathcal{T}_k)\Big) \|v\|_{H^1(\Omega)} .$$

Using the convergence of u_k to u_∞ in $H^1(\Omega)$ and λ_k to λ_∞ in $\mathbb{R}$ from Theorem 4.1, and the convergence of the global estimator to zero from Theorem 4.2, we conclude that

$$|a(u_\infty, v) - \lambda_\infty b(u_\infty, v)| = 0,$$

and the proof is completed. □

4.4 Convergence to the solution set

We conclude this section by stating a general convergence result, which is a consequence of the previous results. We remark that since there exists a set $M(\lambda)$ of eigenfunctions associated to a same eigenvalue λ, the error in the eigenfunction is given by

$$\operatorname{dist}_{H_0^1(\Omega)}(u_k, M(\lambda)) := \min_{u\in M(\lambda)} \|u - u_k\|_{H_0^1(\Omega)}.$$

Theorem 4.4. *Let $\{(\lambda_k, u_k)\}_{k\in\mathbb{N}_0}$ denote the whole sequence of discrete eigenpairs obtained through the adaptive loop stated in Section 3. Then, there exists an eigenvalue λ of the continuous problem (2.1) such that*

$$\lim_{k\to\infty} \lambda_k = \lambda \qquad \textit{and} \qquad \lim_{k\to\infty} \operatorname{dist}_{H_0^1(\Omega)}(u_k, M(\lambda)) = 0.$$

Proof. By Theorem 4.1, taking $\lambda := \lambda_\infty$, we have that $\lim_{k\to\infty} \lambda_k = \lambda$, and by Theorem 4.3, λ is an eigenvalue of the continuous problem (2.1). In order to prove that $\text{dist}_{H_0^1(\Omega)}(u_k, M(\lambda)) \to 0$ as $k \to \infty$ we argue by contradiction. If the result were not true, then there would exist a number $\epsilon > 0$ and a subsequence $\{u_{k_m}\}_{m\in\mathbb{N}_0}$ of $\{u_k\}_{k\in\mathbb{N}_0}$ such that

$$\text{dist}_{H_0^1(\Omega)}(u_{k_m}, M(\lambda)) > \epsilon, \qquad \forall\, m \in \mathbb{N}_0. \tag{4.6}$$

It is possible to extract a subsequence of $\{u_{k_m}\}_{m\in\mathbb{N}_0}$ which still converges to some function $\tilde{u}_\infty \in \mathbb{V}_\infty$. By the arguments of Sections 4.2 and 4.3, $\tilde{u}_\infty$ is an eigenfunction of the continuous problem (2.1) corresponding to the same eigenvalue λ. That is, a subsequence of $\{u_{k_m}\}_{m\in\mathbb{N}_0}$ converges to an eigenfunction in $M(\lambda)$, which contradicts (4.6) and completes the proof. □

In our algorithm we assumed that each of the discrete eigenvalues λ_k is the j-th eigenvalue of the corresponding discrete problem. The result, as stated above, only guarantees that λ_k converges to one eigenvalue λ of the continuous problem, possibly larger than the j-th eigenvalue. We can be sure that we approximate the j-th eigenvalue of the continuous problem under any of the following assumptions:

- *A Non-Degeneracy Assumption.* No eigenfunction is equal to a polynomial of degree $\leq \ell$ on an open region of Ω. This assumption holds for a large class of problems. More precise sufficient conditions on problem data $\mathcal{A}$ and $\mathcal{B}$ to guarantee that this assumption holds will be stated below.

- The meshsize of the initial triangulation is small enough. This assumption goes against the spirit of adaptivity and a posteriori analysis, since we cannot quantify what *small enough* means. But we state it for completeness, because in some (nonlinear) problems there may be no way to overcome this.

Theorem 4.5 (General convergence result)**.** *Let us suppose that the continuous problem (2.1) satisfies the Non-Degeneracy Assumption above, and let $\{(\lambda_k, u_k)\}_{k\in\mathbb{N}_0}$ denote the whole sequence of discrete eigenpairs obtained through the adaptive loop stated in Section 3 and λ denote the j-th eigenvalue of the continuous problem (2.1). Then,*

$$\lim_{k\to\infty} \lambda_k = \lambda \qquad \textit{and} \qquad \lim_{k\to\infty} \text{dist}_{H_0^1(\Omega)}(u_k, M(\lambda)) = 0.$$

Before embarking into the proof of this theorem, it is worth mentioning that the model case of $\mathcal{A} \equiv I$ and $\mathcal{B} \equiv 1$ satisfies the Non-Degeneracy Assumption, due to the fact that the eigenfunctions of the Laplacian are analytic. A weaker assumption on the coefficients $\mathcal{A}$ and $\mathcal{B}$ that guarantee non-degeneracy of the problem are given in the following Lemma, which is a consequence of the regularity results stated in Han (1994).

Lemma 1. *If $\mathcal{A}$ is continuous, and piecewise $\mathcal{P}_1$, and $\mathcal{B}$ is piecewise constant, then problem (2.1) satisfies the Non-Degeneracy Assumption.*

We believe that in the assumptions of the previous lemma, $\mathcal{A}$ can be allowed to be piecewise continuous with discontinuities along Lipschitz interfaces. The only thing needed is a proof of the fact that solutions to elliptic problems with coefficients like these cannot vanish in an open subset of Ω unless they vanish over all Ω. We conjecture that this could be proved using Han's result (Han, 1994) in combination with Hopf's lemma (Gilbarg and Trudinger, 1983), but it will be subject of future work.

We now proceed to prove Theorem 4.5, which will be a consequence of the following lemma.

Lemma 2. *Let $\{h_k\}_{k\in\mathbb{N}_0}$ denote the sequence of meshsize functions obtained through the adaptive loop stated in Section 3. If the continuous problem (2.1) satisfies the* Non-Degeneracy Assumption*, then $\|h_k\|_{L^\infty(\Omega)} \to 0$ as $k \to \infty$.*

Proof. We argue by contradiction. If $\|h_k\|_{L^\infty(\Omega)}$ does not tend to zero, then there exists $k_0 \in \mathbb{N}_0$ and $T \in \mathcal{T}_k$, for all $k \geq k_0$. Since $\|u_{k_m} - u_\infty\|_{L^2(T)} \to 0$ as $m \to \infty$, and $u_{k|_T} \in \mathcal{P}_\ell(T)$, for all $k \geq 0$, using that $\mathcal{P}_\ell(T)$ is a finite dimensional space we conclude that

$$u_{\infty|_T} \in \mathcal{P}_\ell(T). \tag{4.7}$$

Theorem 4.3 claims that u_∞ is an eigenfunction of (2.1) and thus (4.7) contradicts the Non-Degeneracy Assumption. □

It is important to notice that the convergence of h_k to zero is not an assumption, but a consequence of the fact that a subsequence is converging to an eigenfunction u_∞ and the Non-Degeneracy Assumption.

Proof of Theorem 4.5. In view of Theorem 4.4 it remains to prove that λ_k converges to the j-th eigenvalue of (2.1). By Lemma 2 the result follows from (2.3). □

We conclude this section with several remarks.

Remark 1. At first sight, the convergence of $\|h_k\|_{L^\infty(\Omega)}$ to zero looks like a very strong statement, especially in the context of adaptivity. But the uniform convergence of the meshsize to zero should not be confused with quasi-uniformity of the sequence of triangulations $\{\mathcal{T}_k\}_{k\in\mathbb{N}_0}$, the latter is not necessary for the former to hold. Thinking about this more carefully, we realize that if we wish to have (optimal) convergence of finite element functions to some given function in $H^1(\Omega)$, then h_k *must tend to zero everywhere* (pointwise) unless the objective function is itself a polynomial of degree $\leq \ell$ in an open region of Ω. We have that the convergence of h_k to zero is also uniform, and this does not necessarily destroy optimality (Cascon et al., 2008; Stevenson, 2007; Garau et al., 2008b).

Remark 2. A sufficient condition to guarantee that we converge to the desired eigenvalue is to assume that $h_k \to 0$ as $k \to \infty$. This condition is weaker than the Non-Degeneracy Assumption, but it is in general impossible to prove a priori.

Remark 3. Another option to guarantee convergence to the desired eigenvalue is to start with a mesh which is sufficiently fine. In view of the minimum-maximum principles, it is sufficient to start with a triangulation $\mathcal{T}_0$ that is sufficiently fine to guarantee that $\lambda_{j,\mathcal{T}_0} < \lambda_{j_0}$, where $j_0 > j$ is the minimum index such that $\lambda_{j_0} > \lambda_j$. This condition is verifiable a posteriori if we have a method to compute eigenvalues approximating from below. Some ideas in this direction are presented in Armentano and Durán (2003), where the effect of mass lumping on the computation of discrete eigenvalues is studied.

5 QUASI-OPTIMALITY OF THE ADAPTIVE FINITE ELEMENT METHOD

Let $N \in \mathbb{N}$ and let $\mathbb{T}_N$ be the set of all possible conforming triangulations generated by at most N bisections of $\mathcal{T}_0$, i.e.,

$$\mathbb{T}_N := \{\mathcal{T} \in \mathbb{T} \mid \#\mathcal{T} - \#\mathcal{T}_0 \leq N\}.$$

The elements $\mathcal{T}$ of this set will be called triangulations of *complexity* $\leq N$. For an eigenvalue λ of the continuous problem (2.1), the minimum error attainable with triangulations of complexity $\leq N$ is given by

$$\sigma(\lambda, \mathbf{D}; N) := \inf_{\mathcal{T}\in\mathbb{T}_N} \inf_{\substack{v\in\mathbb{V}_\mathcal{T}\\ \|v\|_b=1\\ \mu\in\mathbb{R}}} \operatorname{error}_\mathcal{T}(\mu, v),$$

where $\mathbf{D} := \{\mathcal{A}, \mathcal{B}, \Omega\}$ is the problem data set and the total error $\operatorname{error}_\mathcal{T}(\mu, v)$ is given by

$$\operatorname{error}_\mathcal{T}(\mu, v) := \left(|\lambda-\mu|^2 + \operatorname{dist}_{H_0^1(\Omega)}(v, M(\lambda))^2 + \operatorname{osc}_\mathcal{T}(\mu, v)^2\right)^{1/2}.$$

We say that $(\lambda, \mathbf{D}) \in \mathbb{A}_s$ if the error of the best approximation in $\mathbb{T}_N$ decreases as

$$\sigma(\lambda, \mathbf{D}; N) = O(N^{-s}).$$

It can be proved (Garau et al., 2008b) that when the eigenvalue λ is simple and $(\lambda, \mathbf{D}) \in \mathbb{A}_s$, the sequence of meshes $\mathcal{T}_k$ and discrete eigenpairs (λ_k, u_k) obtained by our adaptive algorithm using Dörfler's marking strategy[3] satisfy

$$\operatorname{error}_{\mathcal{T}_k}(\lambda_k, u_k) = O((\#\mathcal{T}_k - \#\mathcal{T}_0)^{-s}),$$

for all $k \in \mathbb{N}$. In other words, the adaptive algorithm produces a sequence of meshes and approximate solutions with the same complexity as the optimal ones.

6 NUMERICAL EXPERIMENTS

We conclude this article illustrating the advantages in using adaptive refinement instead uniform one when we are approximating eigenfunctions having singularities of different strength.

We consider the computation of the eigenpairs of Laplace's operator given by

$$\begin{cases} -\Delta u &= \lambda u, \qquad \text{in } \Omega, \\ \quad u &= 0, \qquad \text{on } \partial\Omega, \end{cases}$$

where Ω consists of three quarters of a circle, described in polar coordinates as

$$\Omega := \{(\rho, \varphi) \mid 0 < \rho < 1, \quad 0 < \varphi < \tfrac{3}{2}\pi\} \subset \mathbb{R}^2.$$

The variational form is the classical problem:

$$\begin{cases} \int_\Omega \nabla u \cdot \nabla v = \lambda \int_\Omega uv, \qquad \forall\, v \in H_0^1(\Omega), \\ \|u\|_\Omega = 1, \end{cases}$$

which correspond to the choices $\mathcal{A} \equiv I$ and $\mathcal{B} \equiv 1$ in the problem (2.1), and the convergence result of the Theorem 4.5 holds. The exact eigenvalues and eigenfunctions of this problem are known, and we will use them to test the behavior of the adaptive algorithm and investigate experimentally the regularity and optimal order of adaptive approximation.

In order to implement the adaptive algorithm described in Section 3, we use the finite element toolbox ALBERTA (Schmidt and Siebert, 2005), and consider the approximation of the first and the second eigenpairs.

[3] Based upon the a posteriori error indicators $\{\eta_k(T)\}_{T\in\mathcal{T}_k}$ and the marking parameter $\theta \in (0, 1]$, the Dörfler property consists in select a minimal subset of marked elements $\mathcal{M}_k \subset \mathcal{T}_k$ satisfying

$$\sum_{T\in\mathcal{M}_k} \eta_k(T)^2 \geq \theta^2 \sum_{T\in\mathcal{T}_k} \eta_k(T)^2.$$

Approximation of the first eigenvalue

The first eigenvalue is $\lambda_1 \approx 11.394747279$ and its corresponding eigenfunction is given in polar coordinates by

$$u_1(\rho, \varphi) = c_1 J_{\frac{2}{3}}(\sqrt{\lambda_1}\rho) \sin\left(\tfrac{2}{3}\varphi\right),$$

where $J_{\frac{2}{3}}$ is a Bessel function of the first kind, and c_1 is a constant chosen to achieve $\|u_1\|_\Omega = 1$.

Figure 1 shows the energy error $\|u_1 - u_{1,\mathcal{T}}\|_{H^1(\Omega)}$ (left) and the eigenvalue error $|\lambda_1 - \lambda_{1,\mathcal{T}}|$ (right) versus the degrees of freedom (DOFs) when using Lagrange finite element spaces of degree $\ell = 1$ (top) and $\ell = 2$ (bottom). The energy error decay when using Global (uniform) Refinement is approximately of order $N^{-0.33}$ (where N denotes the number of degrees of freedom) indicating that the function u_1 belongs to the Sobolev space $H^{1+\frac{2}{3}}(\Omega)$. Nevertheless, when using adaptive refinement, either the Maximum Strategy (MS) or Dörfler's Strategy (DS), the orders are approximately $N^{-1/2}$ when using linears and N^{-1} for quadratic elements. These rates are the same that would be obtained if $u \in H^2(\Omega)$ for $\ell = 1$ or if $u \in H^3(\Omega)$ for $\ell = 2$ with uniform refinement, and are usually called *optimal convergence rates*.

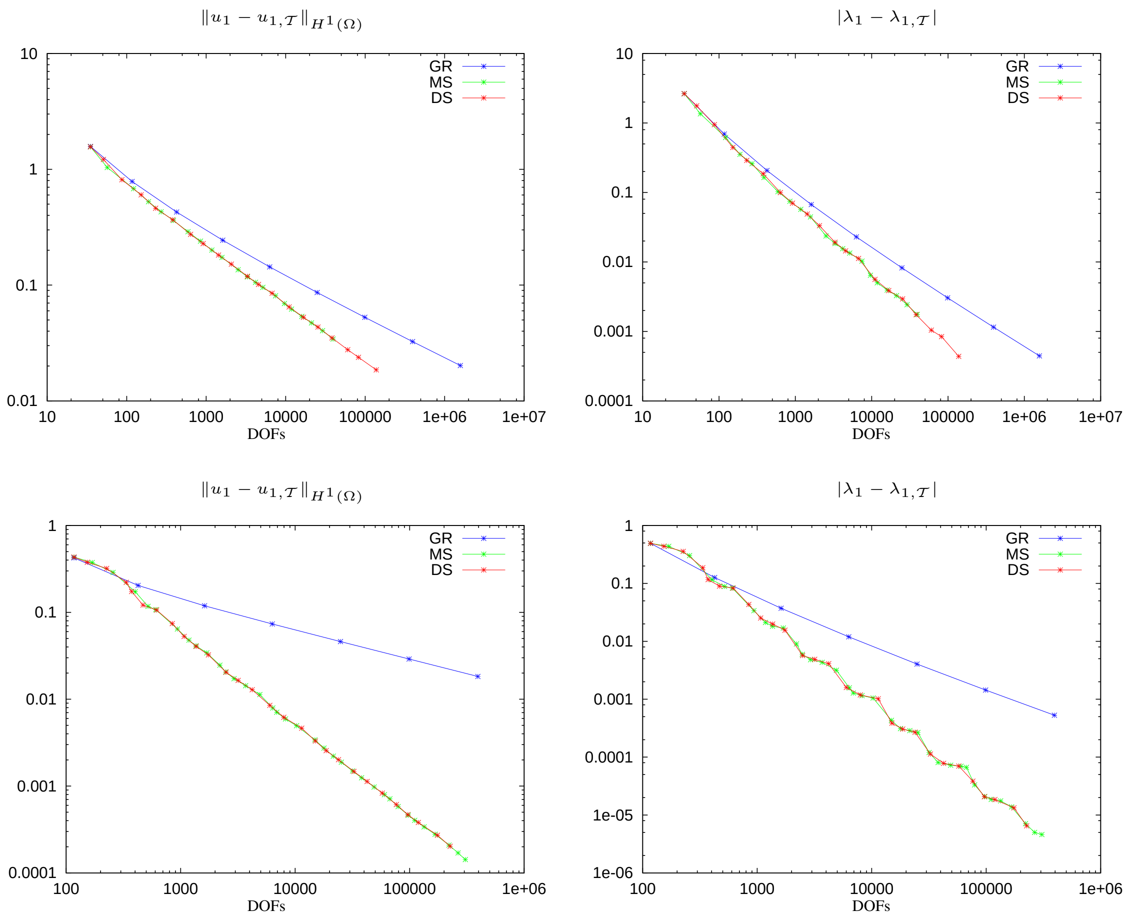

Figure 1: Error decay for the computation of the first eigenvalue with linear (top) and quadratic finite elements (bottom). When considering the energy error $\|u_1 - u_{1,\mathcal{T}}\|_{H^1(\Omega)}$ (left), the decay with Global Refinement (GR) is approximately $N^{-0.33}$ in both cases, and the decay with adaptive refinement, either with the Maximum Strategy (MS) or with Dörfler's (DS) is $N^{-1/2}$ for linears and N^{-1} for quadratics. When considering the error $|\lambda_1 - \lambda_{1,\mathcal{T}}|$, the decay with global refinement is approximately $N^{-0.67}$ in both cases, and the decay with the considered adaptive methods is approximately N^{-1} for linears and $N^{-1.5}$ for quadratics.

These rates are in complete agreement with the theory, since the eigenfunction u_1 is equal to a positive power of ρ times an $H^\infty(\Omega)$ function and according to the results of Gaspoz and Morin (2008) the order of adaptive approximation should be $N^{-\frac{\ell}{d}}$, as observed in the experiments.

The error $|\lambda_1 - \lambda_{1,\mathcal{T}}|$ in the approximation of the eigenvalue is comparable with the error of the eigenfunctions measured in $L^2(\Omega)$. It is approximately $N^{-0.67}$ when using global refinements and linear or quadratic elements, whereas it is approximately N^{-1} and $N^{-1.5}$ when using adaptivity with linears and quadratics respectively.

Approximation of the second eigenvalue

The second eigenvalue is $\lambda_2 \approx 18.278538262$ and its corresponding eigenfunction is given in polar coordinates by

$$u_2(\rho, \varphi) = c_2 J_{\frac{4}{3}}(\sqrt{\lambda_2}\rho)\sin(\tfrac{4}{3}\theta),$$

where $J_{\frac{4}{3}}$ is a Bessel function of the first kind, and c_2 is the constant to achieve $\|u_2\|_\Omega = 1$. This second eigenfunction belongs to $H^{2+\frac{1}{3}}(\Omega)$ and is thus regular for its approximation with linear elements. That is, global refinement yields $\|u_2 - u_{2,\mathcal{T}}\|_{H^1(\Omega)} = O(N^{-1/2})$ for linear elements. Adaptive refinement also presents the same decay, and this is corroborated by the error curves depicted in Figure 2 (top). On the other hand, when considering the approximation with quadratic finite elements, the order of approximation using global refinements is approximately $N^{-2/3}$, not reaching the optimal rate N^{-1} because $u_2 \notin H^3(\Omega)$. However, the adaptive methods are able to capture the singularity and yield the optimal convergence rate $\|u_2 - u_{2,\mathcal{T}}\|_{H^1(\Omega)} = O(N^{-1})$, which is also predicted by the theory of Gaspoz and Morin (2008) and Garau et al. (2008b).

An interesting conclusion that can be drawn from these observations is the following: Despite the fact that it is not worth increasing the polynomial degree when using global refinement and the function is not regular, increasing the polynomial degree can drastically improve the performance of adaptive methods, leading to quasi-optimal convergence rates. More specifically, given a fixed polynomial degree, using adaptive methods on singular solutions leads to the same order of convergence that is obtained when using global refinements on regular solutions.

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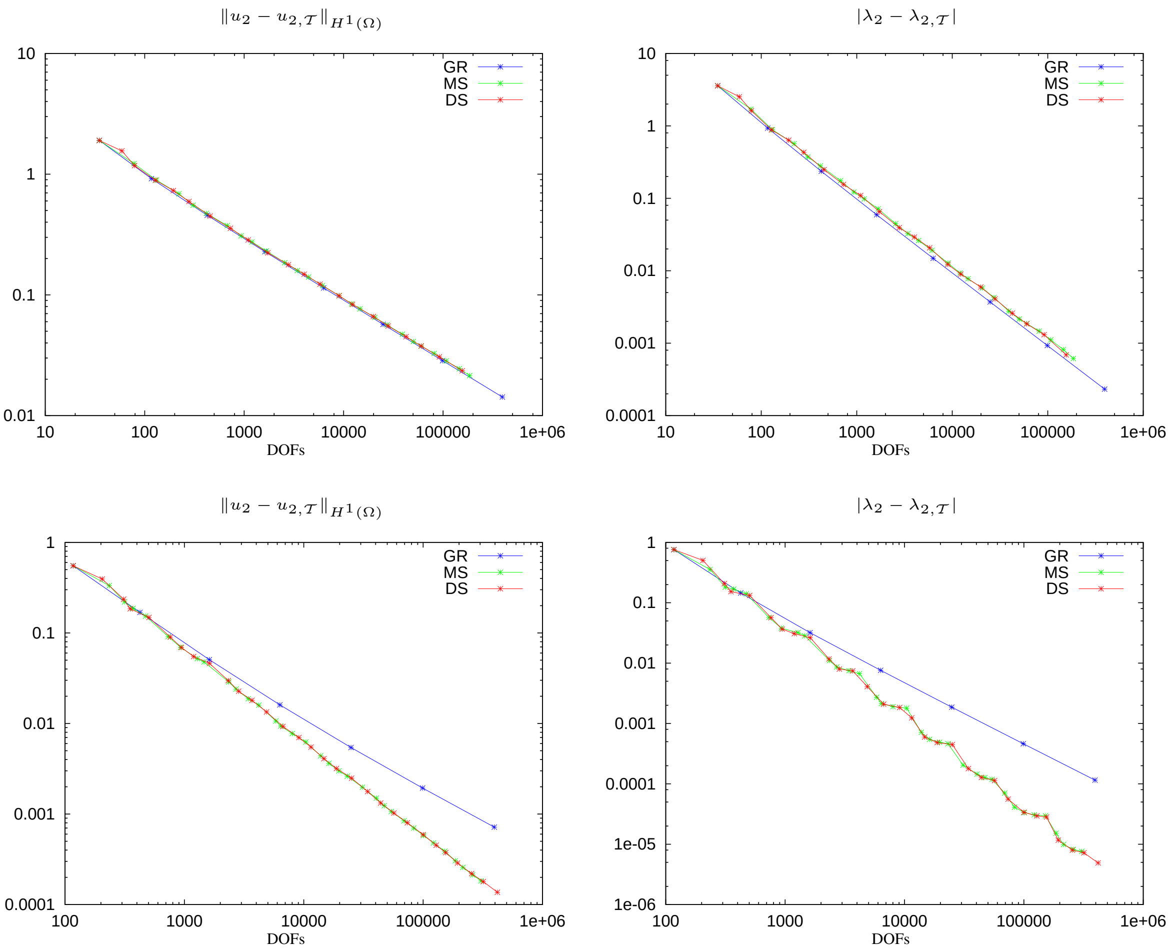

Figure 2: Error decay for the computation of the second eigenvalue with linear (top) and quadratic finite elements (bottom). When considering linears the decay for energy error $\|u_2 - u_{2,\mathcal{T}}\|_{H^1(\Omega)}$ (left) is $N^{-1/2}$ and for $|\lambda_2 - \lambda_{2,\mathcal{T}}|$ is N^{-1}, with both global and adaptive refinement. When considering quadratics the order for the energy error is $N^{-2/3}$ for global and N^{-1} for adaptive refinement, indicating that the eigenfunction u_2 belongs to $H^{2+1/3}(\Omega)$, and that nevertheless adaptive refinement leads to quasi-optimal meshes. The error decay for $|\lambda_2 - \lambda_{2,\mathcal{T}}|$ is the same that one obtains for $\|u_2 - u_{2,\mathcal{T}}\|_{L^2(\Omega)}$, and equals N^{-1} for global and $N^{-1.5}$ for adaptive refinement.

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