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AN INVERSE METHOD FOR THE DESIGN OF STRUCTURES

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Abstract. Inverse methods are powerful design tools that allow engineers to obtain efficient designs at much lower costs than the ones normally involved in experimental and direct computational design. Here we present an inverse method which allows the efficient design of deformable structures or components, such for example airplane wings. The inverse method proposed here allows the engineers to obtain the actual unloaded geometry they should request to a manufacture department, so as to obtain a structure or piece that under the work loads will deform to a pre-specified ideal shape. In the case of an airplane wing or an airfoil, for example, the engineer will be able to obtain the undeformed geometry of the wing such when it is flying at cruise speed subject to the aerodynamics loads, the wing will naturally deform to a pre-specified (and aerodynamically desired) shape. The presented inverse method is based on a novel formulation builded in terms of the finite element method.

1 INTRODUCTION

Computational Mechanics is nowadays an important part of engineering and deals with the simulation of solid and fluid physical phenomena by computers. The task of researchers working in the area is to develop software to predict the behavior of such materials under specific conditions. With the predictions the engineers can, for example, design structures, using the software to determine what will be the behavior, i.e. the deformations and the stresses, of the pieces forming the structure. The majority of computational mechanics software work as *direct analysis* programs (Limache and Idelsohn, 2007), (Limache and Cliff, 2000). When using these tools, the engineer proposes a first model for the geometry of the piece or structure to be designed (for example with CAD) and he will use the software to determine what will be the deformation of the structure when subject to external loads. Then, using his experience he will need to modify the geometry of the initial structure many time until a desired behavior is obtained. Direct analysis software allows a considerable reduction of costs since one does not have to actually manufacture the different possible shapes of the structure. For example, let us suppose that the engineer works for a prestigious aeronautical company in the design of a wing for a new airplane. It would be extremely expensive to propose a wing geometry, manufacture it and then test it in a wind tunnel, to find out that the wing deformation generates aerodynamic forces that affect the stability of the airplane, having to restart the process over and over. A good direct analysis software will help him to considerably reduce these problems since the wing can be modelled in a computer without the need to actually build it.

However, there exists other computational programs that are even more powerful and efficient than the classical direct analysis ones, those are called *inverse design* programs (Limache, 1995), (Limache, 1996) . Inverse design programs reverse the behavior of direct analisis tools. With them the engineer can set the external conditions (loads) and specify the behavior he wants the piece to have, then the inverse program will determine what is the geometry or structural shape that generates such desired response. Many inverse design programs are related to optimization techniques (Anderson and Ventakakrishnan, 1997). The structural shape gets adjusted so as to minimize a functional in which the desired properties are imposed.

As an example of inverse design, let us consider again the case of the airplane wing design, let us suppose the engineer wants to design a wing whose geometry generates a specified pressure distribution so as to generate high lift at low speeds. With an inverse design tool the engineer will input as data the desired pressure distribution and the program will computationally determine the wing shape that will be able to generate such desired pressure distribution. Note that the inverse technique is much powerful than the direct program, since it does not requires that the engineer be generating countless numbers of wing geometries until obtaining the expected distribution.

Of course generating an inverse method program is much complex than developing a direct analysis program. Furthermore, there no exist general inverse programs they usually can be developed for a certain range of applications. However, we must stress out that *inverse programs* like their *direct analysis* counterparts have the physical laws included in their formulation. Furthermore, a direct analysis program can be used to verify that, indeed, the inverse designed geometry satisfies the specified requirements.

In the present article, an inverse method for hyperelastic structural design is presented. The method can be used to determine the unloaded (unstressed) shape of an structure so as to when subject to specific loads the structure takes an ideal pre-specified shape. The method uses the most general equations of continuum mechanics and can be applied to general constitutive equations. It can be used in problems dealing with large rotation and large deformations and satisfies objectivity. The obtained formulation requires the solution of non-linear equations. The applications of the method are wide, particularly in manufacture processes of deformable structures (Limache and Idelsohn, 2007), (Fachinotti et al., 2007). As useful examples, we will consider the problems of wing design and beam design.

In the case of airplane wing design, an engineer would like to simply use a wing whose shape has in theory well-known aerodynamic properties. However, if he just sends this ideal shape to be manufactured, once in flight, the strong aerodynamic forces would deform the wing changing the initial ideal shape. Ought to the deformation, the wing would loose all of its desired aerodynamic characteristics. As a consequence, the designer will need to remodel new unloaded wings until the resulting loaded wing adquires good performance characteristics. This is a extremely costly solution.

Ideally, one would like to know what will the geometry of the undeformed wing would need to be so as to when in flight the wing will deform and adquire the aerodynamically ideal shape. Well, this is precisely what the inverse method proposed here can do.

In the next section, the equations of large deformations for continuum mechanics are presented. Then, the method is described in the context of the Finite Element Method. The article is closed with numerical results using MulPhys (Limache, 2008). MulPhys is a numerical program developed at CIMEC that can be used for the simulation of non-linear solid mechanics with finite-deformations. The inverse formulation has been integrated into the core package, producing a new module that we call MulPhys-Inverse.

2 CONTINUUM MECHANICS FRAMEWORK

2.1 Reference and Current Configurations

In continuum mechanics, the equations describing the dynamics and motion of material bodies are defined. A material body is formed by a compact set of material particles. We can represent the body by the specification of its geometrical and material properties at a certain moment when the body was available to us for inspection. This specification is called the *reference configuration*, and at such configuration we can assign a one to one correspondence between each particle χ and the vector position $\boldsymbol{x}^0$ defining the particle's location with respect to the *reference frame* defined at the moment of inspection:

$$\boldsymbol{x}^0 = \tilde{\boldsymbol{x}}_{\mathbf{0}}(\chi) \tag{1}$$

At the reference configuration, we know the volume or region v^0 occupied by the body and we can get any of its physical properties. These physical properties can be scalar, vector or tensor quantities. For example, we can obtain its density field ρ^0

$$\rho^0 = \overline{\rho}(\boldsymbol{x}^0) \tag{2}$$

or its stress state field, the Cauchy Stress $\boldsymbol{\sigma}^0$:

$$\boldsymbol{\sigma}^0 = \overline{\boldsymbol{\sigma}}(\boldsymbol{x}^0) \tag{3}$$

Here the reference configuration will be choosen as the *undeformed configuration*, i.e. the state of zero stress:

$$\boldsymbol{\sigma}^0 \equiv \mathbf{0} \tag{4}$$

Note that the physical quantities in the reference configuration are denoted with an upper-index "0".

To define the equilibrium state of our material body. Let us assume that the body has moved, rotated and deformed from the reference configuration to a new configuration, called *current configuration*. We can fully describe the body's new configuration by describing the motion of the material particles in terms of their *current* vector positions $\boldsymbol{x}^1$ in physical space.

$$\boldsymbol{x}^1 = \tilde{\boldsymbol{x}}_{\boldsymbol{1}}(\chi) \tag{5}$$

Combining eqs. (1)-(5) we can account for the configuration changes between the reference positions $\boldsymbol{x}^1$ and the current positions $\boldsymbol{x}^1$ by defining the deformation map:

$$\boldsymbol{x}^{\boldsymbol{1}} = \overline{\boldsymbol{x}}(\boldsymbol{x}^0) = \tilde{\boldsymbol{x}}_{\boldsymbol{1}}\left(\tilde{\boldsymbol{x}}_{\boldsymbol{0}}{}^{-1}(\boldsymbol{x}^0)\right) \tag{6}$$

The function $\overline{\boldsymbol{x}}$ is called *deformation map* because it defines the deformation of the body from the reference configuration into the current configuration. In the current configuration, the physical quantities are denoted using an upper index "1". Then the volume occupied by the body will be denoted by v^1 and its density field and stress field will be denoted by ρ^1 and $\boldsymbol{\sigma}^{\boldsymbol{1}}$, respectively.

A measure of the change of configuration is given by the *deformation gradient* tensor:

$$\boldsymbol{F}_0^1 = \frac{\partial\overline{\boldsymbol{x}}(\boldsymbol{x}^{\boldsymbol{0}})}{\partial\boldsymbol{x}^{\boldsymbol{0}}} = \frac{\partial\boldsymbol{x}^1}{\partial\boldsymbol{x}^0} \tag{7}$$

Note that $\boldsymbol{F}$ defines how a material differential changes:

$$d\boldsymbol{x}^1 = \boldsymbol{F}_0^1 \cdot d\boldsymbol{x}^0 = \frac{\partial\boldsymbol{x}^1}{\partial\boldsymbol{x}^0} \cdot d\boldsymbol{x}^0 \tag{8}$$

2.2 Strong Form of Equations of Continuum Mechanics

The equations of conservation of mass and momentum define the dynamics, deformation and motion of material bodies. They can be written either in the reference configuration or in the current configuration. Most people is used to see these equations written in the current configuration, they are:

$$\frac{D\rho^1}{Dt} + \rho^1\,\mathrm{div}(\boldsymbol{v}^1) = 0 \tag{9}$$

$$\rho^1\frac{D\boldsymbol{v}^1}{Dt} = \nabla^1 \cdot \mathrm{div}(\boldsymbol{\sigma}^1) + \rho^1\boldsymbol{b}^1 + \boldsymbol{f}^1 \tag{10}$$

where $\frac{D}{Dt}()$ denotes the material time-derivative, ∇^1 denotes de nabla operator in the current configuration, $\boldsymbol{b}$ denotes the external body forces per unit of mass and $\boldsymbol{f}$ denotes any other external force applied to the body. However, when dealing with general mechanics it is most convenient to represent these equations in the reference configuration (Gurtin, 1981), they are:

$$\rho^1 J_0^1 = \rho^0 \tag{11}$$

$$\rho^0\frac{D\boldsymbol{v}^1}{Dt} = \nabla^0 \cdot (\boldsymbol{P}_0^1) + \rho^0\boldsymbol{b}_0^1 + \boldsymbol{f}_0^1 \tag{12}$$

where J_0^1 stands for the determinant of the deformation gradient:

$$J_0^1 = \det(\boldsymbol{F}_0^1) \tag{13}$$

and where $\boldsymbol{P}_0^1$ represents the First Piola-Kirchhoff Stress Tensor:

$$\boldsymbol{P}_0^1 = J_0^1 \boldsymbol{\sigma}_0^1 \cdot (\boldsymbol{F}_0^1)^{-T} \tag{14}$$

The above equations are complete when the Cauchy stress is linked to a deformation measure through a constitutive equation. The constitutive equation defines the material characteristics and here, we will only discuss the case of an hyperelastic material (Ottosen and Ristinmaa, 2005) whose constitutive equation is given by:

$$\boldsymbol{\sigma}_0^1 = \boldsymbol{F}_0^1 \cdot \boldsymbol{C} : \boldsymbol{E}_0^1 \cdot (J_0^1)^{-1} (\boldsymbol{F}_0^1)^T \tag{15}$$

where $\boldsymbol{C}$ is the fourth order isotropic material tensor and $\boldsymbol{E}$ is the Green Lagrangean Strain Tensor (Ogden, 1984).

The above system of equations (11)-(15) are fully objective and they let us to describe the dynamics and finite deformations of such hyperelastic material body. Since we are interested in determine equilibrium states only, we can drop the acceleration and velocity terms from eq. (12):

$$\rho^1 J_0^1 = \rho^0 \tag{16}$$

$$\nabla^0 \cdot (\boldsymbol{P}_0^1) + \rho^0 \boldsymbol{b}_0^1 + \boldsymbol{f}_0^1 = \boldsymbol{0} \tag{17}$$

The above equations will be solved using a Finite Element Method (FEM) approach, the discretization methodology will be presented in the next section. The developed formulation is general and not limited to classical small-deformations or linear-elasticity theory (Oñate, 1995).

3 A FINITE ELEMENT METHOD FORMULATION

3.1 Weak Form

FEM is based on a weak form obtained by integration of the differential equations. Using a weighting functions defined in the reference domain:

$$W = W(\boldsymbol{x}^0) \tag{18}$$

we get that:

$$W\rho^1 J_0^1 - W\rho^0 J_0^0 = 0 \tag{19}$$

$$W\nabla^0 \cdot (\boldsymbol{\sigma}_0^1 \cdot J_0^1 \boldsymbol{F}_0^{1^{-T}}) + W(\rho^0 \boldsymbol{b}_0^1 + \boldsymbol{f}_0^1) = \boldsymbol{0} \tag{20}$$

Integration over v^0, leads to:

$$\int_{v^0} W\rho^1 J_0^1 dv^0 - \int_{v^0} W\rho^0 J_0^0 dv^0 = 0 \tag{21}$$

$$\int_{v^0} W\nabla^0 \cdot (\boldsymbol{\sigma}_0^1 \cdot J_0^1 \boldsymbol{F}_0^{1^{-T}}) dv^0 + \int_{v^0} W(\rho^0 \boldsymbol{b}_0^1 + \boldsymbol{f}_0^1) dv^0 = \boldsymbol{0} \tag{22}$$

Integration by parts of the stress terms in the momentum equation leads to

$$-\int_{v^0}(\boldsymbol{\sigma}_0^1\cdot J_0^1\boldsymbol{F}_0^{1-T})\cdot\nabla^0 W dv^0+\int_{a^1}W(\boldsymbol{\sigma}_0^1\cdot\boldsymbol{n}^1)da^1+\int_{v^0}W(\rho^0\boldsymbol{b}_0^1+\boldsymbol{f}_0^1)dv^0=\boldsymbol{0} \qquad (23)$$

using the external traction forces:

$$\boldsymbol{t}^1=\boldsymbol{\sigma}_0^1\cdot\boldsymbol{n}^1 \qquad \text{on traction surfaces } a_{\boldsymbol{t}}^1$$

and the compatibility condition of weighting functions on the remaining boundary surface we get that:

$$\boldsymbol{R}_{\text{mass}}=\int_{v^0}W\rho^1J_0^1dv^0-\int_{v^0}W\rho^0J_0^0dv^0=0 \qquad (24)$$

$$\boldsymbol{R}_{\text{mom}}=-\int_{v^0}(\boldsymbol{\sigma}_0^1\cdot J_0^1\boldsymbol{F}_0^{1-T})\cdot\nabla^0 W dv^0+\int_{a_{\boldsymbol{t}}^1}W\boldsymbol{t}^1da^1+\int_{v^0}W(\rho^0\boldsymbol{b}_0^1+\boldsymbol{f}_0^1)dv^0=\boldsymbol{0} \qquad (25)$$

3.2 Spatial Discretization - Galerkin Approach

The reference domain v^0 is discretized in a mesh formed by the union of tetrahedron (triangular) elements in 3D (2D), the mesh is defined through the position coordinates $\hat{\boldsymbol{x}}^0$ of its nodes. In such elements it is possible to defined piece-wise linear functions which form the basis functions of weighting functions $W$and of interpolation functions N. Then, denoting any nodal value by α we have that these functions can be written as:

$$W=W_\alpha(\hat{\boldsymbol{x}}^0;\boldsymbol{x}^0) \qquad (26)$$

$$N=N_\alpha(\hat{\boldsymbol{x}}^0;\boldsymbol{x}^0) \qquad (27)$$

where in the above eqs. we have written explicitly the dependence of these piece-wise linear functions in terms of the nodal coordinates (compare eq. (26) with eq. (18)). Now, when subject to external loads and body forces the structure initially defined in the domain v^0 deforms occupying the deformed domain v^1. Such deformed domain is defined by the union of the deformed tetrahedron elements which result from the mapping of the reference nodal positions $\hat{\boldsymbol{x}}^0$ into deformed nodal positions $\hat{\boldsymbol{x}}^1$. Mathematically we have:

$$\boldsymbol{x}^1=N_\alpha\hat{\boldsymbol{x}}_\alpha^1 \qquad (28)$$

If we make explicit the dependence on nodal values of the interpolating function (eq. (27)), we have that:

$$\boldsymbol{x}^1=\boldsymbol{x}^1(\hat{\boldsymbol{x}}^0,\hat{\boldsymbol{x}}^1;\boldsymbol{x}^0) \qquad (29)$$

From eq. (29), it follows that all physical quantities (including the deformation mapping $\boldsymbol{F}_0^1$ and the Cauchy stress $\boldsymbol{\sigma}_0^1$) intervening in the eqs. of equilibrium (24)-(25) have the same type of functional dependence. Since eqs. (24)-(25) are integrated over the reference space $\boldsymbol{x}^0$, the functional dependence on this variable dissapears and we can write them in the following compact way:

$$\boldsymbol{R}_{\text{mass}}(\hat{\boldsymbol{x}}^0,\hat{\boldsymbol{x}}^1)=0 \qquad (30)$$

$$\boldsymbol{R}_{\text{mom}}(\hat{\boldsymbol{x}}^0,\hat{\boldsymbol{x}}^1)=\boldsymbol{0} \qquad (31)$$

4 NUMERICAL SOLUTIONS

4.1 The standard methods: direct analysis tools

In the standard methods the initial undeformed configuration of the structure is specified and we want to determine how much and how the structure will deform when certain external loads are applied. This means that eqs. (30)-(31) are of the type:

$$\boldsymbol{R}_{\text{mass}}(\hat{\boldsymbol{x}}^1) = 0 \tag{32}$$

$$\boldsymbol{R}_{\text{mom}}(\hat{\boldsymbol{x}}^1) = \boldsymbol{0} \tag{33}$$

The above non-linear equations can be solved numerically (using Newton's Method) to determine the deformation positions $\hat{\boldsymbol{x}}^1$ and density ρ^1 and from these the remaining physical variables in the deformed configuration. In particular from eq. (15), we can determine the Cauchy stresses and from them the corresponding von-Mises stresses.

Let us consider two examples.

The first example consists of a perfectly rectangular hyperelastic beam with a squared section of sides S=1.0m and lenght L=4.0m which is clamped to the wall on its left side as Figure 1a, shows. The flexible beam is loaded with two nodal forces acting on the lower corners of the right side. The problem consists of predicting what will be the deformation of the beam as a consequence of the imposed loads. The results are shown in Figure 1b, and were obtained using the `MulPhys`' static-solver (Limache, 2008; Alves de Queiroz and Limache, 2007). Because of the external forces the flexible beam curves down having a deflection at the tip of $\delta = -1.035$. Figures 1a and 1b also show the initial and final von-Mises stresses, respectively, predicted by the numerical simulation.

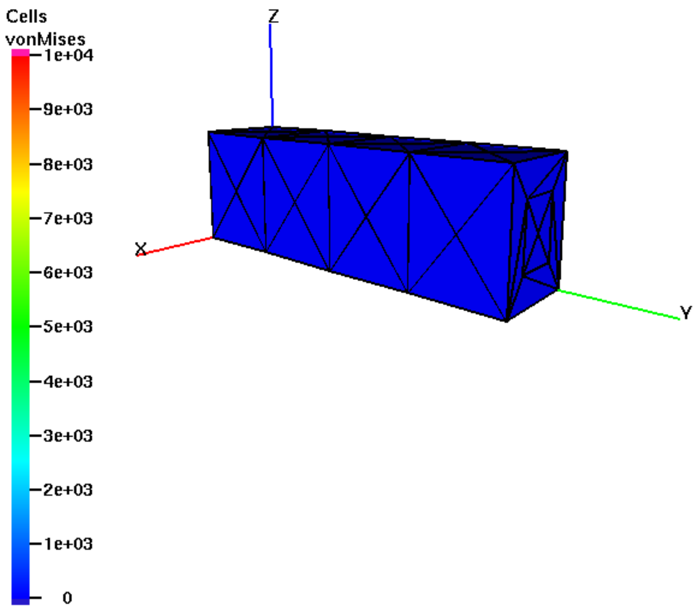
The second example consist of an airplane wing formed by NACA 0012 airfoils of chord = 1.0m and a wing span of 5.0m, as shown in Figure 2a. To simulate the application of aerodynamic forces the wing was loaded with two downside external forces acting on the the wing tips. The problem consists of predicting what will be the deformation of the wing as a consequence of the imposed loads. The results are shown in Figure 2b, and where obtained using the static solver of `MulPhys`. As it can be seen because of the external forces the flexible wing combs down along its tips. Figures 2a and 2b also show the initial and final von-Mises stresses, respectively, predicted by the numerical simulation.

4.2 The inverse method: Test Cases

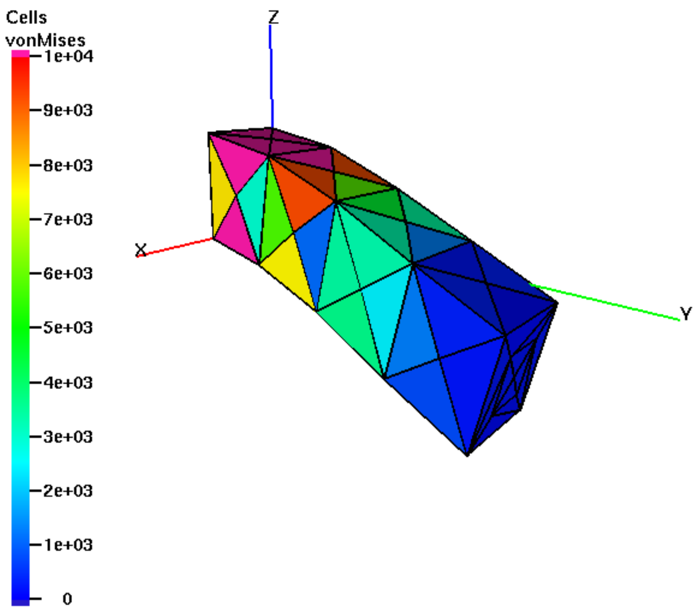
In the inverse method presented in this article, one specifies the deformed configuration and the developed formulation allows one to determine what is the corresponding unloaded (undeformed) geometry. Here, we will test the inverse design method by checking that when the loaded configurations (i.e. Figures 1b and 2b) are given as input to the inverse program, the output will be the corresponding unloaded structures (i.e. Figures 1a and 2a). Mathemathically speaking, this means that we switch the state variables in eqs. (30)-(31). In the inverse case, we have that $\hat{x}^1$ are specified and $\hat{\boldsymbol{x}}^0$ become the unknown state. This means that eqs. (30)-(31) become of the type.

$$\boldsymbol{R}_{\text{mass}}(\hat{\boldsymbol{x}}^0) = 0 \tag{34}$$

$$\boldsymbol{R}_{\text{mom}}(\hat{\boldsymbol{x}}^0) = \boldsymbol{0} \tag{35}$$

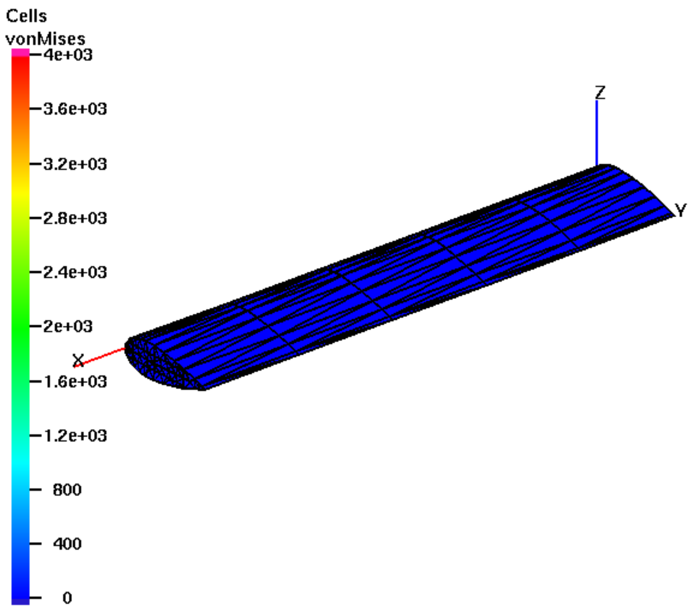


(a) INPUT: Unloaded straight beam

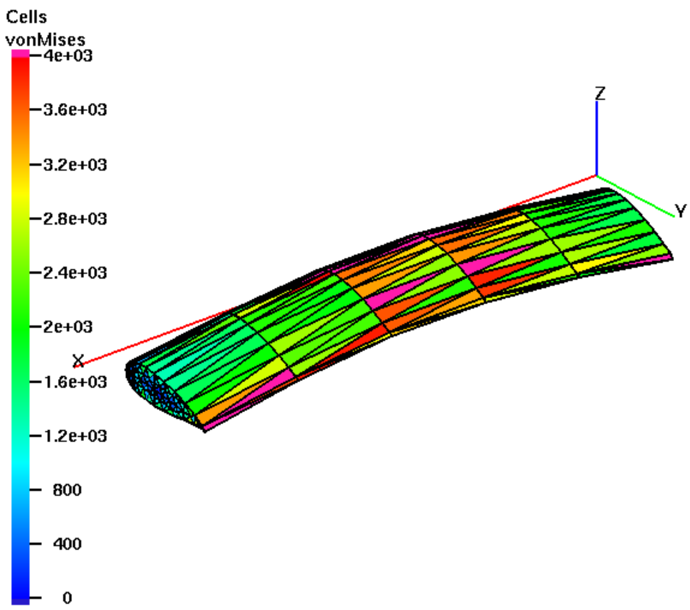


(b) OUTPUT: Loaded bended beam

Figure 1: Direct Analysis Beam Problem.



(a) INPUT: Unloaded straight wing



(b) OUTPUT: Loaded combed wing

Figure 2: Direct Analysis Wing Problem

Again, these non-linear equations can be solved using multi-dimensional root finding algorithms like Newton's Method. The mathematical formulation of this procedure has been implemented into `MulPhys`, so the user can change from direct analysis to inverse design, quickly and easily. The non-linear equations are solved in `MulPhys` using Newton's method and a mixed scheme for computing the involved Jacobians. The numerical results for the proposed test cases are shown in Figures 3 and 4. Figure 3b shows how the straight bar defined in Figure 1a is recovered when the loaded structure is supplied (Figure 3a). Note in Figure 3a that the input stress for the algorithm are initially zero. Figure 4b shows how the straight wing defined in Figure 2a is recovered when the loaded combed upwards wing structure is supplied (Figure 4a). Note in Figure 4a that the input stress for the algorithm are initially zero. As can be seen the method works successfully.

4.3 The inverse method: Design Cases

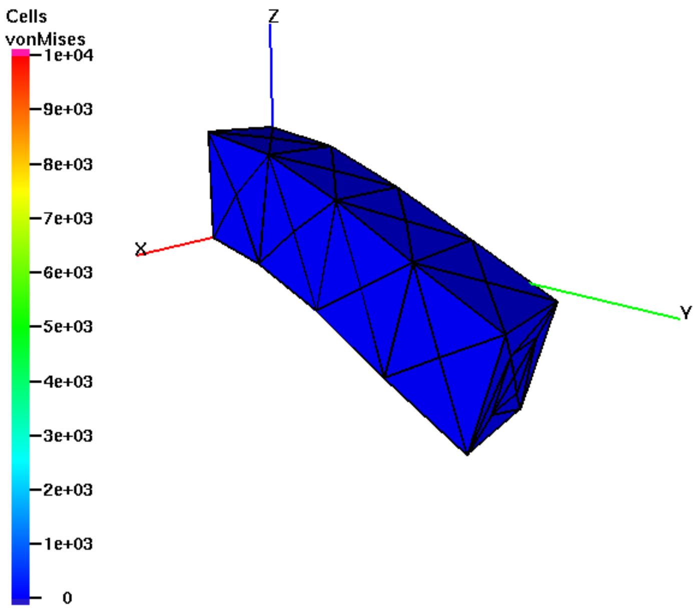
Here, we present two more interesting and practical design cases derived of the examples discussed above. In practical manufacturing processes usually one wants to have structures that once loaded and deformed by external forces, adquires predefined shapes.

Let us consider first the clamped beam problem described in Sections 4.1. Most probably a civil engineer would like that the beam have a straight rectangular shape once it is loaded. He can not build directly the rectangular beam shown in Figure 1a, because once loaded the beam will bend aquiring the shape shown in Figure 1b.

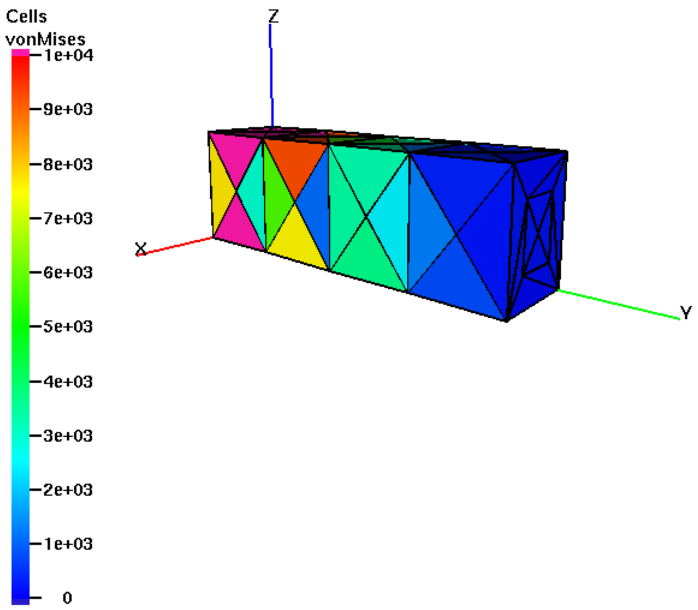
The question is: *what would have to be the shape of a clamped beam, so as to when loaded with specific external forces will adquire the form of a perfectly rectangular structure?*. Using the inverse method presented in this article, one can easily determined such shape. The procedure is simple. Set the *inverse flag* in `MulPhys` and provide, as INPUT, the desired shape with the desired density distribution. In our case, the desired shape is the rectangular beam shown in Figure 5a. Then the program determines for us what is the actual shape of the beam that needs to be manufactured. In our case the computed shape is shown in Figure 5b. A direct analysis evaluation of the bended-up beam shown in Figure 5b proves that in fact when such beam gets loaded the straight rectangular beam is obtained.

Finally let us consider a structural wing desing problem. An aeronautical engineer would like to have, under cruise flight conditions, a constant chord wing made of NACA 0012 airfoils. He can not build directly this ideal wing (shown in Figure 2a), because under the aerodynamic forces will bend or twist aquiring a deformed shape like the one shown in Figure 2b.

The question is: *what would have to be the geometry of the wing that needs to be builded, so as to when loaded with specific external forces will adquire the form of a straight wing with constant chord NACA 0012 airfoils?*. Using the inverse method presented in this article, one can easily determined such shape. The procedure is simple. Set the *inverse flag* in `MulPhys` and provide, as INPUT, the desired wing shape with the desired density distribution. In our case the desired shape is the straight wing shown in Figure 6a. Then the program determines for us what should be the geometry that will need to be manufactured. For our example, the designed geometry is shown in Figure 6b. A direct analysis evaluation of the combed-up wing shown in Figure 6b proves that in fact when such beam gets loaded the straight wing is obtained.

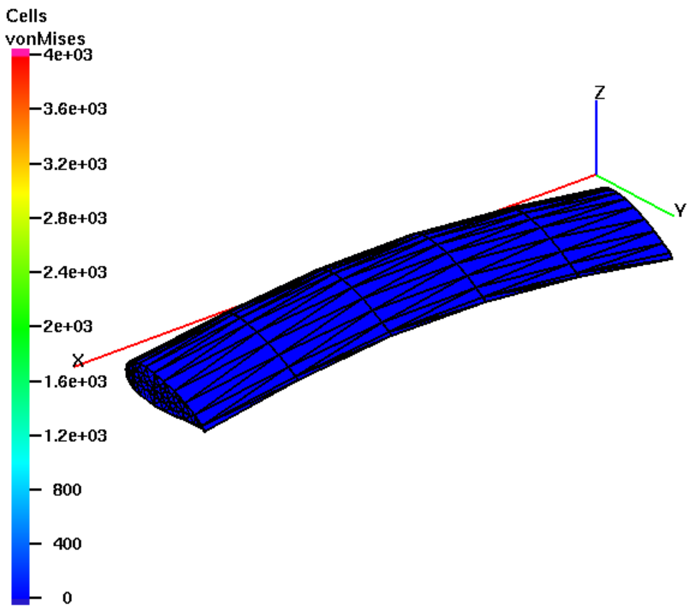


(a) INPUT: loaded down-bended beam

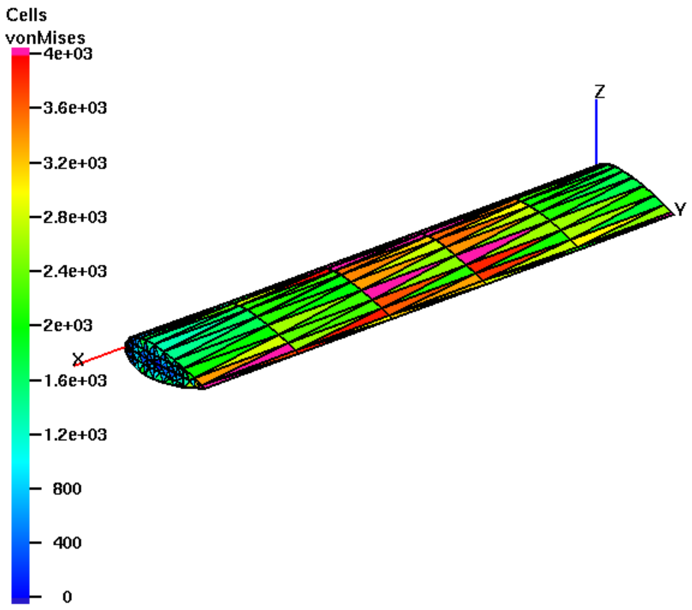


(b) OUTPUT: Unloaded rectangular beam

Figure 3: Inverse Method Beam Test Problem

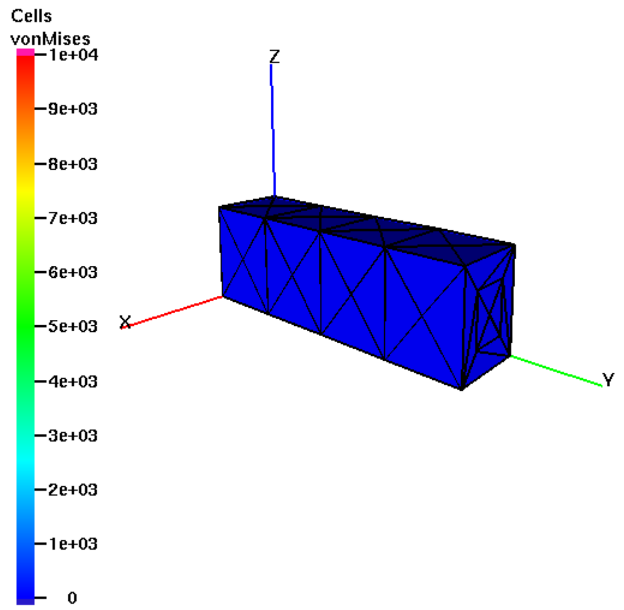


(a) INPUT: loaded downwards-combed wing

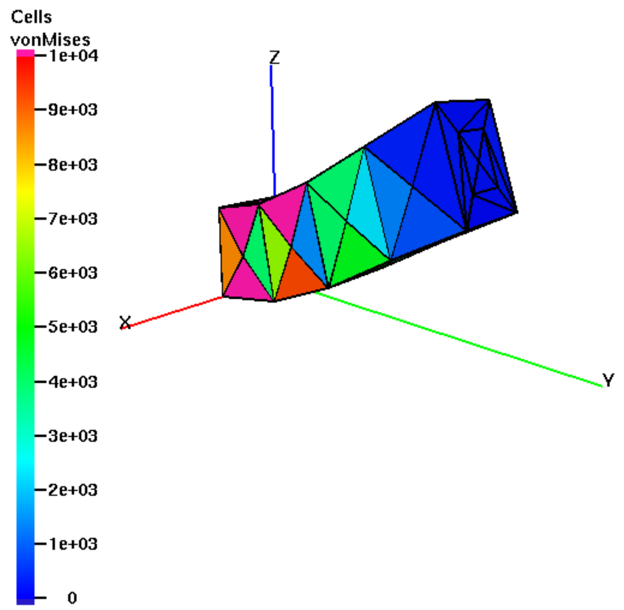


(b) OUTPUT: Unloaded straight wing

Figure 4: Inverse Method Wing Test Problem

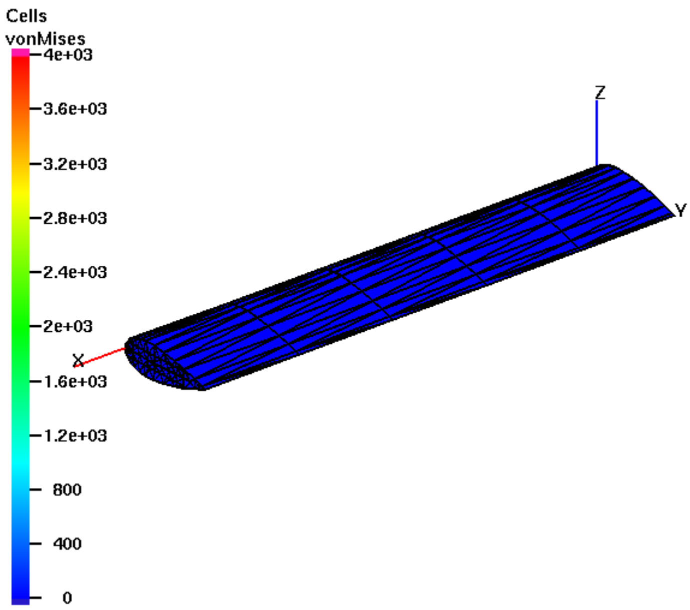


(a) INPUT: loaded straight beam

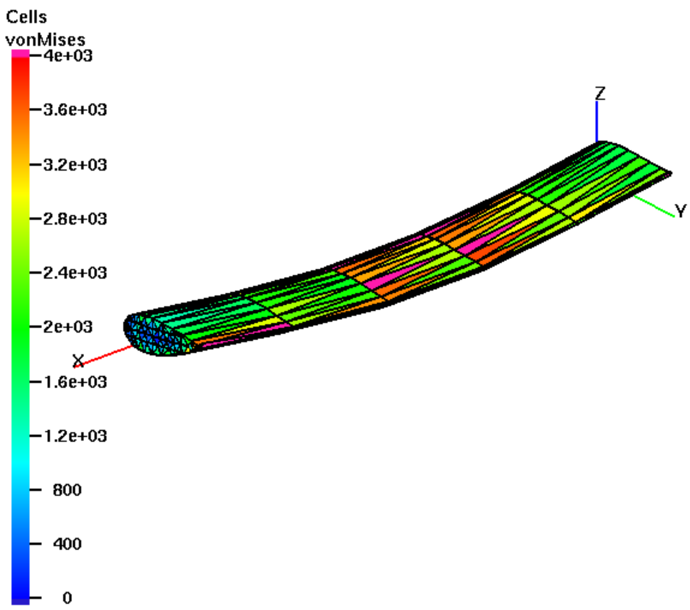


(b) OUTPUT: Unloaded upwards-bended beam

Figure 5: Inverse Method Beam Design Problem



(a) INPUT: loaded straight wing



(b) OUTPUT: unloaded upwards-combed wing

Figure 6: Inverse Method Wing Design Problem

5 CONCLUSIONS

An inverse method for the design of structures has been presented. The method is based on a novel FEM formulation. General continuum mechanics has been incorporated to the formulation, so the method is not restricted to the classical small perturbation, linear elasticity theory and can handle large deformable structural design. The method is a extremely promising tool for multiple engineering areas and manufacturing processes. In this article examples have been presented oriented to the design of beams and airplane wings.

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