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A FULLY COUPLED FORMULATION FOR INCOMPRESSIBLE FLUID-ELASTIC STRUCTURE-INTERACTIONS

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Abstract. *We present a general formulation for analysis of fluid flows with structural interactions using the particle finite element method (PFEM). The fluids are fully coupled to the structures that can undergo highly non-linear response due to large deformations. The key feature of the PFEM is the use of an updated Lagrangian description to model the motion of nodes(particles) in both the fluid and the structure domains. A mesh connects the nodes defining the discretized domains where the governing equations, expressed in an integral form are solved as in the standard FEM. A fractional step scheme for the transient coupled fluid-structure solution is described. Examples of application of the PFEM method to solve a number of fluid-structure interaction problems including free-fluid-surfaces, breaking waves and fluid particle separation may be easily solved with this formulation are presented.*

1 INTRODUCTION

The analysis of multiphysics problems, and specifically the solution of fluid-structure interactions, has been given increased attention during recent years. There are many problems where a direct fully coupled analysis is needed to model the physics of a fluid-structure interaction (FSI) problem accurately. This is particularly the case when the structure undergoes large deformations in the interaction with the fluid. In this paper, a Lagrangian method will be used together with a particular form of finite element method (FEM) called the Particle FEM (PFEM).[1,2] The approach presented in this paper solves the FSI in a monolithic fashion. The method combines the best features of particle (meshless) techniques and finite element methods. In the next section the basis of the Lagrangian and Eulerian descriptions are briefly presented. Next the constitutive equations are derived. Then the discretization of these equations in space and time is then discussed. Finally, numerical results in two dimensions problems are presented.

2 EQUATIONS OF MOTION

Neglecting thermomechanical effects, the motion of the elemental particles of any continuum material (fluid, solid or gas) can be determined by the equations of conservation of mass and momentum. These equations can be postulated as the continuum form of classical Newton's equations of motion and are independent of the material constitution or geometry. Using an Eulerian description, i.e. looking at the elemental particles as they pass thru fixed positions in the physical space, the equations of conservation of mass and momentum can be stated in local form as:

$$\frac{D\rho\left(\boldsymbol{x},t\right)}{Dt}=-\rho\left(\boldsymbol{x},t\right)\operatorname{div}\boldsymbol{V}\left(\boldsymbol{x},t\right) \tag{1}$$

$$\rho\left(\boldsymbol{x},t\right)\frac{D\boldsymbol{V}\left(\boldsymbol{x},t\right)}{Dt}=\operatorname{div}\sigma\left(\boldsymbol{x},t\right)+\rho\left(\boldsymbol{x},t\right)\mathbf{b}\left(\boldsymbol{x},t\right) \tag{2}$$

where $\sigma\left(\boldsymbol{x},t\right)$ is the Cauchy Stress, and, $\rho\left(\boldsymbol{x},t\right)$ and $\boldsymbol{V}\left(\boldsymbol{x},t\right)$ are the density and the velocity vector of the elemental particles that at time t pass thru the position $\boldsymbol{x}$, respectively. $\frac{D\phi}{Dt}$ denotes the total or material time-derivative of ϕ. In the absence of couple stresses the Cauchy Stress is a second order symmetric tensor. In the Eulerian description the divergence div is taken with respect to the spatial coordinates $\boldsymbol{x}$. For example, div computed in cartesian coordinates $\boldsymbol{x}=x_k\boldsymbol{e}_k$ is:

$$\operatorname{div}\boldsymbol{V}=\frac{\partial V_k}{\partial x_k}$$

At any time t equations (1) and (2) can be solved on the whole volume Ω_t occupied by the material body, provided one knows the external body forces $\mathbf{b}$ (per unit of mass) and surface forces $\bar{\boldsymbol{t}}$.

We can write equivalent conservation equations using a Lagrangian Formulation. In the Lagrangian description, one essentially follows the history of individual particles. Consequently, two independent variables are taken: the time t of evolution and a vector-type label associated to each particle. The label can be conveniently taken as the reference position vector $\boldsymbol{X}$ of

the particle at some initial reference time $t = 0$. In this description, then, any quantity M is expressed as $M = M(\boldsymbol{X}, t)$. In particular, the particle position is written as

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t) \tag{3}$$

which represents the location at t of the particle whose position was $\boldsymbol{X}$ at $t = 0$.

The knowledge of the deformation map χ for all the elemental particles completely defines the motion and deformation of the material body. In particular the velocities can be computed by:

$$\boldsymbol{V}(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{\chi}(\boldsymbol{X}, t)}{\partial t} = \frac{D\boldsymbol{x}}{Dt} \tag{4}$$

Assuming the existence of the deformation map it is possible to rewrite the conservation equations (1) and (2) in terms of the reference coordinates $\mathbf{X}$:

$$\rho(\boldsymbol{x}, t) = \rho\left(\boldsymbol{\chi}(\boldsymbol{X}, t)\right) = \frac{\rho_0(\boldsymbol{X}, t)}{J(\boldsymbol{X}, t)} \tag{5}$$

$$\rho_0(\boldsymbol{X}, t)\frac{D\boldsymbol{V}(\boldsymbol{X}, t)}{Dt} = \mathrm{Div}\, \boldsymbol{P}(\boldsymbol{X}, t) + \rho_0 \boldsymbol{f}(\boldsymbol{X}, t) \tag{6}$$

where $\rho_0(\boldsymbol{X}, t)$ is the density of the material in the reference configuration, $J(\boldsymbol{X}, t)$ is the Jacobian

$$J(\boldsymbol{X}, t) = \det \boldsymbol{F}(\boldsymbol{X}, t) \tag{7}$$

of the deformation gradient

$$\boldsymbol{F}(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \mathrm{Grad}(\boldsymbol{\chi}(\boldsymbol{X}, t)) \tag{8}$$

and $\boldsymbol{P}(\boldsymbol{X}, t)$ is the First Piola-Kirchhoff Stress tensor

$$\boldsymbol{P}(\boldsymbol{X}, t) = J(\boldsymbol{X}, t)\boldsymbol{\sigma}(\boldsymbol{\chi}(\boldsymbol{X}, t)) \cdot \boldsymbol{F}^{-T}(\boldsymbol{X}, t) \tag{9}$$

Note that the differential operators divergence and gradient are written in lower-case letters $(\mathrm{div}, \mathrm{grad})$ when computed with respect to spatial coordinates x_k and in capital letters $(\mathrm{Div}, \mathrm{Grad})$ when computed with respect to material coordinates X_k.

From now on, we simplify the notation by writing the physical quantities without referencing their explicit dependence on time and on material or spatial coordinates. Then, the conservation equations (5) and (6) can be rewritten simply as:

$$\rho = \frac{\rho_0}{J} \tag{10}$$

$$\rho_0 \frac{D\boldsymbol{V}}{Dt} = \mathrm{Div}\, \boldsymbol{P} + \rho_0 \boldsymbol{f} \tag{11}$$

with the Jacobian given by:

$$J = \det \boldsymbol{F} \tag{12}$$

and the First Piola-Kirchhoff Stress by:

$$\boldsymbol{P} = J\boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} \tag{13}$$

Observe that in the Lagrangian formulation, conservation of mass (10) is a direct consequence of a straightforward computation of the density in terms of the Jacobian and of the initial density configuration ρ_0. The momentum equation has to be solved in conjunction with the boundary conditions representing the external loads and motion restrictions under which the material body is moving. In most of the cases one specifies surface forces t at certain surfaces Γ_σ of the boundary Γ of the body and impose prescribed displacements in the remaining parts $\Gamma_{\boldsymbol{U}}$ of the material's boundary:

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{t}} \,\text{on}\, \Gamma_{\boldsymbol{\sigma}} \tag{14}$$

$$\boldsymbol{U} = \bar{\boldsymbol{U}} \,\text{on}\, \Gamma_{\boldsymbol{U}} \tag{15}$$

3 TIME-DISCRETIZATION OF EQUATIONS OF MOTION USING AN LAGRANGIAN FORMULATION

To obtain numerical solutions to equations (10)-(15) we use a Lagrangian Formulation. The formulation can be made either Total Lagrangian or Updated Lagrangian. In the former the reference configuration is the initial one, in the second the reference configuration is chosen to be the most recent computated configuration and it is updated every time step. In this paper we will use the Updated Lagrangian approach only. Having chosen a reference configuration which corresponds to a volume Ω_0 with a density distribution ρ_0 a fully implicit time-discretization of equations (10) and (11) is:

$$\rho^{(n+1)} = \frac{1}{J^{(n+1)}}\rho_0 \tag{16}$$

$$\rho_0 \frac{\boldsymbol{V}^{(n+1)} - \boldsymbol{V}^{(n)}}{\triangle t} = \text{Div}\left(J^{(n+1)}\boldsymbol{\sigma}^{(n+1)} \cdot \boldsymbol{F}^{-T(n+1)}\right) + \rho_0 \boldsymbol{b}^{(n+1)} \tag{17}$$

The above equations are solved in the reference volume Ω_0 and the boundary conditions imposed on the reference volume's boundary $\Gamma_0 = \Gamma_{0\boldsymbol{U}} + \Gamma_{0\boldsymbol{\sigma}}$.

Equations (10)-(15) or their discrete form (16)-(17) are independent of the material, so we need to incorporate the information of the specific material we want to model. This is done by means of specifying the material's constitutive equation, which basically links the stress tensor to the deformation. In the context of the present paper we treat the problem of simulating incompressible newtonian fluids and isotropic hypoelastic solids either separately or in the form of a fluid-structure interaction. Accordingly in the next section we discuss their constitutive equations.

4 CONSTITUTIVE EQUATIONS

4.1 Constitutive Equation of Incompressible or Nearly-Incompressible Newtonian Fluids

We treat the incompressible fluid as the limit case of a nearly-incompressible newtonian flow. The constitutive equation of a newtonian fluid is a linear relationship between the Cauchy Stress $\boldsymbol{\sigma}$ and the Deformation rate tensor $\boldsymbol{D}$:

$$\boldsymbol{\sigma} = -p\boldsymbol{I} - \frac{2}{3}\mu_f(\operatorname{tr}\boldsymbol{D})\boldsymbol{I} + 2\mu_f\boldsymbol{D} \tag{18}$$

where μ_f is the fluid viscosity and p is the thermodynamic (or hydrostatic) pressure of the fluid. By definition $\boldsymbol{D}$ is the symmetric part

$$\boldsymbol{D} = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T) \tag{19}$$

of the velocity gradient tensor:

$$\boldsymbol{L} = \operatorname{grad}\boldsymbol{V} \tag{20}$$

Note that:

$$\operatorname{tr}\boldsymbol{D} = \operatorname{div}\boldsymbol{V} = \boldsymbol{\varepsilon_v} \tag{21}$$

If $\boldsymbol{D}$ is decomposed in its volumetric and its deviatoric components:

$$\boldsymbol{D} = \left[\frac{1}{3}(\operatorname{tr}\boldsymbol{D})\boldsymbol{I} + \acute{\boldsymbol{D}}\right] = \left[\frac{1}{3}\varepsilon_v\boldsymbol{I} + \acute{\boldsymbol{D}}\right] \tag{22}$$

We can write (19) as:

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu_f\acute{\boldsymbol{D}} \tag{23}$$

Following reference,[3] if the newtonian flow is nearly-incompressible then density changes are related to pressure changes by the following relationship:

$$dp = \frac{K_f}{\rho}d\rho \tag{24}$$

where K_f is the elastic bulk modulus of the fluid. Then applying the material time derivative to the equation above and using the equation of conservation of mass we get:

$$\frac{Dp}{Dt} = \frac{K_f}{\rho}\frac{D\rho}{Dt} = - \quad K_f \operatorname{div}\boldsymbol{V} \tag{25}$$

4.2 Time-Discretization of the Constitutive Equation of a Newtonian Fluid

A implicit discretization of the fluid's constitutive equation (23) is:

$$\boldsymbol{\sigma}^{(n+1)} = -p^{(n+1)}\boldsymbol{I} + 2\mu_f \acute{\boldsymbol{D}}^{(n+1)} \tag{26}$$

The discretized version of equation (25) for nearly incompressible flows would be:

$$\frac{p^{(n+1)} - p^{(n)}}{\triangle t} = \quad -K_f \operatorname{div}^{(n+1)} \boldsymbol{V}^{(n+1)} \tag{27}$$

So:

$$p^{(n+1)} = \quad p^{(n)} - \triangle t K_f \varepsilon_v^{(n+1)} \tag{28}$$

4.3 Constitutive Equation of Hypoelastic Solids

A isotropic hypoelastic solid is a material whose constitutive equation is given by a linear isotropic relationship between the stress rate and the deformation rate:

$$\overset{\triangle}{\boldsymbol{\tau}} = 2\mu_h \boldsymbol{D} + \lambda_h (\operatorname{tr} \boldsymbol{D})\boldsymbol{I} \tag{29}$$

where $\overset{\triangle}{\boldsymbol{\tau}}$ is the Truesdell rate

$$\overset{\triangle}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \boldsymbol{L}\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot\boldsymbol{L}^T = \boldsymbol{F}\cdot\dot{\boldsymbol{S}}\cdot\boldsymbol{F}^T \tag{30}$$

of the Kirchhoff stress tensor $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ and $\dot{\boldsymbol{S}}$ is the material time-derivative of the Second Piola-Kirchhoff Stress:

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\cdot\boldsymbol{P} = J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T} \tag{31}$$

4.4 Time-discretization of the constitutive equation of an hypoelastic solid

A implicit discretization of the constitutive rate equation is:

$$\overset{\triangle}{\boldsymbol{\tau}}^{(n+1)} = 2\mu_h \boldsymbol{D}^{(n+1)} + \lambda_h (\operatorname{tr} \boldsymbol{D}^{(n+1)})\boldsymbol{I} \tag{32}$$

The Truesdell rate of the Kirchhoff stress is written within the interval as

$$\overset{\triangle}{\boldsymbol{\tau}}^{(n+1)} = \boldsymbol{F}^{(n+1)}\cdot\dot{\boldsymbol{S}}^{(n+1)}\cdot\boldsymbol{F}^{T(n+1)} \tag{33}$$

where $\boldsymbol{F}^{(n+1)}$ is the deformation gradient up to time $t^{(n+1)}$. Applying the generalized midpoint rule for $\dot{\boldsymbol{S}}^{(n+1)}$

$$\dot{\boldsymbol{S}}^{(n+1)} = \frac{\boldsymbol{S}^{(n+1)} - \boldsymbol{S}^{(n)}}{\triangle t} \tag{34}$$

we get

$$\boldsymbol{F}^{(n+1)} \cdot \left(\boldsymbol{S}^{(n+1)} - \boldsymbol{S}^{(n)} \right) \cdot \boldsymbol{F}^{T(n+1)} = 2\mu \triangle t \boldsymbol{D}^{(n+1)} + \lambda \triangle t (\operatorname{tr} \boldsymbol{D}^{(n+1)}) \boldsymbol{I} \tag{35}$$

using that $\boldsymbol{F} \cdot \boldsymbol{S} \cdot \boldsymbol{F}^T = J\boldsymbol{\sigma}$ and the so-called incremental deformation tensor $\boldsymbol{f}^{(n+1)}$ defined as:

$$\boldsymbol{f}^{(n+1)} \cdot \boldsymbol{F}^{(n)} = \boldsymbol{F}^{(n+1)} \tag{36}$$

we get:

$$\begin{aligned} J^{(n+1)} \boldsymbol{\sigma}^{(n+1)} = J^{(n)} \boldsymbol{f}^{(n+1)} \cdot \boldsymbol{\sigma}^{(n)} \cdot \boldsymbol{f}^{T(n+1)} + \\ \lambda_h \triangle t (\operatorname{tr} \boldsymbol{D}^{(n+1)}) \boldsymbol{I} + 2\mu_h \triangle t \boldsymbol{D}^{(n+1)} \end{aligned} \tag{37}$$

The equation (37) can be applied for both formulations. However in this paper we will use the Updated Lagrangian approach, then it suffices to replace $\boldsymbol{f}^{(n+1)}$ by $\boldsymbol{F}^{(n+1)}$ and $J^{(n)}$ by 1. Defining

$$\bar{\mu}_h^{(n+1)} = \frac{\mu_h \triangle t}{J^{(n+1)}}, \bar{\lambda}_h^{(n+1)} = \frac{\lambda_h \triangle t}{J^{(n+1)}} \tag{38}$$

and the tensor

$$\widehat{\boldsymbol{\sigma}}_h^{(n+1)} = \frac{1}{J^{(n+1)}} \boldsymbol{F}^{(n+1)} \cdot \boldsymbol{\sigma}^{(n)} \cdot \boldsymbol{F}^{T(n+1)} \tag{39}$$

we obtain the stress update expression:

$$\boldsymbol{\sigma}^{(n+1)} = \hat{\boldsymbol{\sigma}}_h^{(n+1)} + \bar{\lambda}_h^{(n+1)} (\operatorname{tr} \boldsymbol{D}^{(n+1)}) \boldsymbol{I} + 2\bar{\mu}_h^{(n+1)} \boldsymbol{D}^{(n+1)} \tag{40}$$

Decomposing the deformation rate tensor in terms of their deviatoric and volumetric part in the above equation, we obtain the following alternative expression:

$$\boldsymbol{\sigma}^{(n+1)} = \hat{\boldsymbol{\sigma}}_h^{(n+1)} + \left(\bar{\lambda}_h^{(n+1)} + \frac{2}{3} \bar{\mu}_h^{(n+1)} \right) (\boldsymbol{\varepsilon_v}^{(n+1)}) \boldsymbol{I} + 2\bar{\mu}_h^{(n+1)} \acute{\boldsymbol{D}}^{(n+1)} \tag{41}$$

Defining a hypoelastic volumetric viscosity:

$$\kappa_h = \bar{\lambda}_h + \frac{2}{3} \bar{\mu}_h \tag{42}$$

We get:

$$\boldsymbol{\sigma}^{(n+1)} = \hat{\boldsymbol{\sigma}}_h^{(n+1)} + \kappa_h^{(n+1)} \boldsymbol{\varepsilon_v}^{(n+1)} \boldsymbol{I} + 2\bar{\mu}_h^{(n+1)} \acute{\boldsymbol{D}}^{(n+1)} \tag{43}$$

If we decompose $\hat{\boldsymbol{\sigma}}_h^{(n+1)}$ in its deviatoric and volumetric part

$$\hat{\boldsymbol{\sigma}}_h^{(n+1)} = -\hat{p}_h^{(n+1)} \boldsymbol{I} + \acute{\hat{\boldsymbol{\sigma}}}_h^{(n+1)} \tag{44}$$

We have:

$$\boldsymbol{\sigma}^{(n+1)} = -\hat{p}_h^{(n+1)} + \kappa_h^{(n+1)} \boldsymbol{\varepsilon_v}^{(n+1)} \boldsymbol{I} + 2\bar{\mu}_h^{(n+1)} \acute{\boldsymbol{D}}^{(n+1)} + \acute{\hat{\boldsymbol{\sigma}}}_h^{(n+1)} \tag{45}$$

From the application of the trace operator to the equation above we get the following useful relationship:

$$p^{(n+1)} = \hat{p}_h^{(n+1)} - \kappa_h^{(n+1)} \boldsymbol{\varepsilon_v}^{(n+1)} \tag{46}$$

where $p^{(n+1)}$ is the mean pressure:

$$p = -\frac{1}{3} \operatorname{tr} \boldsymbol{\sigma}$$

of the stress tensor $\boldsymbol{\sigma}^{(n+1)}$. Then,

$$\boldsymbol{\sigma}^{(n+1)} = -p^{(n+1)} \boldsymbol{I} + 2\bar{\mu}_h^{(n+1)} \acute{\boldsymbol{D}}^{(n+1)} + \acute{\hat{\boldsymbol{\sigma}}}_h^{(n+1)} \tag{47}$$

This is a discrete version of the constitutive equation of an hypoelastic material.

4.5 A unique constitutive formulation for a Newtonian Fluid and a Hypoelastic Solid

Now we make use of the similarity between the constitutive equation of the hypoelastic solid (47) and the constitutive equation of a Newtonian fluid (26) so as to write a unique constitutive equation for both materials. The only differences between the constitutive equations are the additional term $\acute{\hat{\boldsymbol{\sigma}}}_h^{(n+1)}$ for the hypoelastic solid and the fact that their Lamé Coefficients are not longer constant. As a consequence we can write the following general expressions for both materials:

$$\boldsymbol{\sigma}^{(n+1)} = -p^{(n+1)} \boldsymbol{I} + 2\mu^{(n+1)} \acute{\boldsymbol{D}}^{(n+1)} + \acute{\hat{\boldsymbol{\sigma}}}^{(n+1)} \tag{48}$$

with

$$p^{(n+1)} = \hat{p}^{(n+1)} - \mathcal{K} \boldsymbol{\varepsilon_v}^{(n+1)} \tag{49}$$

where

$$\mu^{(n+1)} = \begin{cases} \bar{\mu}_h = \frac{\mu_h \triangle t}{J^{(n+1)}} \text{ if Hypoelastic solid} \\ \\ \mu_f^{(n+1)} \text{ if Newtonian fluid} \end{cases}$$

$$\widehat{p}^{(n+1)} = \begin{cases} \hat{p}_h^{(n+1)} \text{ if Hypoelastic solid} \\ \\ p^{(n)} \text{ if Newtonian fluid} \end{cases}$$

$$\mathcal{K} = \begin{cases} \frac{\lambda_h \triangle t}{J^{(n+1)}} + \frac{2}{3} \frac{\mu_h \triangle t}{J^{(n+1)}} \text{ if Hypoelastic solid} \\ \\ \triangle t K_f \text{ if Newtonian fluid} \end{cases}$$

$$\hat{\boldsymbol{\sigma}}^{(n+1)} = \begin{cases} \frac{1}{J^{(n+1)}} \boldsymbol{F}^{(n+1)} \cdot \boldsymbol{\sigma}^{(n)} \cdot \boldsymbol{F}^{T(n+1)} \text{ if Hypoelastic solid} \\ p^{(n)} \boldsymbol{I} \text{ if Newtonian fluid} \end{cases} \tag{50}$$

$$\hat{\acute{\boldsymbol{\sigma}}}^{(n+1)} = \begin{cases} \hat{\acute{\boldsymbol{\sigma}}}_h^{(n+1)} \text{ if Hypoelastic solid} \\ 0 \text{ if Newtonian fluid} \end{cases}$$

5 INTEGRATION OF MOTION EQUATIONS

With the discretization of the constitutive equations we can now proceed further with the scheme of numerical integration. Inserting the general constitutive equation (48) into equation (17) and assuming an Updated Lagrangian Approach (i.e. $\rho_0 = \rho^{(n)}$)we get:

$$\begin{aligned} \boldsymbol{V}^{(n+1)} = \boldsymbol{V}^{(n)} - \triangle t \rho^{-1(n)} \operatorname{Div} \left(J^{(n+1)} p^{(n+1)} \boldsymbol{I} \cdot \boldsymbol{F}^{-T(n+1)} \right) + \\ \triangle t \rho^{-1(n)} \operatorname{Div} \left(J^{(n+1)} 2\mu^{(n+1)} \acute{\boldsymbol{D}}^{(n+1)} \cdot \boldsymbol{F}^{-T(n+1)} \right) + \\ \triangle t \rho^{-1(n)} \operatorname{Div} \left(J^{(n+1)} \hat{\acute{\boldsymbol{\sigma}}}^{(n+1)} \boldsymbol{F}^{-T(n+1)} \right) + \triangle t \boldsymbol{b}^{(n+1)} \end{aligned} \tag{51}$$

Now in order to ease readiness, we will drop the upper-index $(n+1)$ from the unknown quantities. So for example $\boldsymbol{F}^{-T(n+1)}$ will be written simply as $\boldsymbol{F}^{-T}$. Note that we keep the index(n) of the known quantities. Using this convention we can rewrite the above equation as:

$$\begin{aligned} \boldsymbol{V} = \boldsymbol{V}^{(n)} - \triangle t \rho^{-1(n)} \operatorname{Div} \left(J p \boldsymbol{I} \cdot \boldsymbol{F}^{-T} \right) + \\ \triangle t \rho^{-1(n)} \operatorname{Div} \left(J 2\mu \acute{\boldsymbol{D}} \cdot \boldsymbol{F}^{-T} \right) + \\ \triangle t \rho^{-1(n)} \operatorname{Div} \left(J \hat{\acute{\boldsymbol{\sigma}}} \boldsymbol{F}^{-T} \right) + \triangle t \boldsymbol{b} \end{aligned} \tag{52}$$

Using (49),we have:

$$\begin{aligned} \boldsymbol{V} = \boldsymbol{V}^{(n)} - \triangle t \rho^{-1(n)} \operatorname{Div} \left(J \hat{p} \boldsymbol{I} \cdot \boldsymbol{F}^{-T} \right) + \triangle t \rho^{-1(n)} \operatorname{Div} \left(J \mu \left(\operatorname{grad} \boldsymbol{V} + \operatorname{grad}^T \boldsymbol{V} \right) \cdot \boldsymbol{F}^{-T} \right) + \\ \triangle t \rho^{-1(n)} \operatorname{Div} \left(J \left(\mathcal{K} - \frac{2}{3} \mu \right) \varepsilon_v \boldsymbol{I} \cdot \boldsymbol{F}^{-T} \right) + \triangle t \rho^{-1(n)} \operatorname{Div} \left(J \hat{\acute{\boldsymbol{\sigma}}} \boldsymbol{F}^{-T} \right) + \triangle t \boldsymbol{b} \end{aligned} \tag{53}$$

5.1 Fractional Step Integration

The last equation is a fully-coupled equation involving four degrees of freedom by node. To reduce the complexity we use a Fractional-Step Method similar to the one proposed in.[4] This basically consists in splitting each time step in two steps as follows.

If we define the variable $\boldsymbol{V}^*$ such that:

$$\begin{aligned}\boldsymbol{V}^* = \boldsymbol{V}^{(n)} - \Delta t \rho^{-1(n)} \operatorname{Div}\left(J\hat{p}\boldsymbol{I}\cdot\boldsymbol{F}^{-T}\right) + \Delta t \rho^{-1(n)} \operatorname{Div}\left(J\mu \operatorname{grad}\boldsymbol{V}\cdot\boldsymbol{F}^{-T}\right) + \\ \Delta t \rho^{-1(n)} \operatorname{Div}\left(J\mu \operatorname{grad}^T\boldsymbol{V}\cdot\boldsymbol{F}^{-T}\right) + \Delta t \rho^{-1(n)} \operatorname{Div}\left(J\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{F}^{-T}\right) + \Delta t\boldsymbol{b}\end{aligned} \tag{54}$$

Then we can write (54) as:

$$\boldsymbol{V} = \boldsymbol{V}^* + \Delta t \rho^{-1(n)} \operatorname{Div}\left(J\left(\mathcal{K} - \frac{2}{3}\mu\right)\varepsilon_v \boldsymbol{I}\cdot\boldsymbol{F}^{-T}\right) \tag{55}$$

We proceed to uncouple the field pressure, by applying the Eulerian divergence div to (55):

$$\operatorname{div}\boldsymbol{V} = \operatorname{div}\boldsymbol{V}^* +$$

$$\Delta t \operatorname{div}\left(\rho^{-1(n)} \operatorname{Div}\left(J\left(\mathcal{K} - \frac{2}{3}\mu\right)\varepsilon_v \boldsymbol{I}\cdot\boldsymbol{F}^{-T}\right)\right) \tag{56}$$

using that

$$\operatorname{div}\boldsymbol{A} = J^{-1}\operatorname{Div}\left(J\boldsymbol{F}^{-1}\boldsymbol{A}\right) = tr\left(\operatorname{Grad}\boldsymbol{A}\boldsymbol{F}^{-1}\right)$$

for any vector $\boldsymbol{A}$ and the fact that $\operatorname{div}\boldsymbol{V} = \varepsilon_{\boldsymbol{v}}$ we get:

$$\varepsilon_{\boldsymbol{v}} = J^{-1}\operatorname{Div}\left(J\boldsymbol{F}^{-1}\boldsymbol{V}^*\right) +$$

$$\Delta t J^{-1} \operatorname{Div}\left(J\boldsymbol{F}^{-1}\rho^{-1(n)} \operatorname{Div}\left(J\left(\mathcal{K} - \frac{2}{3}\mu\right)\varepsilon_v \boldsymbol{I}\cdot\boldsymbol{F}^{-T}\right)\right) \tag{57}$$

Making use of (49) we get

$$-\frac{J}{\mathcal{K}}(p-\hat{p}) + \Delta t \operatorname{Div}\left(J\boldsymbol{F}^{-1}\rho^{-1(n)} \operatorname{Div}\left(J\left(1 - \frac{2\mu}{3\mathcal{K}}\right)(p-\hat{p})\boldsymbol{I}\cdot\boldsymbol{F}^{-T}\right)\right) = \tag{58}$$

$$\operatorname{Div}\left(J\boldsymbol{F}^{-1}\boldsymbol{V}^*\right) \tag{59}$$

Once the intermediate velocity $\boldsymbol{V}^*$ has been computed from equation (54). The pressure is compute from (59) and the end step velocity $\boldsymbol{V}$ is obtained by adding to $\boldsymbol{V}^*$ the dynamical effect of the computed pressure p.

6 FEM: SPATIAL DISCRETIZATION

Multiplying the velocity equations by the velocity weight function W and the pressure equation by the pressure weight function q , the Galerkin weighted residual method to solve the split

equations the following integrals are obtained:

$$\int_{\Omega_n} \boldsymbol{V}^* \cdot \boldsymbol{W} dV_n - \int_{\Omega_n} \Delta t \rho^{-1(n)} \operatorname{Div}\left(J\mu \operatorname{Grad} \boldsymbol{V}^* \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-T}\right) \cdot \boldsymbol{W} dV_n =$$
$$\int_{\Omega_n} \boldsymbol{V}^{(n)} \cdot \boldsymbol{W} dV_n + \int_{\Omega_n} \Delta t \rho^{-1(n)} \operatorname{Div}\left(J\mu \operatorname{Grad}^T \boldsymbol{V}^* \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-T}\right) \cdot \boldsymbol{W} dV_n$$
$$+ \int_{\Omega_n} \Delta t \rho^{-1(n)} \operatorname{Div}\left(J\acute{\hat{\boldsymbol{\sigma}}} \cdot \boldsymbol{F}^{-T}\right) \cdot \boldsymbol{W} dV_n -$$
$$\int_{\Omega_n} \Delta t \rho^{-1(n)} \operatorname{Div}\left(\hat{p}\boldsymbol{I} \cdot \boldsymbol{F}^{-T}\right) \cdot \boldsymbol{W} dV_n + \int_{\Omega_n} \Delta t \boldsymbol{b} \cdot \boldsymbol{W} dV_n +$$
$$\int_{\Gamma_{\sigma n}} \Delta t \rho^{-1(n)} J \left(\boldsymbol{\sigma} - \left(-\hat{p}\boldsymbol{I} + \mu \operatorname{Grad} \boldsymbol{V}^* \cdot \boldsymbol{F}^{-1} + \mu \operatorname{Grad}^T \boldsymbol{V}^* \cdot \boldsymbol{F}^{-1} + \acute{\hat{\boldsymbol{\sigma}}}\right)\right) \cdot \boldsymbol{F}^{-\boldsymbol{T}} \cdot \boldsymbol{N} \cdot \boldsymbol{W} d\Gamma_n \tag{60}$$

$$- \int_{\Omega_n} \frac{1}{\mathcal{K}} (p - \hat{p}) \, q J dV_n +$$
$$\int_{\Omega_n} \Delta t \operatorname{Div}\left(J\rho^{-1(n)} \boldsymbol{F}^{-1} \cdot \operatorname{Div}\left(J\left(1 - \frac{2\mu}{3\mathcal{K}}\right)(p - \hat{p})\, \boldsymbol{I} \cdot \boldsymbol{F}^{-T}\right)\right) q dV_n = \tag{61}$$
$$\int_{\Omega_n} \operatorname{Div}\left(J\boldsymbol{F}^{-1} \cdot \boldsymbol{V}^*\right) q dV_n + \int_{\Gamma_{Vn}} \left(\bar{\boldsymbol{V}} J\boldsymbol{F}^{-T} \cdot \boldsymbol{N} - \boldsymbol{V}^* \cdot J\boldsymbol{F}^{-T} \cdot \boldsymbol{N}\right) q d\Gamma_n$$

$$\int_{\Omega_n} \boldsymbol{V} \cdot \boldsymbol{W} dV_n = \int_{\Omega_n} \boldsymbol{V}^* \cdot \boldsymbol{W} dV_n -$$
$$\int_{\Omega_n} \Delta t \rho^{-1(n)} \left(1 - \frac{2\mu}{3\mathcal{K}}\right) \operatorname{Div}\left(J\,(p - \hat{p})\, \boldsymbol{I} \cdot \boldsymbol{F}^{-T}\right) \cdot \boldsymbol{W} dV_n + \tag{62}$$
$$\int_{\Gamma\sigma_n} \Delta t \rho^{-1(n)} J \left(1 - \frac{2\mu}{3\mathcal{K}}\right)(p - \hat{p})\, \boldsymbol{I} \cdot \boldsymbol{F}^{-T} \cdot \boldsymbol{N} \cdot \boldsymbol{W} d\Gamma_n$$

Integrating by parts some of the terms, the above equations become:

$$\int_{\Omega_n} \boldsymbol{V^*} \cdot \boldsymbol{W} dV_n + \int_{\Omega_n} \Delta t \rho^{-1(n)} J\mu \operatorname{Grad}\left(\boldsymbol{V}^*\right) \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-T} \cdot \operatorname{Grad}\left(\boldsymbol{W}\right) dV_n =$$
$$\int_{\Omega_n} \boldsymbol{V}^{(n)} \cdot \boldsymbol{W} dV_n - \int_{\Omega_n} \Delta t \rho^{-1(n)} J\mu \operatorname{Grad}^T\left(\boldsymbol{V}^*\right) \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-T} \cdot \operatorname{Grad}\left(\boldsymbol{W}\right) dV_n -$$
$$\int_{\Omega_n} \Delta t \rho^{-1(n)} J\acute{\hat{\boldsymbol{\sigma}}} \cdot \boldsymbol{F}^{-T} \cdot \operatorname{Grad}\left(\boldsymbol{W}\right) dV_n + \tag{63}$$
$$\int_{\Omega_n} \Delta t \rho^{-1(n)} J\hat{p}\boldsymbol{I} \cdot \boldsymbol{F}^{-T} \cdot \operatorname{Grad}\left(\boldsymbol{W}\right) dV_n + \int_{\Omega_n} \Delta t \boldsymbol{b} \cdot \boldsymbol{W} dV_n +$$
$$\int_{\Gamma_{\sigma n}} \Delta t \rho^{-1(n)} J\boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} \cdot \boldsymbol{N} \cdot \boldsymbol{W} d\Gamma_n$$

$$\int_{\Omega_n} \frac{1}{\mathcal{K}} (p - \hat{p})\, qJ dV_n + \int_{\Omega_n} \Delta t \rho^{-1(n)} J^2 \left(1 - \frac{2\mu}{3\mathcal{K}}\right) \mathrm{Grad}\left((p - \hat{p})\right) F^{-1} : \mathrm{Grad}\,(q)\, F^{-1} dV_n = \int_{\Omega_n} J \boldsymbol{V^*} \cdot \boldsymbol{F}^{-T} \cdot \mathrm{Grad}\,(q)\, dV_n - \int_{\Gamma_{Vn}} \bar{\boldsymbol{V}} qJ \cdot \boldsymbol{F}^{-T} \cdot \boldsymbol{N} d\Gamma_n \tag{64}$$

$$\int_{\Omega_n} \boldsymbol{V} \cdot \boldsymbol{W} dV_n = \int_{\Omega_n} \boldsymbol{V}^* \cdot \boldsymbol{W} dV_n + \int_{\Omega_n} \Delta t \rho^{-1(n)} J \left(1 - \frac{2\mu}{3\mathcal{K}}\right) (p - \hat{p})\, \boldsymbol{I} \cdot \boldsymbol{F}^{-T} \cdot \mathrm{Grad}\,(\boldsymbol{W})\, dV_n \tag{65}$$

The full version of the discrete problem takes place when the unknown functions are approximated on the finite elements. Functions are approximated using an equal order interpolation for all variables:

$$\begin{aligned} V_j &= \sum W_i\,(X,t)\, V_{i,j} \\ p &= \sum W_i\,(X,t)\, P_i \end{aligned} \tag{66}$$

where W_i are the MFEM shape functions and $\boldsymbol{V}$ and p the nodal values of the three components of the unknown velocity and the pressure respectively. More details of the mesh discretization process and the choice of shape functions are given in section 8 .Substituting the approximation (66) into (63)-(65) leads to the following systems of equations:

$$(\boldsymbol{M} + \Delta t \boldsymbol{K})\, V^\star = \boldsymbol{M} V^{(n)} - \Delta t \boldsymbol{D} \hat{\boldsymbol{\sigma}} - \Delta t \widetilde{\boldsymbol{K}} V^\star + \Delta t \boldsymbol{D} \hat{p} + \Delta t \boldsymbol{F} \tag{67}$$

$$\left(\widehat{\boldsymbol{M}} + \Delta t \left(1 - \frac{2\mu}{3\mathcal{K}}\right) \widehat{\boldsymbol{K}}\right) \Delta p = \boldsymbol{D} V^* - \boldsymbol{G_V} \tag{68}$$

$$\boldsymbol{M} V = \boldsymbol{M} V^\star + \Delta t \left(1 - \frac{2\mu}{3\mathcal{K}}\right) \boldsymbol{D} \Delta p \tag{69}$$

where the previous matrices are:

$$
\begin{aligned}
M_{ij} &= \int_{\Omega_n} W_i W_j dV_n \\
\widehat{M_{ij}} &= \int_{\Omega_n} \frac{1}{\mathcal{K}} W_i W_j J dV_n \\
K_{ij} &= \int_{\Omega_n} J\rho^{-1(n)} \mu \, \mathrm{Grad}(W_i) \boldsymbol{F}^{-1} \cdot \mathrm{Grad}(W_j) \boldsymbol{F}^{-1} dV_n \\
\widehat{K_{ij}} &= \int_{\Omega_n} J^2 \rho^{-1(n)} \, \mathrm{Grad}(W_i) \boldsymbol{F}^{-1} \cdot \mathrm{Grad}(W_j) \boldsymbol{F}^{-1} dV_n \\
\widetilde{K_{ij}} &= \int_{\Omega_n} J\rho^{-1(n)} \, \mathrm{Grad}^T(W_i) \boldsymbol{F}^{-1} \cdot \mathrm{Grad}(W_j) \boldsymbol{F}^{-1} dV_n \\
D_{ij} &= \int_{\Omega_n} J W_i \boldsymbol{F}^{-T} \, \mathrm{Grad}\,(W_j)\, dV_n \\
F_i &= \int_{\Omega_n} \boldsymbol{b} W_i dV_n \\
G_{Vi} &= \int_{\Gamma_{\sigma n}} \overline{\boldsymbol{V}} W_i J \cdot \boldsymbol{F}^{-T} \cdot \boldsymbol{N} d\Gamma_n
\end{aligned}
\tag{70}
$$

7 SUMMARY OF A FULL ITERATIVE TIME STEP

The solution of the equation (67)-(69) involves linearization .The successive iteration algorithm has been chosen for the present analysis . Furthermore, during a time step, the mesh can suffer severe distortion due to the properties of a fluid to resist only to deformation rates. In order to preserve the positivity of the Jacobian, the time step is limited by the element diameter and velocity.
A full time step may be described as follows: starting with the known values $\boldsymbol{X}^{(n)}$, $\boldsymbol{V}^{(n)}$ and $\boldsymbol{\sigma}^{(n)}$ in each particle, the computation of the new particle position involve the following steps:

1- Initialize $\boldsymbol{F}^{(n+1,0)} = \delta_{ij} \boldsymbol{ei} \otimes \boldsymbol{ej}$,$J^{(n+1,0)} = 1$, $\boldsymbol{x}^{(n+1,0)} = \boldsymbol{X}^{(n)}$
2- While(not converge)
a)Evaluate the velocity $\boldsymbol{V}^{\star(n+1,i+1)}$

$$
\begin{aligned}
\left(\boldsymbol{M} + \Delta t \boldsymbol{K}\right) V^{\star(n+1,i+1)} = \boldsymbol{M} V^{(n)} - \Delta t \boldsymbol{D} \hat{\boldsymbol{\sigma}}^{(n+1)} - \\
\Delta t \widetilde{\boldsymbol{K}} V^{(n+1,i)} + \Delta t \boldsymbol{D} \hat{p}^{(n+1)} + \Delta t \boldsymbol{F}^{(n+1)}
\end{aligned}
\tag{71}
$$

b)Evaluate the pressure $p^{(n+1,i+1)}$

$$
\left(\widehat{\boldsymbol{M}} + \Delta t \left(1 - \frac{2\mu}{3\mathcal{K}}\right) \widehat{\boldsymbol{K}}\right) \Delta p^{(n+1,i+1)} = \boldsymbol{D} V^{\star(n+1,i+1)} - \boldsymbol{G_V} \overline{V}^{(n)} \tag{72}
$$

c)Evaluate the velocity $\boldsymbol{V}^{(n+1,i+1)}$

$$\boldsymbol{M}V^{(n+1,i+1)} = \boldsymbol{M}V^{\star(n+1,i+1)} + \Delta t \left(1 - \frac{2\mu}{3\mathcal{K}}\right) \boldsymbol{D}\Delta p^{(n+1,i+1)} \tag{73}$$

d)Move the particles to the $x^{(n+1,i+1)}$ position with

$$\boldsymbol{x}^{(n+1,i+1)} = \boldsymbol{X}^{(n)} + \triangle t \frac{\boldsymbol{V}^{(n+1,i+1)} + \boldsymbol{V}^{(n)}}{2} \tag{74}$$

e)Compute the deformation gradient $\boldsymbol{F}^{(n+1,i+1)}$ from (8) and $J^{(n+1,i+1)}$ from (7)
3-If converge, update the mesh nodes with

$$\boldsymbol{X}^{(n+1)} = \boldsymbol{x}^{(n+1,i+1)} \tag{75}$$

4-Evaluate the $\boldsymbol{\sigma}^{(n+1)}$

$$\boldsymbol{\sigma}^{(n+1)} = -p^{(n+1)}\boldsymbol{I} + 2\mu^{(n+1)}\acute{\boldsymbol{D}}^{(n+1)} + \acute{\hat{\boldsymbol{\sigma}}}^{(n+1)} \tag{76}$$

5-Generate the new mesh
6-Compute the new time step

8 MESH GENERATION

A key point for the success of the Lagrangian FSI formulation is the fast regeneration of a mesh at every time step on the basis of the position of the nodes in the space domain. The mesh is generated using the so called extended Delaunay tesselation (EDT) presented in[5–7] .The C^0 continuous shape functions of the elements can be simply obtained using the so called meshless finite element interpolation (MFEM). Details of the mesh generation procedure and derivation of the MFEM shape functions can be found in.[5–7] One of the main problems in mesh generation is the correct definition of the boundary domain. The use of the extended Delaunay partition makes it easier to recognize boundary nodes.
Considering that the particles follow a variable $h(x)$ distribution, where $h(x)$ is the minimum distance between two particles, the following criterion has been used: All particles on an empty sphere with a radius $r(x)$ bigger than $h(x)$, are considered as boundary nodes. α is a parameter close to, but greater than one. Note that this criterion is coincident with the Alpha Shape concept [9].
In this work, the boundary surface is defined by all the polyhedral surfaces(or polygons in 2D) having all their nodes on the boundary and belonging to just one polyhedron.
The method described also allows one to identify isolated fluid particles outside the main fluid domain. These particles are treated as part of the external boundary where the pressure is fixed to the atmospheric value.

9 EXAMPLE: DAM BREAKING IN AN HYPOELASTIC CONTAINER

The example shows a hypoelastic solid recipient in which the water is contained.

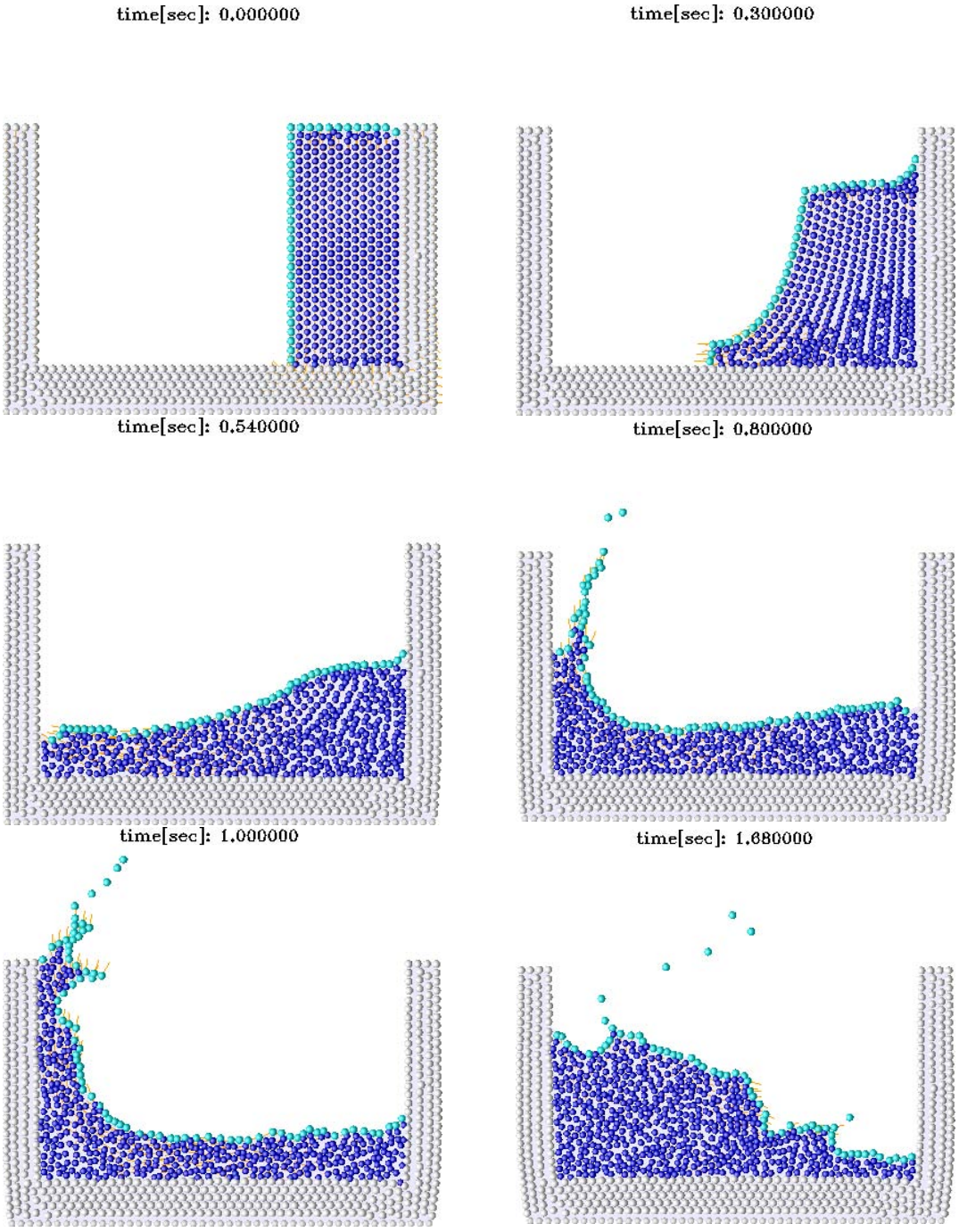

Figure 1: Dam breaking in an hypoelastic container

The water is initially located on the right supported by a removable board. The collapse starts at time t=0, when the removable board is slip-up. The water falls and runs along the bottom wall until, near 1 sec., it impinges on the left vertical wall which deforms by the resultant of the pressure and the viscous forces of the fluid. Breaking waves appear at 1 sec. Around t=1.5 sec. the main water wave reaches the right wall again. Figure 1 shows the domain profile at different time steps. In this example the power of the method to represent breaking waves and flow separation for a very complicated and random problem is verified .

10 CONCLUSIONS

In this paper, a Particle Finite Element Method applied to solve Fluid-Structure Interaction problems in a monolithic numerical scheme was presented. The method takes advantage of the similarity between the discrete constitutive equations of the materials under consideration: incompressible newtonian fluid and hypoelastic solid. The fractional step approach presented here has proved to be an efficient procedure for solving the Lagrangian equations. Two essential ingredients of the numerical approach are: the Delaunay triangulation algorithm which provides a very efficient connectivity scheme, and, the identification of boundary nodes using an Alpha-Shape type technique.

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