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THERMOMECHANICAL FATIGUE ANALYSIS BASED ON CONTINUUM MECHANICS

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RESUMEN

Las estructuras y piezas mecánicas sometidas a cargas cíclicas pueden fallar aún cuando las tensiones máximas generadas por estas cargas sean inferiores al límite de resistencia del material. A este modo de rotura se denomina falla por fatiga y su análisis mediante métodos numéricos resulta complejo pero de crucial importancia en muchas y muy diversas áreas de la ingeniería (mecánica, aeronáutica, etc.).

A diferencia de la usual aproximación al tema mediante mecánica de fractura, en el presente trabajo se desarrolla un modelo constitutivo basado en la mecánica de medios continuos que permite analizar el fenómeno de fatiga que se produce bajo cargas cíclicas, tanto mecánicas como térmicas. Este modelo permite el tratamiento conjunto de fenómenos acoplados, combinación de comportamiento mecánico no lineal (elasticidad no lineal, plasticidad y daño) con efectos térmicos y de fatiga. Por último se presenta un ejemplo numérico de aplicación del modelo al análisis de moldes de fundición sometidos a cargas cíclicas térmicas y mecánicas. (die-casting).

ABSTRACT

Time varying cyclic loads produce failure of structural parts for values of stress lower than those obtained in static tests. This phenomenon is called fatigue and its numerical analysis is complex but of crucial importance for most engineering fields (mechanical, aeronautical, etc.).

In this work a thermo-mechanical constitutive model which allows to simulate thermal fatigue is presented. This model is based on the mechanics of the continuous medium and allows to treat coupled phenomena such as fatigue with damage, plasticity, viscosity and temperature in a unified way. The formulation based on the theories of damage and plasticity is developed and the necessary modifications in order to include the fatigue phenomena into these theories are outlined. Finally, the performance of the proposed model is shown via the analysis of an industrial test case for die-casting applications.

INTRODUCTION

The study of fatigue starts from the basic fact that it is not a phenomenon associated with the classic concept of plasticity and/or damage. Failure occurs under load conditions below the elastic limit strength of the material. A progressive loss of strength in the material is verified, which depends on the number of load cycles generating a secondary plasticity and/or damage effect.

Towards 1860, A. Wöhler[1], a Bavarian railway engineer, gave a decisive push to the knowledge of fatigue phenomena by performing numerous tests under diverse loading conditions in order to determine the origin of untimely collapse of railway axles. Results of his studies are the well-known stress-cycle curves (S-N) as well as the concept of fatigue limit strength or endurance.

Wöhler's S-N curves (see figure 1b) are obtained experimentally by subjecting identical specimens to cyclic harmonic stresses and establishing their life span measured by a number of cycles. The curves also depend on the ratio between the lowest and the highest stresses (R=Smin/Smax) and the mean stress.

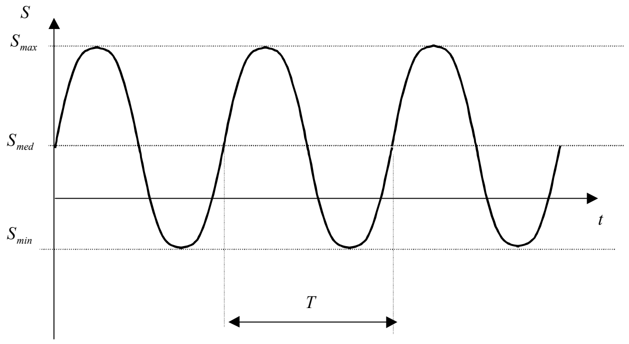


Figure 1a: ***Stress evolution at a point.***

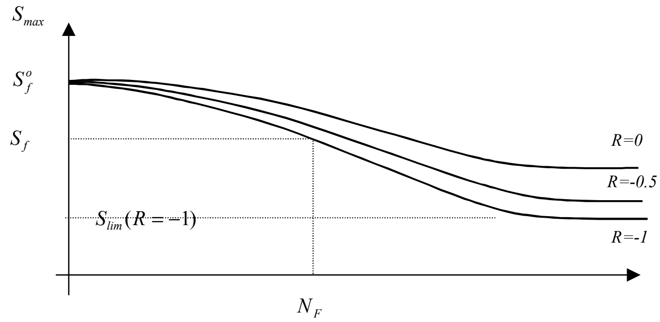


Figure 1b: $S-N$ ***Wöhler's curves***

In the 1950s, Manson[2] and Coffin[3] suggested that the plastic strain is one of the main causes of cyclic damage in metals, and proposed an experimentally derived expression relating the number of cycles to the inelastic strain.

Several authors studied fatigue based on fracture mechanics: Irwin[4] tried to relate fatigue phenomena with fracture via stress intensity factors. Along the same line, years later, Paris, Gomez and Anderson[5] characterized fatigue crack growth by means of the stress intensity factor. They were the first to suggest that crack length increase depend on the fluctuation range of this stress intensity factor. Later Paris[6] found a now notorious mathematical relation between the number of cycles and crack length.

In these last years a fundamental change in focus occurred and Chaboche[7] presented a work that include fatigue into the general damage theory of continuum mechanics. This study was based on the hypothesis that fatigue damage is, essentially, of the same nature as the mechanical damage and can be described via an internal variable allowing the adequate treatment of accumulation and localization of dislocations. This internal variable relates damage of the material with the number of cycles.

FATIGUE AS A CONTINUUM MECHANICS FORMULATION

The theoretical structure of constitutive models such as plasticity and damage (Lubliner[8,9], Malvern[10]) are suitable for the study of non linear fatigue problems. It can be stated that the mechanical effect known as fatigue produces a loss of material strength as a function of the number of cycles, reversion index and load amplitude. This loss of strength subjects the material to inelastic behavior, which may be considered as micro-cracking followed by their coalescence which leads to the final collapse of structural parts.

A model formulated in the framework of continuum mechanics overcomes a series of drawbacks observed in the methods based on fracture mechanics, because:

a.- The classical models proposed for fatigue only forecast the life of a part as a function of the number of cycles of periodical load (Paris[5,6]). The proposed model allows to consider the loss of material strength due to combined effects such us: fatigue, fracture, damage, plasticity, visco-elasticity, temperature, etc. This means that complex phenomena occurring in materials may be forecast and enables the study of their safety at a given moment of their life, before total collapse.

b.- It must be recalled that experimental tests constitute a good tool for specific cases and to obtain parameters, but they could not be extended to more complex situations than those studied in the laboratory. Fracture mechanics does not offer a true solution to this problem since it is extremely complicated to quantify the effect due to complex load conditions. With a continuum mechanics approach, it is possible to take into account combined effects of other factors such as thermal loads, mean stress, multi-axial stress states, plastic damage, etc.

c.- The introduction of a new internal variable related to fatigue allows the treatment of the problem of accumulative damage without formulating a complementary constitutive rule. In this way, the model proposed here is able to take into account nonlinear damage accumulation problems that occur when a structural part is subjected to cycles with different load amplitudes.

Thermo Plastic damage model

This model is based on the hypothesis of the additivity of the elastic Ψ^e and the plastic Ψ^p parts of the free energy. For a given entropy η and temperature θ field the elastic part of the strain is given by $E^e_{ij} = E_{ij} - E^p_{ij} - E^\theta_{ij}$ **;** where the total strain E_{ij} and the strain due to temperature field E^θ_{ij} operate as free field variables **(**Green[11], Lubliner[12] ,Luccioni[13], Oller[14]). The free energy is

$$\Psi = \Psi^e(E^e_{ij}, d, \theta) + \Psi^p(\alpha^p, \theta) = \left\{ (1-d)\frac{\rho^2}{2m^o}\left[E^e_{ij}\mathrm{C}^o_{ijkl}(\theta)E^e_{kl}\right] + \Psi^p(\alpha^p, \theta) \right\} - \theta\eta \qquad (1)$$

where m^o is the density of the material, $E^e_{ij}, E_{ij}, E^p_{ij}, E^\theta_{ij}$ are the strain tensors, $d^{ini} \le d \le 1$ is the mechanical damage variable, d^{ini} is the initial value of damage variable C^0_{ijkl} and C^S_{ijkl} is the constitutive tensors for the original and secant material, S_{ij} is the stress tensor for a single material point and ρ is the plastic degradation variable that take into account the density increment due to the reduction of material's porosity.
Considering the second thermodynamic law (Clasius-Duhem inequality) (Lubliner[8,9], Malvern[10]), the thermo mechanical dissipation is obtained as

$$\Xi = \frac{S_{ij} : \dot{E}^p_{ij}}{m^o} - \frac{\partial \Psi}{\partial \alpha^p}\dot{\alpha} - \frac{\partial \Psi}{\partial d}\dot{d} - \frac{J}{\theta m^o} q_i \nabla\theta \ge 0 \quad , \qquad (2)$$

and the constitutive laws are derived from the dissipation condition:

$$S_{ij} = m^o \frac{\partial \Psi^e}{\partial E^e_{ij}} = (1-d)\rho^2 \mathrm{C}^o_{ijkl}(\theta)\left(E^e_{kl}\right) \quad , \quad \eta = \frac{\partial \Psi}{\partial \theta} \qquad (3a)$$

Also, from the last expression, the other thermo-mechanical variables can be obtained as:

$$\begin{aligned}
&\text{Constitutive tensor :} && C^S_{ijkl}(d,\rho,\theta) = \frac{\partial S_{ij}}{\partial E^e_{kl}} = m^o \frac{\partial^2 \Psi^e}{\partial E^e_{ij}\, \partial E^e_{kl}} \\
&\text{Conjugate thermal expansion coeficient :} && \beta_{ij}(d,\rho,\theta) = -\frac{\partial S_{ij}}{\partial \theta} = -m^o \frac{\partial^2 \Psi^e}{\partial \theta\, \partial E^e_{ij}} \qquad (3b) \\
&\text{Specific Heat :} && c_k = \theta \frac{\partial \eta}{\partial \theta} = -m^o \theta \frac{\partial^2 \Psi}{\partial \theta}
\end{aligned}$$

Thermal Mechanical Coupled Model

Based on the first thermodynamic law and Fourier's law (Lubliner[3,4], Malvern[7]) the heat equation coupled with the mechanical problem can be written in the classical form as

$$Q + J\, div\left(k_i \frac{\partial \theta}{\partial x_i} \right) - \theta \beta_{ij}\, \dot{E}^e_{ij} + D^p - c_k\, m^o\, \dot{\theta} = 0 \tag{4}$$

where Q represents the caloric power, $(\theta\beta_{ij}\,\dot{E}^e_{ij})$ the thermo-elastic coupled term, β_{ij} is the conjugate of the thermal expansion coefficient, D^p the thermo-plastic coupled term and J is the determinant of the Jacobian matrix. Equation (4) together with the properly imposed boundary conditions, can simulate thermo-mechanical processes.

Fatigue formulation for Thermo-Elasto-plastic-damage model

Fatigue theory presented in this work differs from that introduced by Chaboche[7] because the evolution of the internal damage, variable as a function of the number of cycles, is not defined in an explicit form. In this work, a function that modifies the discontinuity threshold (yield or damage) is formulated, producing an "implicit evolution of fatigue variable" included in the internal plastic and/or damage variables of the inelastic model. In addition, the thermal part of the fatigue definition is introduced in the same form as in the mechanical formulation, throughout via an implicit internal damage variable.

The formulation is based on introducing the effect of the number of cycles on the plastic and/or damage discontinuities, and needs the fulfillment of the plastic and/or damage consistency conditions:

$$f(S_{ij}) - \underbrace{\overline{K}(S_{ij},\alpha^p).\, f_{red}(N,S_{med},R,\theta)}_{K(S_{ij},R,N,\theta)} = 0 \;\Rightarrow\; \underbrace{\left(\frac{f(S_{ij})}{f_N(N,S_{med},R)\cdot f_\theta(\theta)} \right)}_{f'(S_{ij},N,R,\theta)} - \overline{K}(S_{ij},\alpha^p) = 0$$

$$\overline{S}(S_{ij}) - \underbrace{\overline{F}^D(S_{ij},d).\, f_{red}(N,S_{med},R,\theta)}_{F^D(S_{ij},R,N,\theta)} = 0 \;\Rightarrow\; \underbrace{\left(\frac{\overline{S}(S_{ij})}{f_N(N,S_{med},R)\cdot f_\theta(\theta)} \right)}_{\overline{S}'(S_{ij},N,R,\theta)} - \overline{F}^D(S_{ij},d) = 0 \tag{5}$$

The reduction function $f_{red}(N,S_{med},R,\theta) = f_N(N,S_{med},R)\cdot f_\theta(\theta)$ makes the plasticity and damage models dependent on the thermal fatigue phenomenon, where N is the current number of cycles, $R = \frac{S_{min}}{S_{max}}$ is the stress reversion factor, S_{med} the mean stress (see figure 2a), $f_N(N,S_{med},R)$ is the reduction function influenced by the number of the cycles and $f_\theta(\theta)$ is the thermal reduction function.

In this way, fulfilling the plastic and damage consistent condition, the tangent constitutive tensor can be rewritten for non degradation of the material stiffness ($\rho = 0$) as,

$$\dot{S}_{ij} = \mathrm{C}^{ep}_{ijkl}\,\dot{E}_{kl} - \mathrm{C}^{\theta}_{ijkl}\,\dot{E}^{\theta}_{kl} \tag{6}$$

where:

$$C^{s}_{ijkl} = (1-d)\,C^{o}_{ijkl}$$

$$C^{e}_{ijkl} = C^{s}_{ijkl} - \frac{1}{(1-d)}\left[\left(\frac{\partial \overline{S'}}{\partial S^{o}_{rs}}\right)C^{o}_{rsij}\right]S_{kl}$$

$$C^{\mathrm{ep}}_{\mathrm{ijkl}} = C^{e}_{ijkl} - \frac{C^{s}_{ijrs}\dfrac{\partial G}{\partial S_{rs}}\dfrac{\partial f'}{\partial S_{mn}}C^{e}_{mnkl}}{-\dfrac{\partial F}{\partial \alpha^{p}_{r}}(h_r)_{tu}\dfrac{\partial G}{\partial S_{tu}} + \dfrac{\partial f'}{\partial S_{mn}}C^{s}_{mnrs}\dfrac{\partial G}{\partial S_{rs}}}$$

$$C^{\theta}_{\mathrm{ijkl}} = C^{s}_{ijkl} - \frac{C^{s}_{ijrs}\dfrac{\partial G}{\partial S_{rs}}\left(\dfrac{\partial F'}{\partial S_{mn}} - \dfrac{\partial f'}{\partial S_{mn}}C^{e}_{mnkl}\right)}{-\dfrac{\partial F}{\partial \alpha^{p}_{r}}(h_r)_{tu}\dfrac{\partial G}{\partial S_{tu}} + \dfrac{\partial f'}{\partial S_{mn}}C^{s}_{mnrs}\dfrac{\partial G}{\partial S_{rs}}}$$

It is also necessary to define an unique global hardening internal variable (plastic and damage) based on the normalized dissipation energy

$$\dot{q} = \Xi_m \,.\, \Re(S_{ij}) = \left(\Xi^p + \Xi^d\right)\left[\frac{r(S_{ij})}{g_f} + \frac{(1-r(S_{ij}))}{g_c}\right] \quad , \quad r(S_{ij}) = \begin{cases} 1 & \text{for pure tension} \\ 0 & \text{for pure compresion} \end{cases} \qquad (7)$$

where Ξ^p, Ξ^d are the plastic and damage energy dissipation and g_f , g_c are the maximum energy limits that can be dissipated at a material point at the end of the inelastic process. It must be observed that this definition ensures the simultaneous fulfillment of the plastic and damage consistency conditions. It can also be seen that the mechanical process described allows rate dependent or rate independent coupled phenomena with the number of cycles. Therefore, the strength of the material comes out from the combination of the two phenomena defined in independent spaces (see figure 2).

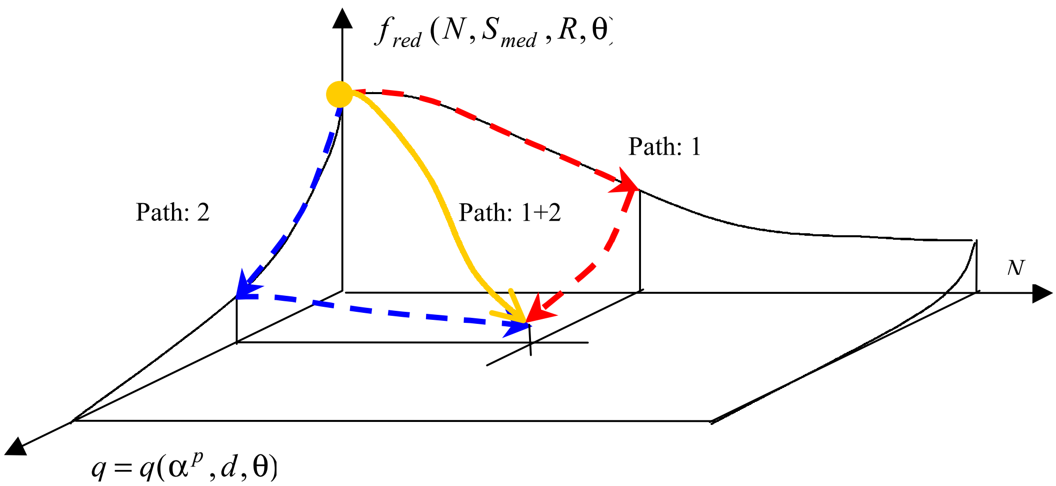


Figure 2: ***Schematic view of the hyper yield-damage surface.***

Particular form of the reduction function $f_{red}(N, S_{med}, R, \theta)$ for the A517 steel

As an example the expressions of the strength material evolution of a A517 steel are presented. For the initial state, i.e. *N=0* cycles, the strength is $S_{max} = S_f^o$, which also corresponds to the strength threshold for a non cyclic problem, and when $N \to \infty$ the strength limit $S_{lim} = S_f^o \left(\frac{1}{1.26} + \frac{R}{3.3} \right)$ is reached using the Wöhler curve. That is, $S_{max} = S_f^o \; e^{-A(\log N)^\alpha}$ Substituting $S_{lim} = S_{max}(N = N_{lim} = 10^7)$ into the last expression, the parameter A can be obtained as

$$S_{lim} = S_f^o \left(\frac{1}{1.26} + \frac{R}{3.3} \right) = S_f^o \; e^{A(\log N)^\alpha} \quad \to \quad A = -\frac{\ln\left(\frac{1}{1.26} + \frac{R}{3.3} \right)}{7^\alpha} \tag{8}$$

For more detail about this application see the scheme represented in figure 3. In order to adjust the Wöhler curves $\alpha = 4.(2^{2R})$ is adopted.

$$S_{lim} = S_f^o \cdot \left(\frac{1}{1.26} + \frac{R}{3.3} \right)$$

If: $S_{APL,max} < S_{lim}$

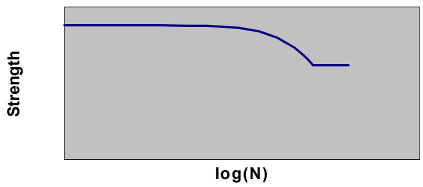
$$\boxed{S_f = S_f^o(q,\theta) \cdot e^{-B_1 \cdot \log(N)^\beta}} \quad N < 10^7$$

$$\alpha = 4 \cdot 2^{2R} \qquad \beta = \mathrm{c} \cdot \alpha$$

$$S_f = S_{f\,\inf} \quad N > 10^7$$

$$S_{f\,\inf} = S_f^o - S_{APLmax} \cdot \left(\frac{S_f^0 - S_{lim}}{S_{lim}} \right)$$

$$B_1 = \frac{-\ln\left(\frac{S_{APLmax}}{S_f^0} \right)}{7^\beta}$$



If: $S_{APL,max} > S_{lim}$

$$\boxed{S_f = S_f^o(q,\theta) \cdot e^{-B_2 \cdot \log(N)^\beta}} \quad \forall N$$

$$B_2 = \frac{-\ln\left(\frac{S_{APLmax}}{S_f^0} \right)}{(\log N_F)^\beta}$$

$$N_F = \log^{-1}\left(7 \cdot \sqrt[\alpha]{\frac{\ln\left(\frac{SAPL_{max}}{S_f^0} \right)}{\ln\left(\frac{1}{1.26} + \frac{R}{3.3} \right)}} \right)$$

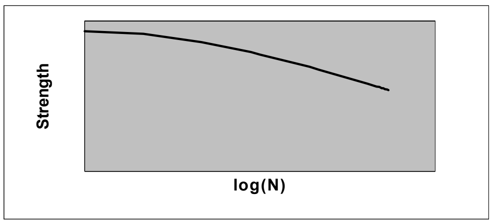


Figure 3: ***Schematic representation of the fatigue strength.***

The expressions before mentioned give good results, and should be extended to any material type. However for any kind of material, experimental Wöhler's curves are necessary in each case. The coefficient $C > 1$ is an adjustment parameter to the experimental Wöhler's curve.

FINITE ELEMENT EQUILIBRIUM EQUATIONS

According to the principle of virtual work and the first law of thermodynamics, the mechanical and thermal equilibrium equations in the referential configuration (Malvern[10]) are ,

$$\begin{cases} \int_V \left(\dot{u}_i \cdot m^o \cdot \ddot{u}_i + S_{ij} \cdot \nabla_i^s \dot{u}_j \right) dV - \int_V m^o b_i \dot{u}_i \, dV - \oint_S t_i \dot{u}_i \, dS = 0 \\ \int_V \theta \, div(q_i) \, dV + \int_V q_i \nabla \theta \, dV - \oint_{S_b} \theta \, q_i n_i \, dS_b = 0 \end{cases} \tag{9}$$

Substituting eq.(6) in the mechanical equation (9a) and eq.(4) in the thermal one (9b), approximating the displacement by $\mathbf{u}(x_i) \cong \mathbf{N}^u(x_i) \cdot \mathbf{U}$,the temperature field by $\theta(x_i) \cong \mathbf{N}^\theta(x_i) \cdot \Theta$ and using finite element procedures the equilibrium equations are obtained

$$\begin{cases} \mathbf{M}_u \cdot \ddot{\mathbf{U}} + \mathbf{f}_u^{\text{int}} + \mathbf{M}_{u-\theta} \cdot \dot{\Theta} - \mathbf{f}_u^{ext} = \mathbf{0} \\ \mathbf{C}_\theta \cdot \dot{\Theta} + \mathbf{M}_{\theta-u} \cdot \dot{\mathbf{U}} + \mathbf{K}_\theta \cdot \Theta + \mathbf{D}^p - \mathbf{f}_\theta = 0 \end{cases} \tag{10}$$

where $\mathbf{N}^u(x_i)$ and $\mathbf{N}^\theta(x_i)$ are the displacement and temperature approximations functions, $\mathbf{U}$ and Θ are the discrete nodal values of displacement and temperature, $\mathbf{M}_u$ is the dynamic mass matrix, $\mathbf{f}_u^{\text{int}} = (f_k)_u^{\text{int}} = \int_V S_{ij} \nabla_i^S N_{jk}^u \, dV$ is the mechanical internal force, $\mathbf{M}_{\theta-u}$ is the thermal stiffness matrix, $\mathbf{f}_u^{ext}$ is the mechanical load vector, $\boldsymbol{C}_\theta$ is the caloric capacity matrix, $\mathbf{M}_{u-\theta}$ is the thermoelastic coupled matrix, $\boldsymbol{K}_\theta$ is the conductivity matrix, $\boldsymbol{D}^p$ is the mechanical dispassion and $\mathbf{f}_\theta$ is the thermal load vector.

APPLICATIONS

In this section an application example using the proposed model is presented. The example consists on a thermo-mechanical coupled test based on a quarter symmetric part of a steel mould (figure 4) The mould works under an imposed cycle of high temperatures and pressures (figure 5). The finite element mesh used in the analysis is shown in figure 6 and results of the analysis are summarized in figures 7 to 9.

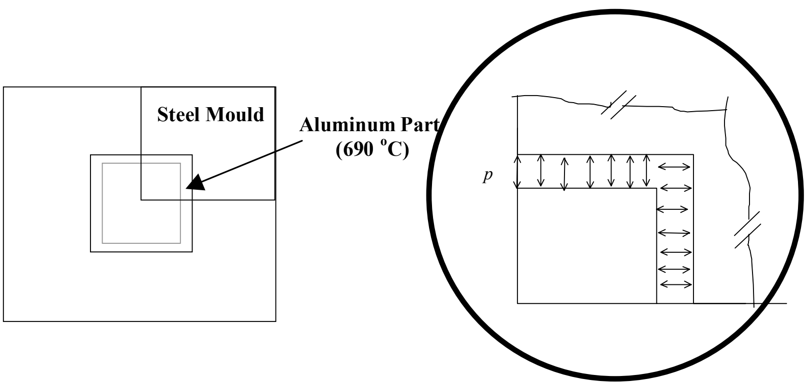


Figure 4: ***Geometry of the die to manufacture an aluminum square tube***

Material characteristics:

Steel material subjected to a micro crack by means of an isotropic damage model.

$E = 2.15\times10^{11}\ N/m^2$,$\nu = 0.3$, $S_o^f = 1.6\times10^9\ N/m^2$,$G_f = G_c = 30\times10^5\ N/m$

$m^o = 7800\,kg/m^3$, $k = 25W/m\,^oC$, $c_k = 460\,J/Kg\,^oC$,$h_{air} = 10\,N/M\,S\,^OC$,

$\alpha^{TER} = 12\times10^{-6}\,1/^o\,C$, $\theta^{ref} = 20\,^oC$

$\theta = 400^oC \quad S^f = 1.3\times10^9\ N/m^2$

$\theta = 500^oC \quad S^f = 1.1\times10^9\ N/m^2 \qquad E = 1.76\times10^{11}\ N/m^2$

$\theta = 600^oC \quad S^f = 0.8\times10^9\ N/m^2 \qquad E = 1.65\times10^{11}\ N/m^2$

$\theta = 650^oC \quad S^f = 0.6\times10^9\ N/m^2$

Aluminum material subjected to isotropic elastic behavior.

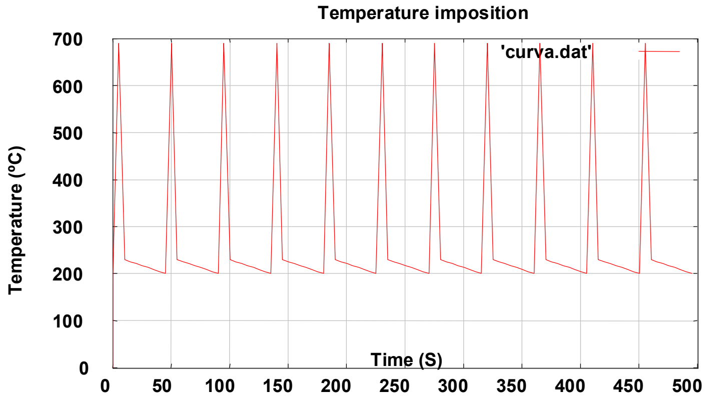
$E = 6.93\times10^{10}\ N/m^2$,$\nu = 0.37$, $m^o = 2650\,kg/m^3$,

$k = 234.46\,W/m\,^oC$, $c_k = 955\,J/Kg\,^oC$,$h_{air} = 10\,N/M\,S\,^OC$,

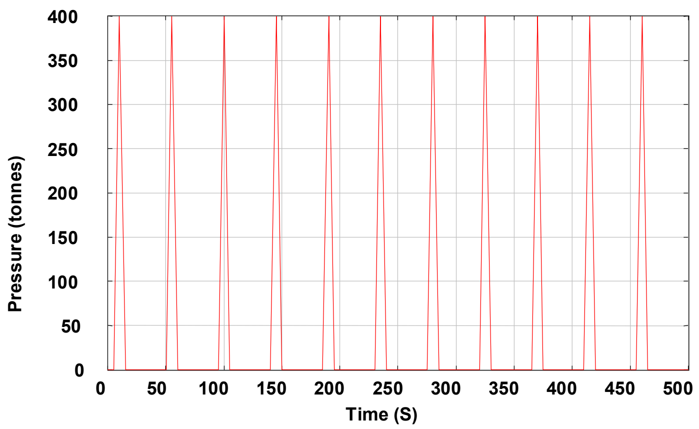
$\alpha^{TER} = 12.19\times10^{-6}\,1/^o\,C$, $\theta^{ref} = 20\,^oC$

*Figure 5: **Load characteristics***

Temperature imposition on the aluminum following the history described on the following function:



Pressure imposition on the aluminum casting part:



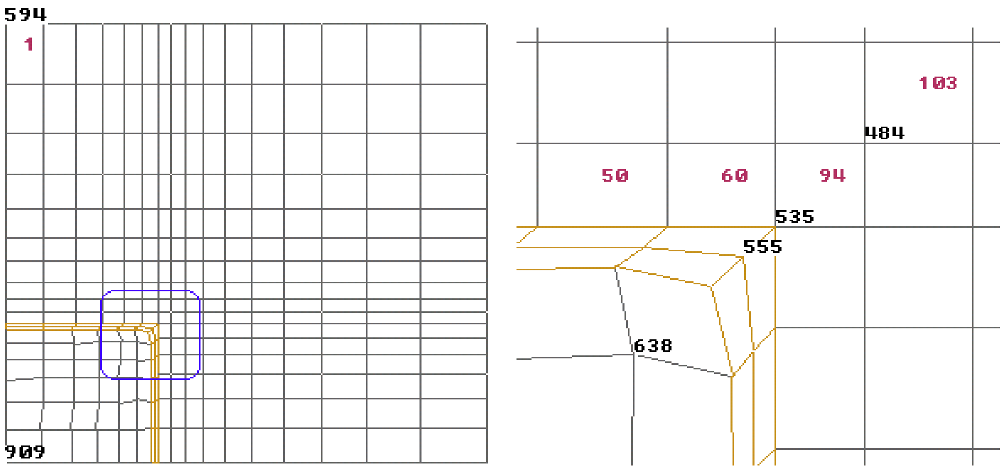


Figure 6: ***Mesh used for the numerical analysis*** (1/4 of the whole problem) and detail of the internal corner. Nodes and elements numbered are those for which plots are presented

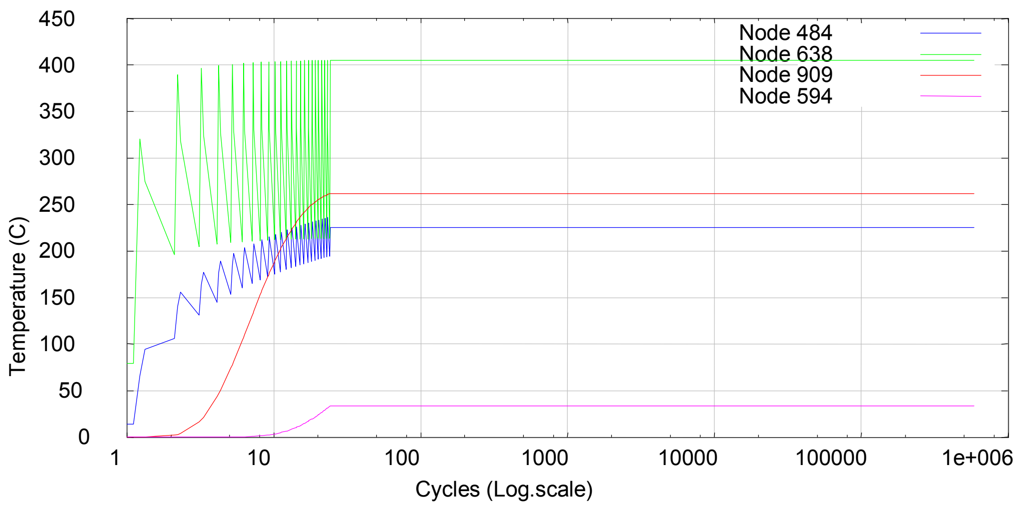


Figure 7: ***Temperature evolution at different points of the mould.***
Cyclic behavior is observed at nodes close to the cast part.

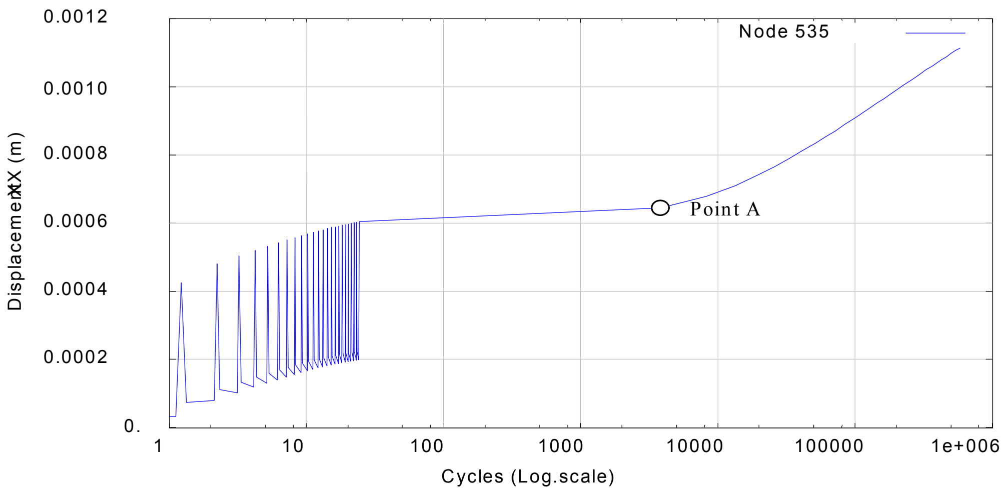


Figure 8: ***Displacement evolution of the internal corner of the mould.***
The point A shows a clear change in the mechanical behavior of the mould. The up-turn of the displacement could be considered as the end of fatigue life of the mould

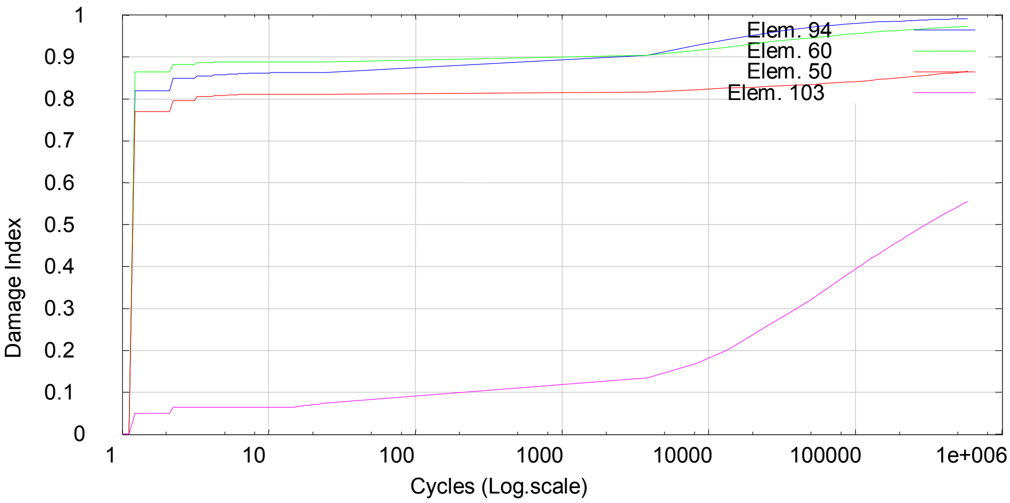


Figure 9: ***Damage index history at the first Gauss Point of the elements close to the internal corner.***

CONCLUSIONES

This work presents a constitutive model, based on the mechanics of a continuous medium that allows to model coupled thermo-mechanical fatigue behavior of materials subjected to periodical loads. The advantages of this formulation are evident, since it allows to consider the solid as a continuum and the combination of many additional phenomena can be included. The formulation of fatigue theory combined with damage, plasticity, viscous phenomena and temperature, make attractive the application of this theory. Another important aspect is that this theory uses the continuum mechanics framework as a basis for the coupled formulation utilizing well-established thermodynamic principles. In this work some results showing the capabilities of the model is also presented.

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