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NUMERICAL SOLUTION OF REAL MAGNETOGASDYNAMICS EQUATIONS

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ABSTRACT

The main objective of this work is to present the results obtained with a computational code which solves the time dependent magnetogasdynamic equations (MGD) in one dimension. This research represents the initial stage towards achieving a comprehensive description of the ablative pulse plasma thruster (APPT) behavior. The equations that govern MGD flows are continuity, momentum, energy and magnetic induction together with a state equation. These equations have two parts: the first one contains the conservation terms and is hyperbolic; the second one has the diffusive terms and is parabolic. The parabolic part of the equations is written in divergence form, so that there is a diffusive flux. The numerical technique used to solve the equations consists of an approximate Riemann solver that calculates the variables inside each cell by evaluating the flux through the contour of the cells. The TVD scheme proposed by Yee, et al. is used to evaluate the numerical flux. The "eigensystem" technique presented by Powell has also been used and eigenvectors normalization has been carried out following the works of Zarachay et al., Roe and Balsara, and Bodgan. To check the accuracy of the computational code a Riemann problem introduced by Brio and Wu has been simulated. The obtained results are in close agreement with those reported by other authors.

RESUMEN

El objetivo principal de este trabajo es presentar los resultados obtenidos con un código computacional que resuelve las ecuaciones de la magnetogasdinámica real dependientes del tiempo (MGD) en una dimensión. Esta investigación representa la fase inicial hacia lograr una descripción de un módulo propulsivo de plasma pulsante ablativo. Las ecuaciones que gobiernan los flujos MGD son continuidad, cantidad de movimiento, energía e inducción magnética junto con una ecuación de estado. Estas ecuaciones poseen dos contribuciones: una contiene los términos conservativos; y la otra está compuesta de los términos difusivos. La parte difusiva o parabólica de las ecuaciones se escribe en forma de conservación, entonces existe flujo difusivo. La técnica numérica empleada consiste de un seguidor aproximado de Riemann que calcula las variables dentro de cada célula en función del flujo a través del contorno de las mismas. El esquema TVD propuesto por Yee, *et al.* se usa para obtener los flujos numéricos, además se ha utilizado la técnica de ocho ondas presentada por Powell y los vectores propios han sido normalizados siguiendo los trabajos de Zarachay *et al.*, Roe y Balsara, y Bodgan. Para verificar el comportamiento del código se ha simulado el problema del tubo de choque magnetogasdinámico introducido por Brio y Wu. Los resultados obtenidos verifican muy bien los publicados por otros autores.

INTRODUCTION

The main objective of this work is to present the results obtained with a computational code developed to solve the solution of real magnetogasdynamic equations (MGD). This research represents the initial stage towards achieving a comprehensive description of the APPT

behavior. There are previous works in which the authors use or develop numerical codes to simulate the flow inside of magnetoplasmadynamics thruster. For example two-dimensional codes have been developed by Toki *et al.* (1982), and Ao and Fujiwara (1984). Effects of the geometry on the performances has been studied by LaPointe (1992) and Mikellides and Turchi (1999) using the codes MACH2 and MACH3 to simulate the non-steady flow in two and three dimensions. None of these works apply techniques classically used in fluid mechanics that allow for the high resolution capture of discontinuities. The present research explores the capacity of these techniques to simulate the plasma flow.

The equations that govern MGD flows are continuity, momentum, energy and magnetic induction together with a state equation. These equations form a hyperbolic-parabolic system, see Sankaran, *et al.* (2000). The hyperbolic terms represent the ideal MGD equations, whilst the diffusive effects are accounted for in the parabolic terms.

A numerical technique is used which consists of an approximate Riemann solver that calculates the variables inside each cell by evaluating the flux through the contour of the cells. The TVD scheme proposed by Yee, *et al.* (1985) is used to evaluate the numerical flux. The "eigensystem" technique presented by Powell has also been used and eigenvectors normalization has been carried out following the works of Zarachay *et al.* (1994), Roe and Balsara (1996), and Bodgan (1999). To check the accuracy of the computational code a Riemann problem introduced by Brio and Wu (1988) has been simulated. The parabolic components are written in conservation form and they are considered as fluxes. To obtain the numerical flux of the parabolic contributions the technique given by Bodgan (1999) is used.

PLASMA FLOW

In many situations the plasma flow can be represented by the equations of the magnetogasdynamic (Dendy, 1999). In this section theses equations are introduced.

Equations of the real MGD

The equations of ideal MGD in conservative form are expressed non-dimensionally as:

$$\frac{\partial U}{\partial t}+\frac{\partial F_h}{\partial x}=\frac{\partial F_p}{\partial x} \tag{1}$$

where U is the conservative state variable vector

$$U=\left(\rho,\rho u_x,\rho u_y,\rho u_z,B_x,B_y,B_z,E\right)^T \tag{2}$$

and F_h is the vector that specifies the hyperbolic fluxes; it can be written for 1-D problems as:

$$F_h=\begin{pmatrix} \rho u_x \\ \rho u_x^2-B_x^2+p+\dfrac{B^2}{2} \\ \rho u_x u_y-B_x B_y \\ \rho u_x u_z-B_x B_z \\ 0 \\ u_x B_y-u_y B_x \\ u_x B_z-u_z B_x \\ \left(E+p+\dfrac{B^2}{2}\right)u_x-\left(u_x B_x+u_y B_y+u_z B_z\right)B_x \end{pmatrix} \tag{3}$$

the density is indicated as ρ; u_x, u_y, u_z are the components of the vector velocity; B_x, B_y, B_z represent the components of the vector magnetic field; p is the pressure; B^2 is defined as:

$$B^2 = B_x^2 + B_y^2 + B_z^2 \tag{4}$$

The energy is expressed as:

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho\left(u_x^2 + u_y^2 + u_z^2\right) + \frac{1}{2}\left(B_x^2 + B_y^2 + B_z^2\right) \tag{5}$$

where γ is the ratio of specific heats
The parabolic fluxes for 1-D problems are given by:

$$F_p = \begin{pmatrix} 0 \\ \frac{4}{3}\frac{\mu}{\mathrm{Re}\,Al}\frac{\partial v_x}{\partial x} \\ \frac{\mu}{\mathrm{Re}\,Al}\frac{\partial v_y}{\partial x} \\ \frac{\mu}{\mathrm{Re}\,Al}\frac{\partial v_z}{\partial x} \\ 0 \\ \frac{\eta_z}{Lu\,Al}\frac{\partial B_y}{\partial x} \\ \frac{\eta_y}{Lu\,Al}\frac{\partial B_z}{\partial x} \\ \frac{1}{Lu\,Al}\left(\frac{\eta_z}{2}B_y^2 + \frac{\eta_y}{2}B_z^2\right) + \frac{\kappa}{Pe\,Al}\frac{\partial T}{\partial x} + \frac{\mu}{\mathrm{Re}\,Al}\frac{\partial\left(\frac{2}{3}v_x + \frac{1}{2}v_y + \frac{1}{2}v_z\right)}{\partial x} \end{pmatrix} \tag{6}$$

where η_x, η_y, η_z are the non-dimensional resistivities, κ is the non-dimensional thermal conductivity and μ is the non-dimensional viscosity. *Lu*, *Re* and *Pe* are the Lunquist, Reynolds and Peclet numbers respectively.
The hyperbolic terms system of equations (1) can be recast in their quasi-linear form:

$$\frac{\partial U}{\partial t} + A_c \frac{\partial U}{\partial x} = 0; \qquad A_c = \frac{\partial F_h}{\partial U} \tag{7}$$

where A_c is the flux Jacobian, the letter "c" indicates that the derivation has been carried out with regard to the conservative state variables. However, the form of the flux Jacobian is simpler as a function of the primitive variables (W):

$$W = \left(\rho, u_x, u_y, u_z, B_x, B_y, B_z, p\right)^T \tag{8}$$

Therefore, Eq. (8) can be rewritten in the following way:

$$\frac{\partial U}{\partial t} + A_p \frac{\partial W}{\partial x} = 0; \qquad A_p = \frac{\partial F_h}{\partial W} \tag{9}$$

The transformation rule between the Jacobian fluxes is:

$$A_p = \left(\frac{\partial U}{\partial W}\right)^{-1} A_c \left(\frac{\partial U}{\partial W}\right) \tag{10}$$

Riemann solvers based on a system of eight waves using the matrix A_p cannot be applied because an eigenvalue is zero and lacks of physical meaning. It is important to notice that the formulations in primitive variables and in conservative variables are equivalent, therefore this null eigenvalue appears in both formulations. To solve this inconvenient an alternative flux Jacobian has been implemented ($A_p^{'}$), as presented by Powell (1995). It is important to note that 1-D problems do not require to use this new matrix. It has nevertheless been implemented in this work because this matrix allows for a straightforward extension to simulations in two or three dimensions. The eigenvalues of the matrix $A_p^{'}$ are:

$$\lambda_e = u_x; \quad \lambda_a = u_x \pm c_a; \quad \lambda_f = u_x \pm c_f; \quad \lambda_s = u_x \pm c_s; \quad \lambda_d = u_x \tag{11}$$

The Alfvén and the slow and fast magnetosonics velocities are, respectively:

$$\begin{aligned} c_a &= \frac{B_x}{\sqrt{\rho}}; \\ c_{f,s}^2 &= 0.5\left[\frac{\gamma p + B^2}{\rho} \pm \sqrt{\left(\frac{\gamma p + B^2}{\rho}\right)^2 - 4\frac{\gamma p B_x^2}{\rho^2}}\right] \end{aligned} \tag{12}$$

where the positive sign corresponds to fast magnetosonics waves.
Finally, the eigenvectors have been normalized to avoid problems due to the system degeneracy.

Transport coefficients

To evaluate the viscosity (μ), electric conductivity (σ) and thermal conductivity (κ) the expressions given by Spitzer (1956) are used:

$$\begin{aligned} \mu &= 2.204x10^{-14}\frac{M_i T^{\frac{5}{2}}}{Z^2 \ln\Lambda}, \\ \sigma &= 2.634x10^{-2}\gamma_E(Z)\frac{T^{\frac{3}{2}}}{z\ln\Lambda}, \\ \kappa &= 1.96x10^{-9}\varepsilon_T(Z)\delta_T(Z)\frac{T^{\frac{5}{2}}}{z\ln\Lambda} \end{aligned} \tag{13}$$

where M_i is the atomic number of the ions in the plasma, T is the temperature, z is the mean ion charge, Λ and is the number of particles in a Deybe sphere. Coefficients $\varepsilon_T(Z), \delta_T(Z)$ and $\gamma_E(Z)$ are correction factors; when $z = 1$ the coefficients are $\gamma_E(Z) = 0.582$, $\varepsilon_T(Z) = 0.419$ and $\delta_T(Z) = 0.225$.

NUMERICAL METHOD

The equations outlined in the previous section are solved using an approximate Riemann solver together with an explicit scheme. To calculate the numerical flux the TVD upwind technique of

Yee, *et al.* (1985) has been implemented, by doing so a second order approach is obtained. This technique is used to calculate the numerical fluxes in all interior cells. The TVD method for the system given in Eq. (1) can be expressed in the following way:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(\overline{F}^n_{j+\frac{1}{2}} - \overline{F}^n_{j-\frac{1}{2}} \right) \tag{15}$$

The function that determines the numerical fluxes is defined as:

$$\overline{F}^n_{j+\frac{1}{2}} = \frac{1}{2}\left(F^n_{j+1} + F^n_j + R^n_{j+\frac{1}{2}} \Phi^n_{j+\frac{1}{2}} \right)_h + \left(F^n_{j+\frac{1}{2}} \right)_p \tag{16}$$

being R the matrix that contains the right eigenvectors of the matrix A_C. Φ is the dissipation vector whose elements for the second order TVD-upwind scheme are expressed by:

$$\phi^l_{j+\frac{1}{2}} = \left(g^l_{j+1} + g^l_j \right) - \sigma\left(\lambda^l_{j+\frac{1}{2}} + \gamma^l_{j+\frac{1}{2}} \right) \alpha^l_{j+\frac{1}{2}} \tag{17}$$

The following limiter function was used in this work:

$$g^l_j = \operatorname{sgn}\left(\lambda^l_{j+\frac{1}{2}} \right) \max\left\{ 0, \min\left[\sigma^l_{j+\frac{1}{2}} \left| \alpha^l_{j+\frac{1}{2}} \right|, \frac{\operatorname{sgn}\left(\lambda^l_{j+\frac{1}{2}} \right)}{2} \sigma^l_{j+\frac{1}{2}} \alpha^l_{j-\frac{1}{2}} \right] \right\} \tag{18}$$

The function $\sigma(z)$ is given by:

$$\begin{aligned} \sigma(z) &= |\lambda|, \quad \text{if} \quad |\lambda| \geq \varepsilon \\ \sigma(z) &= \frac{\lambda^2 + \varepsilon^2}{2\varepsilon}, \quad \text{if} \quad |\lambda| < \varepsilon \end{aligned} \tag{19}$$

and $\gamma^l_{j+\frac{1}{2}}$ is defined as:

$$\begin{aligned} \gamma^l_{j+\frac{1}{2}} &= \frac{\left(g^l_{j+1} - g^l_j \right)}{\alpha^l_{j+\frac{1}{2}}}, \quad \text{if} \quad \alpha^l_{j+\frac{1}{2}} \neq 0 \\ \gamma^l_{j+\frac{1}{2}} &= 0, \quad \text{if} \quad \alpha^l_{j+\frac{1}{2}} = 0 \end{aligned} \tag{20}$$

For the sake of simplicity, the primitive variables have been used to obtain α^l:

$$\alpha^l = L^l_p \cdot \left(W_{i+1} - W_i \right) \tag{21}$$

L^l_p being the left eigenvector of the l^{th} wave.

To obtain the evaluation of parabolic flow is used the finite difference method. Finally, the hyperbolic flow and parabolic flow are added.

RESULTS

In this section the results obtained for the Riemann problem proposed by Brio and Wu (1988) are presented. This is a benchmark widely used by the MGD scientific community with the objective of evaluating the behavior of the numerical techniques and computational codes. Brio and Wu studied the extension to MGD of the classic shock tube used in gas dynamics. This example is denominated coplanar Riemann problem because only components of the velocity and magnetic field vectors in two directions are allowed. It is important to note that Brio and Wu solved this benchmark only for ideal MGD.

The variables are given in non-dimensional form and an unit length of the magnetogasdynamic shock tube is considered. The discontinuity or diaphragm that separates the left and right initial states is located in the middle of the tube. The initial values are:

$$W_l = (1.0,0.0,0.0,0.0,0.75,1.0,0.0,1.0)^T$$
$$W_r = (0.125,0.0,0.0,0.0,0.75,-1.0,0.0,0.1)^T \tag{22}$$

W_l and W_r being the vectors that contain the primitive variables corresponding to both sides of the diaphragm. Figures 1, 2, 3 show the transverse magnetic field (B_y), the velocity in the longitudinal direction (u_x), and the density (ρ) plotted as a function of the longitudinal distance for ideal and real MGD. A fixed mesh with 4000 nodes is used. It can be noticed that the present results agree satisfactorily with those published by Brio and Wu (1988), and Bodgan (1999).

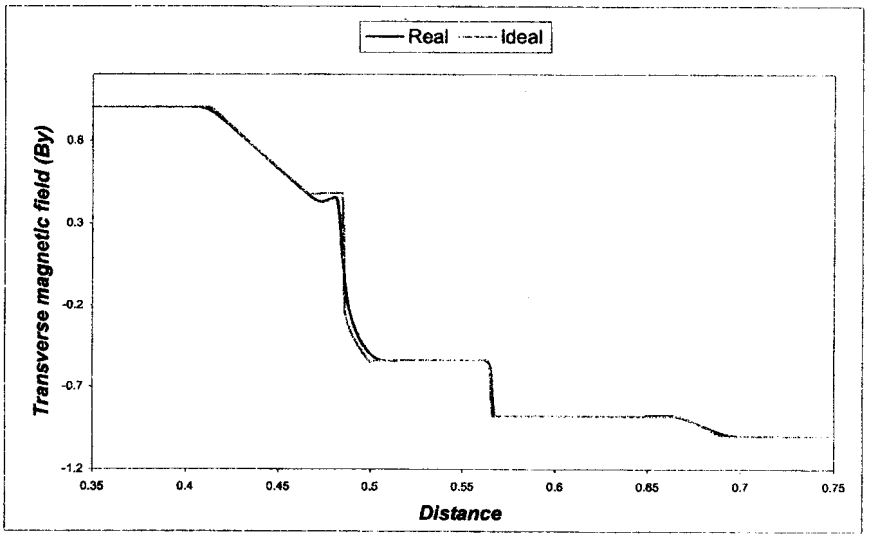

Figure 1. Transverse magnetic field in function to the longitudinal distance.

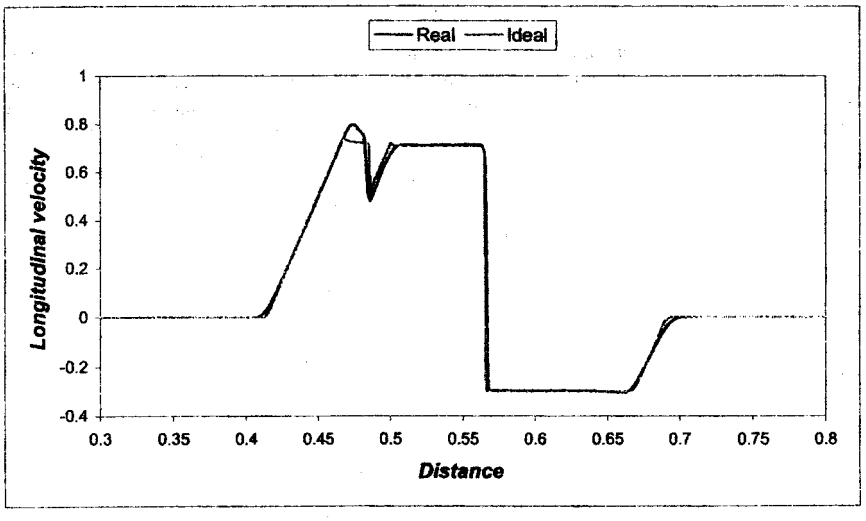

Figure 2. Longitudinal velocity in function to the longitudinal distance.

Finally, one observes that the time required to simulate real MGD flows is greater than the time employed in the simulation of ideal MGD, so to keep computing times comparable, a lower number of nodes should be used. The adopted values of viscosity, electrical resistivity and thermal conductivity are $0.00272728\,Pa.s$, $0.0003 Nm^2 s/C$ and $0.817\,N/s\,^{\circ}K$, respectively.

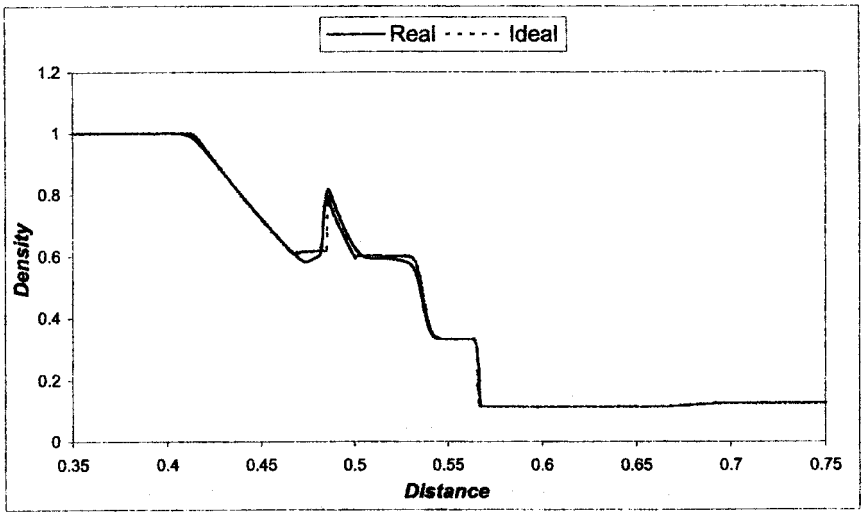

Figure 3. Density in function to the longitudinal distance.

CONCLUSIONS

The main conclusions obtained throughout this research are the followings:
1 - The numerical simulation using an approximate Riemann solver together with the Yee's TVD scheme has shown to be an effective alternative from the point of view of low computacional cost and the accuracy of the results for ideal magnetogasdynamic.
2 - The technique developed by Powell does not introduce modifications in the results as compared to those obtained by using seven waves.
3 - The evaluation of the parabolic terms introduces additional constraints on the CPU time. This is larger than for the ideal MGD.
4 - To reach the solution of the real magnetogasdynamics equations is necessary to use lower CFL numbers than for the ideal magnetogasdynamics equations.
5 - The parabolic contributions lead to smoother distributions of the mechanical and magnetic variables. Therefore, the corresponding waves are not sharply defined.

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