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PERFECT CONTACT, LAPLACE TRANSFORM AND THEIR LINKS

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ABSTRACT

In this paper, we consider a slab represented by the interval $0 < x < 1$, at the initial temperature $u_0(x) = M > 0$ having a positive constant heat flux q on the left face and the contact perfect condition, $u_x(1,t) + \gamma u_t(1,t) = 0$ on the right face $x = 1$. We consider the corresponding heat conduction problem and we assume that the phase-change temperature is 0^0C. We compare estimations of the time of the occurrence of the phase-change obtained by means of Laplace Transform and Method of Lines.

RESUMEN

En este trabajo, nosotros consideramos un material representado por el intervalo $0 < x < 1$, a una temperatura inicial $u_0(x) = M > 0$ con un flujo de calor positivo q en la cara izquierda y la condición de contacto perfecto, $u_x(1,t) + \gamma u_t(1,t) = 0$ en la derecha. Nosotros consideramos el correspondiente problema de conduccion del calor y asumimos que el cambio de fase se produce a 0^0C. Comparamos estimaciones de los tiempos obtenidos por medio de la tranformada de Laplace y el método de líneas.

INTRODUCTION

We consider a one-dimensional slab with its face $y = \ell$ in perfect thermal contact with mass M_f per unit area of a well-stirred fluid (or a perfect conductor) of specific heat c_f. We consider the following heat conduction problem:
Problem P

$$kv_{yy} = \rho c v_\tau, \qquad D = \{(y,t) : 0 \le y \le \ell, \tau > 0\}, \tag{1}$$

$$v(y,0) = V_0 > 0, \qquad 0 \le y \le \ell, \tag{2}$$

$$kv_y(0,\tau) = q_0 > 0, \qquad \tau > 0, \tag{3}$$

$$kv_y(\ell,\tau) + M_f c_f v_\tau(\ell,\tau) = 0, \qquad \tau > 0, \tag{4}$$

We consider the following changes of variables:

$$x = \frac{y}{\ell},$$

$$t = \frac{k\tau}{\rho c \ell^2}.$$

$$v(y,\tau) = cu(x,t)$$

We consider the new problem P1:
Problem P1

$$\begin{aligned}
u_{xx} &= u_t, \qquad D = \{(x,t) : 0 \le x \le 1, t > 0\}, && (5)\\
u(x,0) &= M > 0, \qquad 0 \le x \le 1, && (6)\\
u_x(0,t) &= q > 0, \qquad t > 0, && (7)\\
u_x(1,t) &= \gamma u_t(1,t), \qquad t > 0. && (8)
\end{aligned}$$

Where:

$$M = cV_0,$$

$$q = \frac{c\ell q_0}{k},$$

$$\gamma = -\frac{M_f c_f}{\rho c}.$$

We are interested in obtain estimations of the occurrence of the phase-change for small t (i.e. $t \approx 0$) by means of Laplace Transform (section 1) and Method of Lines(section 2). The Problem P1 holds the following minimum principle that we use in section 1 and 2 in order to consider only the behavior of $u(x,t)$ for $x = 0$.

Lemma 1 *The solution of Problem P1 holds:*

$$u(0,t) \le u(x,t), \qquad 0 \le x \le 1, \qquad t > 0.$$

Proof *We set $v = u_x$, the function $v(x,t)$ satisfies the following heat conduction problem:*

$$\begin{aligned}
v_{xx} &= v_t, \qquad D = \{(x,t) : 0 \le x \le 1, t > 0\}, && (9)\\
v(x,0) &= 0, \qquad 0 \le x \le 1, && (10)\\
v(0,t) &= q > 0, \qquad t > 0, && (11)\\
v(1,t) &= \gamma v_x(1,t), \qquad t > 0. && (12)
\end{aligned}$$

By using the maximum principle for $0 \le x \le 1$ and $t > 0$ we have:

$$\min v(x,t) = \min\{q, 0, v(1,t)\},$$

We suppose that $v(1,t) < 0$ (we remark that $q > 0$), it follows that:

$$\min v(x,t) = v(1,t),$$

by using Hopf lemma we deduce that:

$$v_x(1,t) < 0,$$

which contradicts the condition (12), therefore $u_x(x,t) \ge 0$ from which we obtain the thesis.

THE LAPLACE TRANSFORM

In this section we apply the Laplace Transform at problem considered, we obtain the exact solution of the transformed problem and we use asymptotic behavior to approximate the inverse of the solution of the transformed problem. We approximate the solution of the original problem in order to obtain estimations of the time of the occurrence the change-phase in the material. The Laplace Transform is:

$$U(s,t) = L(u(x,t)) = \int_0^{\infty} u(x,t)e^{-st}dt,$$

where s is a positive parameter. We apply the Laplace Transformation to Problem 1, that is, multiply by e^{-st} and integrate with respect to t from 0 to ∞. This gives:
Problem P2

$$U_{xx}(s,x) - sU(s,x) = -M, \tag{13}$$
$$U_x(s,0) = \frac{q}{s}, \tag{14}$$
$$\gamma sU(s,1) - U_x(s,1) = \gamma M. \tag{15}$$

We remark that $\gamma < 0$.
The solution of Problem P2 is given by:

$$U(s,x) = A(s)\exp(-\sqrt{s}x) + B(s)\exp(\sqrt{s}x) + \frac{M}{s}, \tag{16}$$

where

$$A(s) = \frac{-(\gamma s - \sqrt{s})\exp(\sqrt{s})q}{2s(\gamma s^{\frac{3}{2}}\cosh(\sqrt{s}) - s\sinh(\sqrt{s})}, \tag{17}$$

and

$$B(s) = \frac{(\gamma s + \sqrt{s})\exp(-\sqrt{s})q}{2s(\gamma s^{\frac{3}{2}}\cosh(\sqrt{s}) - s\sinh(\sqrt{s})}. \tag{18}$$

After simple calculation we obtain the following expression for $U(s,x)$:

$$U(s,x) = \left[\frac{-\gamma s\sinh(\sqrt{s}(1-x)) + \sqrt{s}\cosh(\sqrt{s}(1-x))}{\gamma s^{\frac{5}{2}}\cosh(\sqrt{s}) - s^2\sinh(\sqrt{s})}\right] q + \frac{M}{s}, \tag{19}$$

Remark 1 *If we consider $q = 0$, (19) implies that $U(s,x) = \frac{M}{s}$. In this case $u(x,t) = M$ for $0 \leq x \leq 1$ and $t > 0$.*

Lemma 2 *Suppose that the Laplace transform $F(s) = L(f(t))$ has an asymptotic expansion:*

$$F(s) \approx \sum_{\nu=1}^{\infty} a_\nu s^{-\lambda_\nu} \qquad s \to +\infty,$$

where

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$$

then

$$f(t) \approx \sum_{\nu=1}^{\infty} \frac{a_\nu t^{\lambda_\nu - 1}}{(\lambda_\nu - 1)!} \qquad t \to 0.$$

For more details, we refer [1].

Theorem 1 *For the Problem P1 an estimation of the time of change of phase is given by the following equality:*

$$t_{ch} = \left(\frac{\sqrt{\pi}M}{2q}\right)^2.$$

Proof *We need consider only the behavior of (19) for large s and $x = 0$, so that*

$$U(s,0) \approx \frac{M}{s} - \frac{q}{s^{\frac{3}{2}}}, \qquad s \to +\infty. \tag{20}$$

Therefore we can obtain the asymptotic behavior for $u(0,t)$ for small t:

$$u(0,t) \approx M - \frac{2}{\sqrt{\pi}} q t^{\frac{1}{2}}, \qquad t \to 0, \tag{21}$$

We consider the flux q as a variable and the fix initial condition M.
We need $u(0,t) = 0$ in order to have a change phase in the material, that is:

$$M - \frac{2}{\sqrt{\pi}} q t^{\frac{1}{2}} = 0, \tag{22}$$

then the time of change phase holds the following equality:

$$t_{ch} = \left(\frac{\sqrt{\pi}M}{2q}\right)^2. \tag{23}$$

Remark 2 *The estimation given by theorem 1 is valid for $t \approx 0$ which is equivalent to $q \gg M$. We can see that the estimation does not depend on the constant γ.*

Remark 3 *In the problem P1 when $\gamma \to -\infty$ we obtain the following problem*

$$u_{xx} = u_t, \qquad D = \{(x,t) : 0 \leq x \leq 1, t > 0\}, \tag{24}$$
$$u(x,0) = M > 0, \qquad 0 \leq x \leq 1, \tag{25}$$
$$u_x(0,t) = q > 0, \qquad t > 0, \tag{26}$$
$$u(1,t) = C, \qquad t > 0, \tag{27}$$

where C is a constant. This problem was studied in [3] and the authors obtain the following expresion for the time t^* of change of phase

$$t^* \geq \frac{\pi M^2}{4q^2},$$

which is the same that in the theorem two.

In order to obtain an expresion for the time of change of phase which depends on γ, in the next section we will use the method of lines.

METHOD OF LINES

In this section we apply the method of lines to Problem P1 in which the partial differential equation is replaced by a sequence of ordinary differential equations at discrete time levels. For this purpose, we shall define a partition $\{0 = t_0 < t_1 < ... < t_N = T\}$ of $[0, T]$, which for ease of notation is assumed to have equal subintervals $\Delta t = t_i - t_{i-1}$ and $i = 1, .., N$. The simplest, and most commonly used, method of lines approximation for Problem P1 requires the substitution

$$u_t(x, t_n) \approx \frac{u(x, t_n) - u(x, t_{n-1})}{\Delta t},$$

which reduces the partial differential equation (1) to a second order differential equation

$$\Delta t u_n''(x) - u_n(x) = -u_{n-1}(x),$$

for $n = 1, .., N$ and $\Delta t = \frac{T}{N}$, where $u_n = u(x, t_n)$ and $u_n'' = \frac{du_n(x)}{d^2x}$. The boundary conditions are transformed in the following equations:

$$u_n'(0) = q \tag{28}$$

$$-\Delta t u_n'(1) + \gamma u_n(1) = \gamma u_{n-1}(1) \tag{29}$$

The method of lines approximation for the heat conduction problem P1 is given by:
Problem P3(n)

$$\Delta t u_n''(x) - u_n(x) = -u_{n-1}(x), \qquad n = 1, .., N \tag{30}$$

$$u_n'(0) = q, \tag{31}$$

$$-\Delta t u_n'(1) + \gamma u_n(1) = \gamma u_{n-1}(1). \tag{32}$$

The solution of Problem P3(n) has the representation:

$$u_n(x) = A_{n,k} \exp\left(-\frac{1}{k}x\right) + B_{n,k} \exp\left(\frac{1}{k}x\right) + g_n(x), \tag{33}$$

where

$$\begin{aligned} A_{n,k} &= \frac{q(-k+\gamma)\exp\left(\frac{1}{k}\right) - \frac{1}{k}\left(\gamma u_{n-1}(1) + k^2 g_{n-1}'(1) - g_{n-1}(1)\right)}{2\left(\sinh\left(\frac{1}{k}\right) - \frac{\gamma}{k}\cosh\left(\frac{1}{k}\right)\right)}, \\ B_{n,k} &= \frac{-\frac{1}{k}\left(\gamma u_{n-1}(1) + k^2 g_{n-1}'(1) - g_{n-1}(1)\right) - q(k+\gamma)\exp\left(-\frac{1}{k}\right)}{2\left(\sinh\left(\frac{1}{k}\right) - \frac{\gamma}{k}\cosh\left(\frac{1}{k}\right)\right)}, \\ k &= \sqrt{\Delta t}, \end{aligned} \tag{34}$$

and $g_{n,k}(x)$ is a particular solution of Problem P3(n). We remark that A and B depends on n and k. The particular solution $g_{n,k}(x)$ is given by:

$$g_{n,k}(x) = \frac{1}{k}\int_0^x \sinh\left(\frac{1}{k}(s-x)\right) u_{n-1} ds. \tag{35}$$

Henceforth, in order to simplify the notation we omit the indices k. We consider one iteration (i.e. $n = 1$), in this case the solution of problem P3(1) is given by:

$$u_1(x) = A_{1,k} \exp(-\frac{1}{k}x) + B_{1,k} \exp(\frac{1}{k}x) + \frac{M}{k} \int_0^x \sinh(\frac{1}{k}(s-x))ds. \quad (36)$$

We consider this solution only for $x = 0$ (Lemma 1), after some algebraic manipulation, we obtain:

$$u_1(0) = G_\gamma(k)M + F_\gamma(k)q, \quad (37)$$

where

$$F_\gamma(k) = \frac{\gamma \sinh(\frac{1}{k}) - k \cosh(\frac{1}{k})}{\sinh(\frac{1}{k}) - \frac{\gamma}{k}\cosh(\frac{1}{k})}, \quad (38)$$

and

$$G_\gamma(k) = \frac{\sinh(\frac{1}{k}) - k\cosh(\frac{1}{k}) + \frac{1-\gamma}{k}}{\sinh(\frac{1}{k}) - \frac{\gamma}{k}\cosh(\frac{1}{k})}, \quad (39)$$

We look for k satisfying:

$$u_1(0) = G_\gamma(k)M + F_\gamma(k)q = 0,$$

this equation is equivalent to:

$$H_\gamma(k) = -\frac{M}{q}, \quad (40)$$

where

$$H_\gamma(k) = \frac{\gamma \sinh(\frac{1}{k}) - k\cosh(\frac{1}{k})}{\sinh(\frac{1}{k}) - k\cosh(\frac{1}{k}) + \frac{1-\gamma}{k}}. \quad (41)$$

It is easy to verify that $H_\gamma(k)$ possesses the following propertie

Lemma 3 *The function $H_\gamma(k)$ holds the following properties:*

1. $\lim H_\gamma(k) = 0$ *when* $k \to 0^+$ *for all* $\gamma > 0$.
2. $H_\gamma(k) \leq 0$ *for* $k \approx 0$ *for all* $\gamma > 0$.

Proof

$$\begin{aligned} H_\gamma(k) &= \frac{\gamma k \sinh(\frac{1}{k}) - k^2 \cosh(\frac{1}{k})}{k\sinh(\frac{1}{k}) - \cosh(\frac{1}{k}) + 1 - \gamma} \\ &= \frac{(-k^2+\gamma k)e^{1/k} - (k^2+\gamma k)e^{-1/k}}{(k-1)e^{1/k} - (k-1)e^{-1/k} + 2(1-\gamma)}. \end{aligned}$$

Hence, the behavior of $H_\gamma(k)$ for $k \approx 0$ is given by

$$H_\gamma(k) \approx \frac{-k^2+\gamma k}{k+1-\gamma} \qquad k \approx 0, \tag{42}$$

From (42), (1) and (2) holds.

We use the expresion (42) which holds for $k \approx 0$ in order to solve the equation (40). Firstly, we remark that the estimations are comparables. From theorem 1 we have:

$$-\frac{M}{q} = -\sqrt{\frac{4t}{\pi}}, \tag{43}$$

where we note that $t = k^2$. Now we define the following function:

$$R(k) = -\frac{2k}{\sqrt{\pi}} = -\frac{M}{q}. \tag{44}$$

We know from the (40) that:

$$H_\gamma(k) = -\frac{M}{q}, \tag{45}$$

where

$$H_\gamma(k) = \frac{\gamma \sinh\left(\frac{1}{k}\right) - k\cosh\left(\frac{1}{k}\right)}{\sinh\left(\frac{1}{k}\right) - k\cosh\left(\frac{1}{k}\right) + \frac{1-\gamma}{k}}. \tag{46}$$

We may immediately verify the following lemma which implies that the times given by theorem 1 and 2 are comparable

Lemma 4

$$\lim_{k\to 0} \frac{H_\gamma(k)}{R(k)} = \frac{\gamma\sqrt{\pi}}{2}$$

.

In order to adjust the estimation given by the theorem 1 we consider the following function:

$$\varepsilon(\gamma,k) = H_\gamma(k) - R(k), \tag{47}$$

from (42), the behavior for $k \approx 0$ for this function is given by:

$$\begin{aligned} \varepsilon(\gamma,k) &= \frac{-k^2+\gamma k}{k+1-\gamma} + \frac{2k}{\sqrt{\pi}} && (48)\\ &= \varepsilon(\gamma,0) + \varepsilon'(\gamma,0)k + \mathcal{O}(k^2) && (49)\\ &= \left(\frac{2}{\sqrt{\pi}} + \gamma\right)k + \mathcal{O}(k^2). && (50) \end{aligned}$$

By using the last expresion combined with theorem 1, we obtain a new expresion for the time of change of phase in Problem P1 where the parameter γ appears.

Theorem 2 *For the Problem P1 an estimation of the time of change of phase is given by the following equality:*

$$t_{ch} = \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{\gamma^2}\right)\frac{M^2}{q^2}.$$

Remark 4 *If we consider the case where the domain is seminfinity (i.e. $0 < x < +\infty$), then the exact solution is given by:*

$$u(x,t) = M - 2q\sqrt{t} \quad ierfc(\frac{x}{2\sqrt{t}}) \tag{51}$$

where

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)dt,$$

and

$$erfc(x) = 1 - erf(x), \qquad ierfc(x) = \frac{\exp(-x^2)}{\sqrt{\pi}} - xerfc(x).$$

The time of change of phase (i.e. $u(0,t) = 0$) is given by:

$$t = (\frac{\sqrt{\pi}M}{2q})^2, \tag{52}$$

In this case, we consider the following problem $P2_\infty$ (the Laplace Transform):

$$U_{xx}(s,x) - sU(s,x) = -M, \tag{53}$$

$$U_x(s,0) = \frac{q}{s}. \tag{54}$$

Now, the exact solution for problem $P2_\infty$ in $x = 0$ is given by:

$$U(0,s) = -\frac{q}{s^{3/2}} + \frac{M}{s}.$$

Therefore, we obtain that:

$$u(0,t) = -\frac{2q}{\sqrt{\pi}}\sqrt{t} + M.$$

The time of phase change is equal to (52).

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