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UNSATURATED ANALYSIS. A SIMPLIFIED FORMULATION.

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ABSTRACT

This paper shows a practical way to carry out the determination of pore pressure dissipation meanwhile consolidation process is in progress. For the constitutive equation of soil phase was considered valid the elastic field. Water flux was governed by Darcy's law and air flux by Fick's law. Conservation of fluid mass was also preserved. Solid and fluid models were coupled by effective stress parameter, being solid displacement, water and air pore pressure the mains unknowns. For system equation solving purposes, the finite element method was used. Results are presented in two charts, a displacement-time and pore pressure-time. The symmetry that was attained, the simplicity of laboratory evaluation of parameters and an accessible computational implementation, gave enough background for choosing this model.

INTRODUCTION

Theoretical consolidation analysis for soils first steps may belong to the 40's , carrying out the solution of very simple cases. It wasn't but in the 70's, when problems linked to practical engineering where considered whit the seed of numerical standpoint. Nevertheless, the non saturated media consolidation regarding its three phase nature, was recently undertaken sometime between the 80's and 90's.

Along with the introduction of the unsaturated condition in soil mass, an increasing in state variables defining mixture behavior under burden is originated since, besides the solid grains average stress (effective) and the water stress (pore pressure) developed in saturated case, air phase comes up and the evaluation of the mechanical response is required. It must be pointed out that water and air phases, fills the voids between soil particles.

Although saturated condition is the worst from a structural strength standpoint, non saturated soils behavior consideration interest lies in the evaluation of , among many other, deformation due to humidity variation from the initial state, suction influence with underscore in strength evolution, etc, mainly in structures like embankments, earthfill dams and strongly compacted or, conversely, collapsible soils direct foundations.

The main scope of the present paper is to address the mathematical framework for non saturated soils consolidation analysis and its numerical implementation via finite element method. Also, some preliminary results, are showed. Initial suction values for computational implementation, time integration scheme for the aforementioned suction and water and air permeability parameters obtaining, have also been presented. The model hereafter considered, is of simple computational handling in the sense of the amount of relevant coefficients and its overall solution scheme, therefor turns out as an attractive option.

To motivate the description of unsaturated soils mechanical response, we start up from a set of three coupled equation : Momentum Balance, Conservation of fluid mass together with Darcy's law and flux of air phase depicted by Fick's law.

The foregoing set, besides tackling the complex stress evaluation, give rise to the solution of issues related with non stationary flux with the inclusion of variables like soils voids rate and degree of saturation strongly affected with suction and softly with net stress changes.

GOVERNING EQUATIONS

To outline the stress-strain relationship in a unsaturated soil, a volumetric element of soil subject to external pressure σ_{ij} is considered. The stress state inside the element must meet the following equilibrium set deduced from the local version of momentum balance regardless acceleration terms:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 \qquad \text{, con } i=1,2,3 \tag{1}$$

Where σ_{ij} is the total stress,

$$\sigma'_{ij} = \sigma_{ij} - a_1 p_w \delta_{ij} - a_2 p_a \delta_{ij} \tag{2}$$

Where b_i is the body force per unit volume, σ'_{ij} is the effective stress, a_1 and a_2 are the effective stress parameters, δ_{ij} is the Kronecker's delta, p_w and p_a are the pore-water and pore-air pressures

Furthermore, in a elastic isotropic media, making use of stress-strain relationships :

$$\sigma'_{ij} = 2G\varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij} \tag{3}$$

where:

$$G = \frac{E}{2(1+\mu)} \qquad \text{y} \qquad \lambda = \frac{E.\mu}{((1+\mu)(1-2\mu))}$$

Combining (1), (2) y (3), leading:

$$2G\frac{\partial \varepsilon_{ij}}{\partial x_j} + \lambda \frac{\partial \varepsilon_{jj}}{\partial x_i} + a_1 \frac{\partial p_w}{\partial x_i} + a_2 \frac{\partial p_a}{\partial x_i} + F_i = 0 \tag{4}$$

Due to geometric correspondence between strains ε and displacements u_i (linear field), we obtain:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{5}$$

Substituting (5) in (4)

$$G\frac{\partial^2 u_i}{\partial x_j \partial x_j}+(\lambda+G)\frac{\partial^2 u_j}{\partial x_i \partial x_j}+a_1\frac{\partial p_w}{\partial x_i}+a_2\frac{\partial p_a}{\partial x_i}+F_i=0 \tag{6}$$

Above, the local version of momentum balance without acceleration terms regarding existence of air and water phase embedded in soil structure, is determined..

WATER PHASE

Likewise saturated case [4], flux in unsaturated media is described by Darcy's law and conservation of fluid mass.

$$-\frac{1}{\rho_w}\frac{\partial}{\partial x_i}\left(\rho_w\frac{k_{wi}}{\gamma_w}\frac{\partial p_w}{\partial x_i}\right)=-n_w c_f\frac{dp_w}{\partial t}+\frac{1}{V}\frac{\partial V_w}{\partial t} \tag{7}$$

The latter was obtained through a straightforward material derivative of continuity of fluid mass in which, formerly, was included the concepts of relative velocity and water porosity.

Values of k_{wi} throughout process, are yielded by the following relationships[5]:

$$\begin{aligned} k_w &= k_s && \text{for} && (p_a-p_w)\le(p_a-p_w)_b \\ k_w &= k_s.S_e^{\delta} && \text{for} && (p_a-p_w)>(p_a-p_w)_b \end{aligned} \tag{8}$$

where k_s is water permeability with regard to the saturated case, S_e is the effective degree of saturation which is affected by the air entry value $(p_a-p_w)_b$ (relying on soil type and pore size distribution).

AIR PHASE

Air flux is often depicted by Fick's law. According to Fick's law, the rate of transferred mass by a diffusing substance across a unit area is proportional to concentration gradient of the diffusing substance:

$$J_{ai}=-D_i\frac{\partial C}{\partial x_i} \tag{9}$$

where D_i is the coefficient of diffusion, and C is the concentration of the diffusing air
The latter may be described with regard to unit soil volume V:

$$C=\frac{m_a}{V_a/(1-S_r)n} \tag{10}$$

or depending on air density, ρ_a,

$$C=\rho_a(1-S_r)n \tag{11}$$

Being:

m_a air mass in soil element S_r degree of saturation

n porosity

Doing some mathematical transformation with the former and combining it with the air concentration with regard to unit soil volume, a modified Fick's law version suitable for diffusion process inclusion, is furnished

$$:\quad -\frac{1}{\rho_a}\frac{\partial}{\partial x_i}\left(D_i^*\frac{\partial p_a}{\partial x_i}\right)=-\frac{n_a}{P}\frac{dp_a}{\partial t}+\frac{l_a}{V}\frac{dv_a}{dt} \tag{12}$$

where m_a is the mass of air in the soil element, S_r is the degree of saturation, and n is the porosity. It is strongly useful to relate D_i^* with respect to laboratory easily determined soil properties like air phase permeability: $k_a = D_i^*.g$. Its computation may be postulated by:

$$\begin{array}{lll} k_a = 0.0 & \text{for} & (p_a - p_w) \le (p_a - p_w)_b \\ k_a = f(S_e,\lambda) & \text{for} & (p_a - p_w) > (p_a - p_w)_b \end{array} \tag{13}$$

λ is known as pore size distribution index.

Water and air flux trough a unsaturated porous media, is governed by equations (7) and (12). It should be pointed out that we have two equations and four unknowns (p_w, p_a, V_w y V_a), therefor two additional equations are required. This ones may be obtained relating time rate of V_w y V_a to primary variables p_w, p_a y u_i

Governing equation in unsaturated consolidation.

Considering the relationships between $\frac{dV_a}{dt}$, $\frac{dV_w}{\partial t}$ and the relevant fluid variables. Replacing them in (7) y (12), yield:

$$\frac{dV_w}{dV} = a_1\varepsilon_{ii} - [a_{12} + (a_1 - n_w)c_s]dp_w + a_{12}dp_a \tag{14}$$

$$\frac{dV_a}{dV} = a_2\varepsilon_{ii} - [a_{21} + (a_2 - n_a)c_s]dp_a + a_{21}dp_w \tag{15}$$

those, that are respectively substituted in (7) and (12), yielded:

$$-\frac{1}{\rho_w}\frac{\partial}{\partial x_i}\left(\rho_w\frac{k_{wi}}{\gamma_w}\frac{\partial p_w}{\partial x_i}\right)=a_{11}\frac{\partial p_w}{\partial t}-a_{12}\frac{\partial p_a}{\partial t}-a_1\frac{\partial^2 u_i}{\partial t\partial x_i} \tag{16}$$

$$-\frac{1}{\rho_a}\frac{\partial}{\partial x_i}\left(D_i^*\frac{\partial p_a}{\partial x_i}\right)=a_{22}\frac{\partial p_a}{\partial t}-a_{21}\frac{\partial p_w}{\partial t}-a_2\frac{\partial^2 u_i}{\partial t\partial x_i} \tag{17}$$

Equations (16), (17) and (6), form the general set of differential equations governing consolidation in unsaturated porous media [6], in comparison with some others, provide us a symmetric, and thereby simple from a computational standpoint, system of equation.

Initial values special considerations.

The initial inner stress of a unsaturated soil mass will depend on degree of saturation through the influence it exerts over matrix suction (pa-pw). An initial effective stress may be obtained from equilibrium equation wherein it is substituted the air pressure by atmospheric one and the water pore pressure by initial suction furnished form:

$$(p_a - p_w) = \frac{\left\{\frac{1}{2}.\log\left[\left(\left(1-Sr\right)/_{a'}\right)+1\right]-\frac{1}{2}\log\left[-\left(\left(1-Sr\right)/_{a'}\right)+1\right]\right\}}{b'} \tag{18}$$

Above, a' and b' are degree of saturation state surface parameters [9]:

$$Sr = 1 - (a' + d'\sigma_{net}).tgh(b'(p_a - p_w)) \tag{19}$$

Thereby, the water pore tensile stress due to menisci formation that identified unsaturated soil structure behavior, is included

ADOPTED MODEL

From the foregoing set of differential equations, namely (16), (17) and (6); one can obtained a variational version using the weight residual methods and afterwards, by finite element method application, an algorithmic version of the overall problem can be furnished allowing us a spatial description. To avoid mathematical drawbacks for displacements unknowns eight nodes elements were implemented, meanwhile for pore pressures, four nodes ones were used.

$$\begin{bmatrix} \underset{\sim}{K} & \underset{\sim}{C}_{sw} & \underset{\sim}{C}_{sa} \\ \underset{\sim}{C}_{ws} & \underset{\sim}{P}_{ww} & \underset{\sim}{Q}_{wa} \\ \underset{\sim}{C}_{as} & \underset{\sim}{Q}_{aw} & \underset{\sim}{P}_{aa} \end{bmatrix} \begin{Bmatrix} \dot{\underset{\sim}{u}} \\ \dot{\underset{\sim}{p}}_w \\ \dot{\underset{\sim}{p}}_a \end{Bmatrix} + \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{H}_{ww} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{H}_{aa} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{u} \\ \underset{\sim}{p}_w \\ \underset{\sim}{p}_a \end{Bmatrix} = \begin{Bmatrix} \dot{\underset{\sim}{F}}_s \\ \dot{\underset{\sim}{F}}_w \\ \dot{\underset{\sim}{F}}_a \end{Bmatrix}$$

$$\begin{bmatrix} \underset{\sim}{K} & \underset{\sim}{C}_{sw} & \underset{\sim}{C}_{sa} \\ \underset{\sim}{C}_{ws} & \underset{\sim}{P}_{ww} + \underset{\sim}{H}_{ww}.\alpha.\Delta t & \underset{\sim}{Q}_{wa} \\ \underset{\sim}{C}_{as} & \underset{\sim}{Q}_{aw} & \underset{\sim}{P}_{aa} + \underset{\sim}{H}_{aa}.\alpha.\Delta t \end{bmatrix} \begin{Bmatrix} \Delta \underset{\sim}{u} \\ \Delta \underset{\sim}{p}_w \\ \Delta \underset{\sim}{p}_a \end{Bmatrix} + \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \Delta t.\underset{\sim}{H}_{ww} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \Delta t.\underset{\sim}{H}_{aa} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{u} \\ \underset{\sim}{p}_w \\ \underset{\sim}{p}_a \end{Bmatrix} = \begin{Bmatrix} \Delta \underset{\sim}{F}_s \\ \Delta \underset{\sim}{F}_w \\ \Delta \underset{\sim}{F}_a \end{Bmatrix}$$

Where

$$\underset{\sim}{K} = \int_\Omega \underset{\sim}{B}^{u^T} \underset{\sim}{D} \underset{\sim}{B}^u \, d\Omega \qquad \underset{\sim}{C}_{sw} = -\int_\Omega \underset{\sim}{B}^{u^T} a_1 \underset{\sim}{N}^p \, d\Omega$$

$$\underset{\sim}{C}_{sa} = -\int_\Omega \underset{\sim}{B}^T a_2 \underset{\sim}{N}^p \, d\Omega \qquad \underset{\sim}{P}_{aa} = -\int_\Omega \underset{\sim}{N}^{p^T} a_{22} \underset{\sim}{N}^p \, d\Omega$$

$$\underset{\sim}{P}_{ws} = -\int_\Omega \underset{\sim}{N}^{p^T} a_{11} \underset{\sim}{N}^p \, d\Omega \qquad \underset{\sim}{P}_{aa} = -\int_\Omega \underset{\sim}{N}^{p^T} a_{22} \underset{\sim}{N}^p \, d\Omega$$

$$\dot{\underset{\sim}{F}}_s = \int_\Omega \underset{\sim}{N}^{u^T} \dot{b} \, d\Omega + \int_{\Gamma_\sigma} \underset{\sim}{N}^{u^T} \dot{t} \, d\Gamma \qquad \underset{\sim}{H}_{ww} = -\int_\Omega \nabla \underset{\sim}{N}^{p^T} \frac{\underset{\sim}{k}_{wi}}{\gamma_w} \nabla \underset{\sim}{N}^p \, d\Omega$$

$$\underset{\sim}{Q}_{wa} = -\int_{\Omega} \underset{\sim}{N}^{p^T} \dot{\underset{\sim}{q}}_w \, d\Omega \qquad \underset{\sim}{F}_w = -\int_{\Gamma_w} \underset{\sim}{N}^{p^T} \dot{q}_w \, d\Gamma$$

$$\underset{\sim}{H}_{aa} = -\int_{\Omega} \nabla \underset{\sim}{N}^{p^T} \frac{D_i}{P}(1 - S_r) n \, \nabla \underset{\sim}{N}^p \, d\Omega \qquad \underset{\sim}{F}_a = -\int_{\Gamma a} \underset{\sim}{N}^{p^T} \dot{q}_a \, d\Gamma$$

adopting:

$$a_{11} = n_w(c_f - c_s) + a_1 c_s + a_{12} \qquad a_1 = \frac{c_m - c_s}{c}$$

$$a_{22} = {}^{n_a}\!/_{p} + (a_2 - n_a)c_m$$

$$a_{21} = a_{12} = (a_2 - n_a)(c_m - c_s) \qquad a_2 = \frac{c - c_m}{c}$$

where:

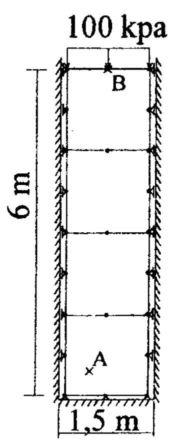
c_m: Compressibility of the soil structure with respect to a change in matric suction
c_s: Compressibility of the soil grains
c: Drained compressibility of the soil structure
c_f: Fluid compressibility

From the already known values of suction and net stress and by using the state surface proposed by [8] and [9], porosity and degree of saturation amounts may be computed.

NUMERICAL EXAMPLE

For this case, for the sake of simplicity in numerical results, the saturation degree remains invariable throughout consolidation process, i.e., tantamount to initial conditions. Even if this is a simplification, several situation in practical engineering, may offer alike situations.

Data:

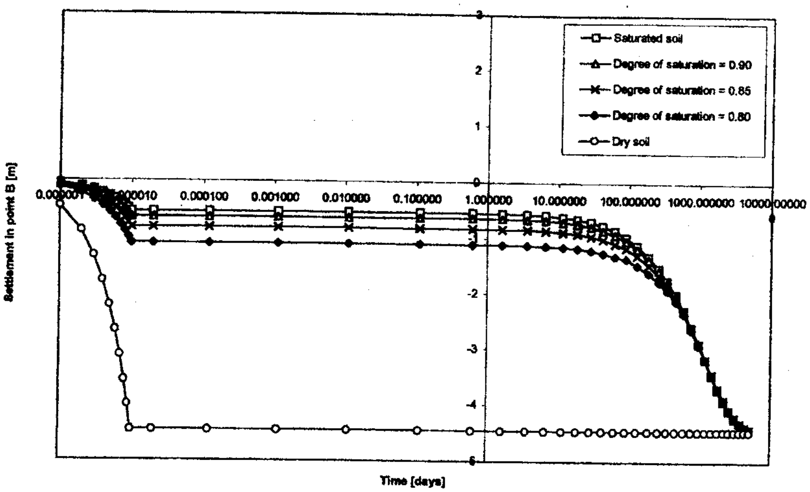


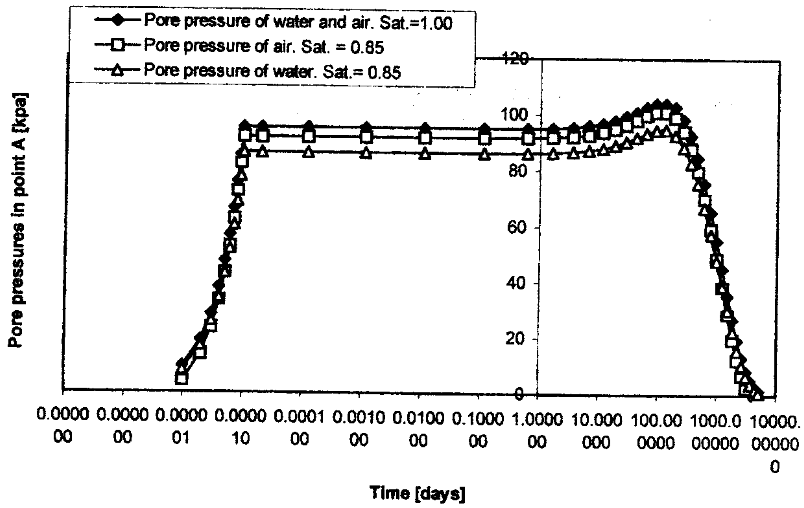
Inner friction angle (radians) ϕ(Rad)	0.1745
Cohesion *C* (Kpa.)	100.00
Initial void rate e_0	0.6
Water compressibility coefficient *Kf.*(Kpa.)	1000000
Grain compressibility coefficient *Ks.* (Kpa.)	1000000

$$e = e_0 + a.(\sigma - p_a) + b.(p_a - p_w) + c.(p_a - p_w).(\sigma - p_a)$$

$$Sr = 1 - (a' + d'\sigma_{net}).\,tgh(b'(p_a - p_w))$$

State surface coefficient for void rate, a	0.00015.
State surface coefficient for void rate, b.	- 0.0001
State surface coefficient for void rate, c	0.000001
State surface coefficient for saturation degree, a'.	0.2
State surface coefficient for saturation degree, b'	0.5
State surface coefficient for saturation degree, d'	-0.001
Slope λ of "effective saturation degree vs. matric suction" plot (also known as "pore size distribution index"	0.25





CONCLUSIONS

- An unsaturated soil consolidation model and its computational implementation, were presented.
- Because of its symmetric nature and a relative low amount of defining parameters, the governing equations remains of simple algorithmic description.
- With light modification of equation (2), physical non linearity may be easily included.
- For further modification of the code, outgrowth of degree of saturation with regards to stress evolution will be considered. It must be warned, that this issue may require special attention because of it involves inverse of hyperbolic tangents with a not well defined image.

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