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STUDY OF FEM ERRORS FOR ELLIPTIC 2D PROBLEMS WITH BORDER SINGULARITIES

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ABSTRACT

In recent years composite mesh methods aiming to construct numerical models where two or more finite element meshes of different granularities are superimposed over the whole domain of the problem were studied by the authors. The focus is stressed on elliptic problems where the domains present border singularities in order to assess the different kinds of error estimations — mixed mesh, Zienkiewicz-Zhu, Zadunaisky's— and their relationships. Estimation of residues and errors for some examples in planar domains is perfomed. Three main illustrating examples are treated here: (1) test problems based on variants of the Poisson equation with boundary conditions of Dirichlet, Neuman and Robin type, (2) elliptic (stationary) advection-diffusion equation with boundary conditions of Dirichlet, Neuman and Robin type; and (3) elliptic problems arising from linear elasticity in plane stress.

RESUMEN

Recientemente los autores han estudiado los métodos de mallas compuestas donde dos o más mallas de elementos finitos de distintas granularidades se superponen en el dominio del problema. Se enfoca en problemas con singularidades de borde con el fin de calibrar los estimadores de error de mallas mezcla, de Zienkiewicz-Zhu, y de Zadunaisky y sus relaciones. Se consideran tres clases principales de ejemplos: (1) problemas test basados en variantes de la ecuación de Poisson con condiciones de borde de Dirichlet, Neumann y Robin (2) versión de estado estacionario de la ecuación de advección-difusión no lineal y (3) problemas elípticos asociados a elasticidad lineal en tensiones planas.

INTRODUCTION

In recent articles we have developed composite finite element models of several problems that are drawn from Mechanical and Chemical Engineering with the aim of providing reasonable *a posteriori* error estimates, and to obtain improved numerical solutions [1] [2] [3] [4]. The main idea of this composite mesh method is to construct a numerical model where two or more finite

element meshes of different granularities (size of the elements) are superimposed over the whole domain of the problem. The motivation of these developments came from the mixture theory of multiphase materials [6] [7]. In this class of materials each component occupies a fraction of the total volume. The physical properties of the composite material is obtained from those of component phases weighted by a participation factor, which is taken as their volumetric fraction or another suitable quantity. In a similar way our numerical model is composed by different finite element meshes and the properties of the whole model is obtained by adding those of the component meshes multiplied by a participation factor. In this case, the different behavior of each phase come from their intrinsic accuracy instead of their physical properties. In this work we study several properties associated to the application of the method to elliptic problems with boundary conditions of Dirichlet, Neumann, and mixed types and showing boundary singularities. The finite element error estimates may be computed a priori or a posteriori. A posteriori error estimates, computed from the numerical solution, are of practical importance and may be categorized under two main subclasses. The first of these is stress recovery, which is also referred to as a postprocessing or flux-projection technique. It was proposed in the context of linear elliptic problems [8]. The second subclass are those of *residual based* estimators, explicit or implicit. The literature covering these topics is vast [9][10]. The error estimates performed by means of the composite mesh fall within the residual based estimators. We also develop in this work multidimensional Zadunaisky's method based error estimators. The composite mesh, proposed in previous papers from the authors, is formed by two finite element meshes sharing the problem domain. The meshes have different element size h, and the connection between components is enforced, for instance, by connecting common nodes. The participation factors α and $(1-\alpha)$ for the fine and coarse meshes, respectively, are defined.
Several illustrating examples are treated in this paper:

1. The test problems based on variants of the elliptic stationary form of the heat equation

$$d\, u_t - \nabla \cdot (c\nabla u) + au = f \tag{1}$$

   with initial values, and boundary conditions of Dirichlet, Neumann and Robin type.

2. The elliptic stationary equation associated to the parabolic advection-diffusion equation

$$u_t - \Delta u + v\nabla u + p(u) = 0 \tag{2}$$

   with initial values, and boundary conditions of Dirichlet, Neumann and Robin type. In this equation p is a polynomial function.

3. Plane strain problems drawn from linear elasticity.

A finite element model of the variational equation is then set, in order to discretize the space coordinates. The task is done in a mixed mesh framework designed to make a posteriori error estimation of approximations and refine the mesh by means of an adaptive algorithm. The next step is then to set a composite mesh finite element. The primary goal of the adaptive composite mesh triangular —or, in some special cases, piecewise bilinear rectangular— finite element method for the stationary problem is to control the space discretization error of the approximate solution as measured from integrals of double mesh residues. The adaptive process is of the h and of the h-r type.
The rest of the paper is devoted to the statement of some conclusions about the implementation details of the proposed method.

NOTATION, MODELS AND METHODS

The test problems we deal with are based on variants of the elliptic stationary form of the heat equation

$$-\nabla \cdot (c\nabla u) + au = f \tag{3}$$

with boundary conditions of Dirichlet ($hu = r$ on $\partial\Omega$), Neumann ($\eta \cdot (c\nabla u) = g$ on $\partial\Omega$) and Robin ($\eta \cdot (c\nabla u) + qu = g$ on $\partial\Omega$) type, where η is the outward unit normal, and g, q, h, and r are functions defined on $\partial\Omega$. Both, the linear and nonlinear cases are treated.
By standard calculations we derive the weak form of the differential equation: Find u such that

$$\int_\Omega ((c\nabla u) \cdot \nabla v + auv - fv)\, dx = \int_{\partial\Omega} (-qu + g)v\, ds, \quad \forall v \tag{4}$$

The stationary elliptic counterpart of the diffusive-advective nonlinear equation case is also treated. The main equation is

$$-\nabla \cdot (c\nabla u) + v\nabla u + p(u) = 0 \text{ in } \Omega \tag{5}$$

where $p(u)$ is a polynomial function in u, with Dirichlet and Robin ($\eta \cdot (c\nabla u) + q(u) = 0$ on $\partial\Omega$) boundary conditions, where $q(u)$ is a polynomial function defined on $\partial\Omega$.
The standard numerical integration of the PDE is performed by the Matlab toolboxes, which are efficient for the classes of problems we deal with (see the Matlab reference books for details on these routines, e.g. [11]). The simultaneous and post processing of these results lead to the error estimations we obtain. We resort to the Matlab Partial Differential Equations Toolbox for the ancillary developments.
A description of the composite mesh concept is given in the next section where we define the double mesh method for error estimation.

The finite element composite mesh

The composite mesh is formed by two (or more) finite element meshes of different accuracy (different element size h), which share the problem domain. Connection between components is enforced (for instance connecting common nodes). A participation factor for each mesh is defined, for details see [4]. This procedure may be applied in several ways, but always with several components, each of different accuracy. The h-version of this procedure uses meshes of different element sizes. The simplest case is that of structured meshes of triangular or rectangular elements, where in the latter case one such element shares the spatial subdomain with four (2D) or eight (3D) elements of the finer mesh. For triangular elements in 2D there are four new ones on the refined mesh. Connection of both meshes is enforced at nodes of the coarser mesh.
The double mesh, therefore, is composed by two finite element meshes of different element size, $h_1 > h_2$, which share the problem domain. In general the second has the additional property of being a refinement of the first. The common nodes connect the two meshes so the complete set of elements are connected. The participation factor of each of the meshes is set as equal, but other arrangements —not addressed in this work— are equally possible.
The operator problem $Lu = f$ can be approximated by the previously explained implementation of the finite element method as

$$L_i u_{h_i} = f_i, \qquad i = 1, 2 \tag{6}$$

where, in general, $h_1 > h_2$, and in practical applications $h_1 = 2h_2$, h_i being the element size of finite element mesh M_{h_i}, $i = 1, 2$. The meaning of symbols h_i are the following: globally they represent the norm of the partition of the domain in elements, but locally they refer to the diameter of the element and say that the elements of size h_2 refine those of size h_1. In the usual case the meshes are connected at common nodes (they are the nodes of the coarser mesh) and one is a refinement of the other.

The mixed mesh solution $u_{h_1h_2}$ is obtained from

$$((1-\alpha)L_{1\to2}+\alpha L_2)u_{h_1h_2}=((1-\alpha)f_{1\to2}+\alpha f_2) \qquad (7)$$

where the symbol $L_{1\to2}$ stands for the immersion of matrix L_1 into the correct places of L_2 padded with zeros. The same for f.
In the case of $\alpha=1/2$, equation (7) reads:

$$(L_{1\to2}+L_2)u_{h_1h_20.5}=(f_{1\to2}+f_2) \qquad (8)$$

We can define the symbol $u_{h_1h_2n}$ for the function, bilinear in the elements, that coincides with $u_{h_1h_2}$ in the coarser nodes (mesh M_{h_1}) and is bilinearly interpolated in the remaining nodes (those of mesh M_{h_2} that are not on mesh M_{h_1}).

Error estimation
We calculate the following residues

$$r_i=L_iu_{h_1h_2}-f_i \qquad (9)$$

the solution $u_{h_1h_2}$ being adapted to the dimension of the matrices involved.
The double mesh solution $u_{h_1h_2}$ lies, in general terms, between the solutions u_{h_i}, $i=1,2$. So the sum of the absolute values of the residues is related and in a direct proportion to the difference between the two approximate solutions and to their absolute errors. The main idea is that the estimation can be done with only one double mesh calculation. In the next section we define the residues.

Zadunaisky's method associated to composite meshes
The method of solving a known solution pseudoproblem associated to the original problem and of similar error production behavior is immersed in the double mesh framework. See [15] for details on the original Zadunaisky's method.

ELLIPTIC TEST PROBLEMS

In this section we treat several elliptic test examples in order to assess the quality of element residues as a posteriori error estimators.
The first and second examples are the double mesh solution of Laplace equation with Dirichlet boundary conditions and a retract angle singularity.

Example 1
The Laplace equation $-\Delta u=0$, in Ω —the unit circle minus the octant defined by lines $y=-x$ and $y=x$, $x<0$— is a first example of domain with a retract corner so the order of approximate solutions is degraded. The boundary conditions of Dirichlet type are $u(x,y)=0$, $y=-x$, $x<0$; $u(x,y)=0$, $y=x$, $x<0$; $u(x,y)=\cos((2/3)\arctan(y/x))$, $x^2+y^2=1$. The retract angle lead to a numerical singularity near the origin of coordinates. The exact solution of this test problem reads $u(x,y)=(x^2+y^2)^{1/3}\cos((2/3)\arctan(y/x))$.
In Fig. 1 the exact double mesh errors $|U_{12}-u|$ are shown. The errors are greater in the neighborhood of the point $(0,0)$. In Fig. 2 the corresponding residues r are plotted. These residues detect the numerical singularity so they allow to refine the mesh in an adaptive manner. Their patterns being similar to those of exact errors. In both cases the errors are located in the neighborhood of point $(0,0)$ where the order of the approximation is degraded by a singularity. The residues allow the adaptive mesh refinement we perform in this example. The mesh is made finer in the neighborhood of point $(0,0)$ (retract angle). In Fig. 3 the exact double mesh errors $|U_{12}-u|$ are shown, but now with a mesh refined in the neighborhood of the singular point.
It is possible to observe the improvement in the quality of the solution due to the adaptive refinement of the mesh.

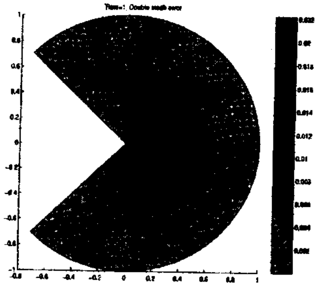

Figure 1: Double mesh errors, ex. 1.

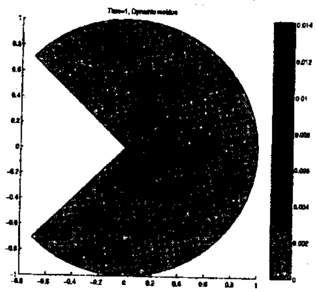

Figure 2: Residues, ex. 1.

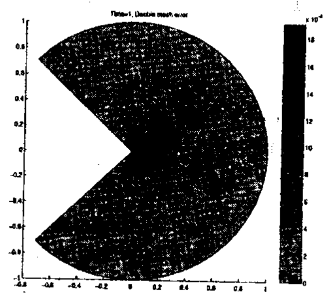

Figure 3: Double mesh errors, ex. 1, adapted mesh.

Example 2

A family of Laplace problems with singularities are studied over the unit square. Here the solutions are

$$u(x,y) = ((x-\frac{1}{2})^2 + (y-\frac{1}{2})^2)^{\frac{\pi}{2\omega}} \cos\frac{\pi}{\omega}\theta, \qquad \theta = \arctan((y-\frac{1}{2})/(x-\frac{1}{2})) \tag{10}$$

$\omega = 2\pi - 2\beta$, and (1) $\beta = \arctan(1)$, (2) $\beta = \arctan(\frac{1}{2})$, and (3) $\beta = \arctan(\frac{1001}{1000})$ (quasi-crack). The case (1) is treated also in the first example, the results for case (2) are omitted for reasons of space. The parameter β is associated to the order of the singularity and is related to the α parameter (for solution improvement).

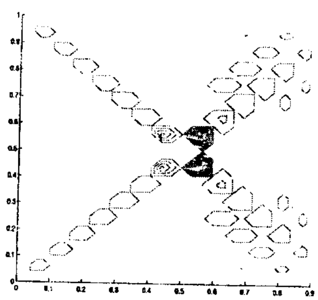

Figure 4: Residues, mixed mesh with weights $\alpha = 0.5$, ex. 2.

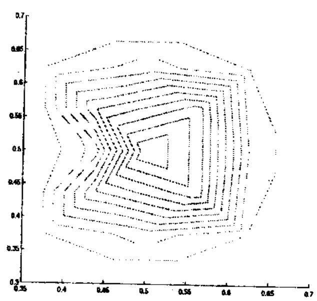

Figure 5: Zienkiewicz-Zhu error estimation, ex. 2.

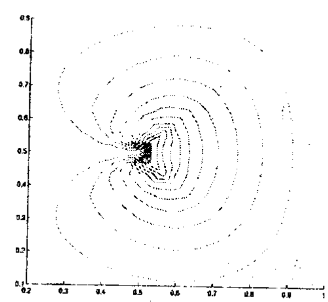

Figure 6: Zadunaisky's method error estimation, ex. 2.

In figures 4, 5, and 6 are respectively plotted the residues, the Zienkiewicz-Zhu error estimators and the Zadunaisky's method error estimator. The double mesh exact errors are omitted. The pattern of these estimators are similar and all detect the corner singularity.

In Table I we show the norms of errors for different mixed mesh solutions for this example. In Fig. 7 the norms of the errors for cases (1): circles for energy norm and pluses for L^∞ norm, and (3): asterisks for energy norm and crosses for L^∞ norm, are plotted against the number of nodes.

In the next section we treat elliptic stationary examples associated with advective and nonlinear terms.

The methodology proposed in this paper is able to perform error estimation in a set of more

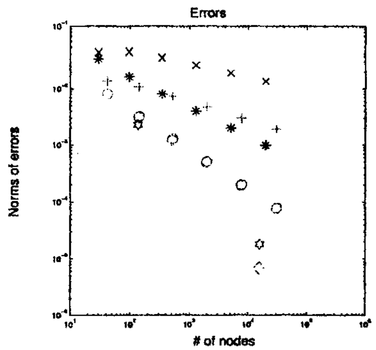

Figure 7: Norms of errors, ex. 2.

Table I: Norms of errors of different mixed mesh solutions for ex. 2 problem, case (3) (2000 nodes aprox.)

α	$\|u-u_d\|_E$	$\|u-u_d\|_{L^2}$	$\|u-u_d\|_{L^\infty}$
0.5	3.6e-3		3.1e-2
1.33	2.6e-3		2.1e-2
1.7	3.0e-3	1.7e-3	1.6e-2
1.9	3.6e-3	1.5e-3	1.2e-2
2	4.0e-3	1.7e-3	1.6e-2

complex problems than those of this section. The main characteristics are: (1) systems of equations, (2) advective terms, and (3) nonlinear coefficients.

NONLINEAR AND ADVECTIVE PROBLEMS

In this section several problems associated to the modeling of catalytic chemical reactors are considered.

Example 3

In this example we treat a non advective non linear elliptic equation associated to a parabolic equation with known exact solution that reads

$$-\Delta u + (1+u)u = (1+x^2+y^2)(x^2+y^2) - 4, \qquad \text{in } \Omega = [0,1]^2 \tag{11}$$

with the boundary conditions $\frac{\partial u}{\partial \eta}(0,y) = 0$, $0 \le y \le 1$, $x = 0$; $\frac{\partial u}{\partial \eta}(x,0) = 0$, $0 \le x \le 1$, $y = 0$; $u(1,y) = (1+y^2)$, $0 \le y \le 1$, $x = 1$; $u(x,1) = (1+x^2)$, $0 \le x \le 1$, $y = 1$; so the exact solution of this test problem is $u(x,y) = (x^2+y^2)$.
This equation is of the same type of models of temperature diffusion in a 2D chemical reactor with Neumann and Dirichlet boundary condition. In Fig. 8 the double mesh error for $\alpha = 0.5$ is shown. In Fig. 9 the corresponding residues are plotted. They detect the change (gradients) in the errors of the methods.

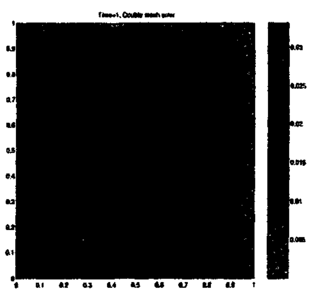

Figure 8: Double mesh errors, ex. 3.

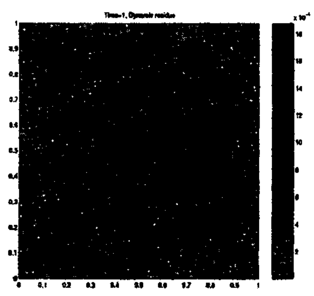

Figure 9: Residues, ex. 3.

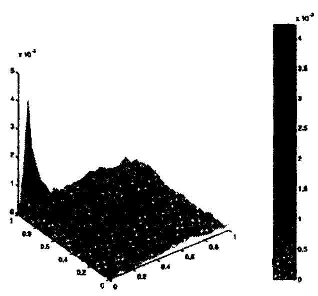

Figure 10: Fine mesh-double mesh difference, ex. 4, adapted mesh.

The next example includes nonlinear and advective terms.

Example 4
In this example the nonlinearities appear in the main body of the equation, an advective term represents the flux of matter inside the 2D reactor and nonlinear (polynomial) boundary conditions represent the catalythic chemical reaction on the top wall of the reactor. The equation is

$$-\Delta u + (1+u)u + 2(1-y)^2\frac{\partial u}{\partial x} = 0, \qquad \text{in } \Omega = [0,1]^2 \tag{12}$$

with the boundary conditions $u(0,y) = 1$, $0 \le y \le 1$, $x = 0$; $\frac{\partial u}{\partial \eta}(x,0) = 0$, $0 \le x \le 1$, $y = 0$; $\frac{\partial u}{\partial \eta}(1,y) = 0$, $0 \le y \le 1$, $x = 1$; $\frac{\partial u}{\partial \eta}(x,1) = -u^2$, $0 \le x \le 1$, $y = 1$;
Based on the residues corresponding to double mesh ($\alpha = 0.5$) solutions we adapt the meshes. Here the exact solution is not known so the reference must be done versus a finer mesh finite element solution so in Fig. 10 the fine-double mesh difference is also shown
The last figures show how the residues detect the zones with greater errors.

LINEAR ELASTICITY EXAMPLE

Example 5
As another example a plane stress distribution on a cracked domain has been also studied. This problem presents a strong singularity. The problem is defined on a square domain, which by symmetry represents a half cracked membrane. The mesh for 32×32 elements, as well as the boundary conditions, is shown in figure 11. The membrane is tractioned by uniform loads applied on the upper bound. The deformed mesh is shown in figure 12. Residuals have been computed for the double mesh with $\alpha = 0.5$. They are drawn in figure 13. It may be observed that errors are concentrated in the region around the crack ti

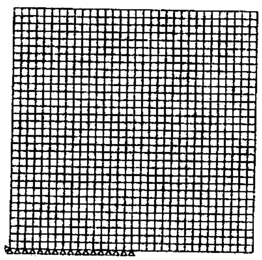

Figure 11: Crack problem: Finite element mesh (32×32 elements).

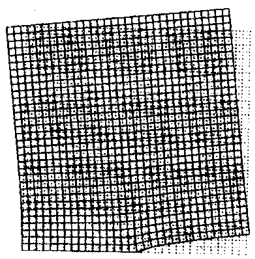

Figure 12: Crack problem: deformed mesh.

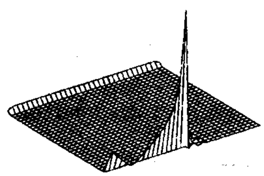

Figure 13: Crack problem: residuals.

CONCLUSIONS

The use of a composite, or mixed, finite element mesh for elliptic problems with border singularities is studied. Two or more finite element meshes are allowed to share the problem domain. These component meshes have different intrinsic accuracies and are affected each by a weight or participation factor. The composite mesh has been used to estimate *a posteriori* discretization errors.

A semiquantitative error estimator based in a double mesh algorithm has been proposed and we have shown that the pattern of this *a posteriori* error is similar to the exact error.
More research is still needed, but the composite mesh method appears to be a powerful and simple tool for obtaining accurate finite element error estimation and allow for adaptivity of meshes.
One of the main improvements we are developing is the complementary analysis of our double mesh and Zadunaisky's estimators.

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