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HAAR LIKE WAVELETS SUPPORTED ON TRIANGLES AND TETRAHEDRA: A MULTIWAVELET APPROACH

Hugo A. Aimar, Ilda C. Hernández
Departamento de Matemática, Fac. de Ingeniería Química-UNL,
Instituto de Matemática Aplicada del Litoral-CONICET
Santiago del Estero 2829 and Güemes 3450, (3000) Santa Fe, Argentina
E-mail: haimar@ceride.gov.ar and ihernan@math.unl.edu.ar

Abstract

The usual dyadic tiling $\mathcal{D}$ of $I\!R^2$ induces a natural triangular tiling of $I\!R^2$, just by dividing each 2-cube $Q \in \mathcal{D}$ into two rectangular triangles. The only difficulty in leading with these geometrical objects is that, even for triangles in the same level, we can not generally obtain any of them by integer translation of a fixed one. Our approach to this situation would be to introduce a new basic transform aside from the usual dilation and integer translation, namely, a "spin". Our aim in this note is to show that the multiwavelet approach solves the problem neatly using only the two traditional transforms.

Resumen

La partición diádica usual $\mathcal{D}$ de $I\!R^2$ induce naturalmente una partición por triángulos, simplemente dividiendo cada cubo $Q \in \mathcal{D}$ en dos triángulos rectágulos. La única dificultad que aparece al trabajar con estos objetos geométricos es que, en general, no podemos obtener cualquiera de ellos por traslaciones enteras de un triángulo fijo y esto aún para triángulos en el mismo nivel. Una solución sería introducir una nueva transformada, la transformada de "spin". El objetivo en estas notas es mostrar que las "multiwavelets" resuelven el problema usando solamente las dos transformadas tradicionales.

1 INTRODUCTION

Let $\mathcal{D}$ be the usual dyadic tiling of $I\!R^2$. Let us induce a triangular tiling of $I\!R^2$ by dividing, for instance by the diagonal whose slope is -1, each 2-cube $Q \in \mathcal{D}$. Unfortunaly these known since Pithagoras geometrical objects do not satisfy the all important similarity property: even for triangles in the same level, we can not generally obtain some of them by integer translation of a fixed one. Instead, two figures will do. Our aim in this note is to show through three special cases that the multiwavelet approach solves the problem nicely using only the two traditional transforms: integer translations and dyadic dilations. Moreover our results can be extended to non-rectangular triangles by changing the dyadic dilations by an adequate dilation matrix A. And even more generally to families of nested partitions satisfying some basic properties.

The existence problem of wavelet bases associated to a MRA of multiplicity r, with arbitrary dilation matrix A and general lattice Γ for cubic fundamental domains, is studied in [2] and [3] (see also [4] and [5]). On the other hand, the existence of Haar like bases on spaces of homogeneous type is considered in [1].

2 FIRST CASE: TILING OF $I\!R^2$ BY RECTANGULAR TRIANGLES

2.1 Domains

Let us start with two rectangular triangles

$$T^1 = \{(x_1, x_2) \in I\!R^2 : \; x_1 \geq 0, \; x_2 \geq 0, \; x_1 + x_2 < 1\}$$
$$T^2 = \{(x_1, x_2) \in I\!R^2 : \; x_1 < 1, \; x_2 < 1, \; x_1 + x_2 \geq 1\}$$

as basic domains, such that $T^1 \cup T^2 = [0,1)^2$ and $T^1 \cap T^2 = \emptyset$. Let us choose the lattice $\Gamma = Z\!\!Z^2$ and the dyadic dilation matrix $A = 2I$. We will denote $T^i_{0,0} = T^i$. Moreover, $T^i_{0,\mathbf{k}} = \mathbf{k} + T^i_{0,0}$ with $\mathbf{k} \in \Gamma$ are the $Z\!\!Z^2$-translates of $T^i_{0,0}$. Let us write $T^i_{j,0} = A^{-j}(T^i_{0,0})$ for the A^{-j}- dilation of $(T^i_{0,0})$. Also $T^i_{j,\mathbf{k}} = A^{-j}(T^i_{0,\mathbf{k}})$ for $i = 1, 2$, $j \in Z\!\!Z$ and $\mathbf{k} \in \Gamma$.

Remark 2.1.1: Observe that these domains do not satisfy two properties that are associated to domains of the type $Q = [0,1)^2$.

- There is no $\mathbf{k} \in \Gamma$ such that $T^i = T^j + \mathbf{k}$ for $i \neq j$ and $i, j = 1, 2$.
- It does not exist a set $\mathcal{K} \subset \Gamma$ such that $A(T^i) = \cup_{\mathbf{k}\in\mathcal{K}} T^i_{0,\mathbf{k}}$.

However, if $x = (x_1, x_2) \in I\!R^2$ then x belongs to exactly one of our triangles. Indeed, if $0 \leq x_i < 1$ for $i = 1, 2$ then

$$x \in \begin{cases} T^1, & \text{if } x_2 < 1 - x_1. \\ T^2, & \text{if } x_2 \geq 1 - x_1. \end{cases}$$

On the other hand, if for $i = 1$ or $i = 2$ we have that $x_i < 0$ or $x_i \geq 1$, then $x = [x_i] + \{x_i\}$ where $[x_i]$ is the integer part and $\{x_i\}$ is the decimal part of x. Again, $x \in T^1_{0,\mathbf{k}}$ or $x \in T^2_{0,\mathbf{k}}$, with $\mathbf{k} = ([x_1], [x_2])$, according to $\{x_2\} \leq 1 - \{x_1\}$ or $\{x_1\} > 1 - \{x_2\}$, respectively.

Moreover, our triangles satisfy two fundamental properties.

Tiling Property The Γ-traslations of T^1, T^2 define a tiling of $I\!R^2$, i.e.,

- $\cup_{\mathbf{k}\in\Gamma}[T^1_{0,\mathbf{k}} \cup T^2_{0,\mathbf{k}}] = I\!R^2$,
- $T^i_{0,\mathbf{k}} \cap T^j_{0,\mathbf{l}} = \emptyset$ for all $\mathbf{k} \neq \mathbf{l}$ and $i, j = 1, 2$.

Quasi-similarity Property For

$$\mathcal{K} = \{\mathbf{k}_1 = (0,0); \; \mathbf{k}_2 = (1,0); \; \mathbf{k}_3 = (0,1); \; \mathbf{k}_4 = (1,1)\}$$

we have $\cup_{i=1}^4(\mathbf{k}_i + A(\Gamma)) = Z\!\!Z^2$ (K is a **digit set for** A **and** Γ). Then

- $A(T^1) = [\cup_{i=1}^3 T^1_{0,\mathbf{k}_i}] \cup T^2_{0,\mathbf{k}_1}$,
- $A(T^2) = [\cup_{i=2}^4 T^2_{0,\mathbf{k}_i}] \cup T^1_{0,\mathbf{k}_4}$.

2.2 Multiresolution Analysis

Let $\phi^1(x) = \chi_{T^1}(x)$ and $\phi^2(x) = \chi_{T^2}(x)$. Denote

$$\phi^i_{j,\mathbf{k}}(x) = 2^j \phi^i(A^j x - \mathbf{k}) = 2^j \chi_{T^i_{j,\mathbf{k}}}(x), \qquad supp(\phi^i_{j,\mathbf{k}}) = T^i_{j,\mathbf{k}};$$
$$\vec{\Phi}(x) = (\phi^1, \phi^2)(x) \text{ and } \vec{\Phi}_{j,\mathbf{k}}(x) = 2^j(\phi^1(A^j x - \mathbf{k}), \; \phi^2(A^j x - \mathbf{k})),$$

for $i = 1, 2$, $\mathbf{k} \in \Gamma$ and $j \in \mathbb{Z}$.

For each $j \in \mathbb{Z}$ we define the following functional spaces:

$\mathbf{V}_j = L^2$-closure of the subspace generated by $\{\phi^i_{j,\mathbf{k}} : i = 1, 2,\ \mathbf{k} \in \Gamma\}$.

Because of the Quasi-similarity Property $\mathbf{V}_0 \subset \mathbf{V}_1$. So each $\phi^i = \chi_{T^i}$ can be expressed as a linear combination of characteristic functions associated to $T^i_{1,\mathbf{k}}$, for $\mathbf{k} \in \mathcal{K}$. In fact,

$$\begin{aligned}\phi^1(x) &= (\chi_{T^1_{1,\mathbf{k}_1}} + \chi_{T^1_{1,\mathbf{k}_2}} + \chi_{T^1_{1,\mathbf{k}_3}} + \chi_{T^2_{1,\mathbf{k}_1}})(x) \\ &= \textstyle\sum_{i=1}^{3} \phi^1(2x - \mathbf{k}_i) + \phi^2(2x - \mathbf{k}_1). \\ \phi^2(x) &= (\chi_{T^2_{1,\mathbf{k}_2}} + \chi_{T^2_{1,\mathbf{k}_3}} + \chi_{T^2_{1,\mathbf{k}_4}} + \chi_{T^1_{1,\mathbf{k}_4}})(x) \\ &= \textstyle\sum_{i=2}^{4} \phi^2(2x - \mathbf{k}_i) + \phi^1(2x - \mathbf{k}_4).\end{aligned}$$

Then we have the following **vectorial scale equation**:

$$\vec{\Phi}(x) = \sum_{i=1}^{4} c_i \vec{\Phi}(2x - \mathbf{k}_i)$$

where $c_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $c_2 = c_3 = Id$ and $c_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$.

It is not hard to prove that ϕ^1, ϕ^2 are the scaling functions of a MRAs (Multiwavelet Multiresolution Analysis, see [3]) associated to A and Γ, i.e., the family $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^2)$ enjoys the following properties

- *Scaling:* $\mathbf{V}_j \subset \mathbf{V}_{j+1}$ for all $j \in \mathbb{Z}$.
- *Separation:* $\cap_{j\in\mathbb{Z}} \mathbf{V}_j = \emptyset$.
- *Density:* $\cup_{j\in\mathbb{Z}} \mathbf{V}_j = \mathbb{R}^2$.
- *Similarity:* $g \in \mathbf{V}_j$ if and only if $g(2\cdot) \in \mathbf{V}_{j+1}$.
- *Basis:* It exists $\phi^i \in L^2(\mathbb{R}^2)$ with $i = 1, 2$ such that $\{\phi^1(\cdot - \mathbf{k}),\ \phi^2(\cdot - \mathbf{k}),\ \mathbf{k} \in \Gamma\}$ is a *bon*$(\mathbf{V}_0)$.

2.3 Wavelet Space

As usual, we define the wavelet spaces $\mathbf{W}_j$ associated to the multiresolution spaces to be the orthogonal complement of $\mathbf{V}_j$ in $\mathbf{V}_{j+1}$, for $j \in \mathbb{Z}$.

Let us first find a basis for $\mathbf{W}_0$. By definition $\mathbf{V}_1 = \mathbf{V}_0 \oplus \mathbf{W}_0$. Since we already have a basis for $\mathbf{V}_0$, then the task is to complete this basis to a basis in $\mathbf{V}_1$. On one hand, $supp(\phi^1_{0,0}) = T^1 = T^1_{1,\mathbf{k}_1} \cup T^1_{1,\mathbf{k}_2} \cup T^1_{1,\mathbf{k}_3} \cup T^2_{1,\mathbf{k}_1}$ and $supp(\phi^2_{0,0}) = T^2 = T^2_{1,\mathbf{k}_2} \cup T^2_{1,\mathbf{k}_3} \cup T^2_{1,\mathbf{k}_4} \cup T^1_{1,\mathbf{k}_4}$. On the other hand, we need to have the function identically $\sqrt{2}$, on each T^i, as an element of the basis. So we must construct three more functions for each triangle T^i, $i = 1, 2$.

This can be done in many ways. A "näive" one is to start with characteristic functions and to orthogonalize with the Gram-Schmidt's method. Let us start with the characteristic functions associated to our triangles T^i. Set

$v^1 = \sqrt{2}\,\chi_{T^1}$

$$v^1_1 = 2\sqrt{2}\,\chi_{T^1_{1,\mathbf{k}_1}} \quad v^1_2 = 2\sqrt{2}\,\chi_{T^1_{1,\mathbf{k}_2}} \quad v^1_3 = 2\sqrt{2}\,\chi_{T^1_{1,\mathbf{k}_3}} \quad v^1_4 = 2\sqrt{2}\,\chi_{T^1_{1,\mathbf{k}_4}}$$

$v^2 = \sqrt{2}\,\chi_{T^2}$

$$v_1^2 = 2\sqrt{2}\chi_{T^2_{1,\mathbf{k}_1}} \quad v_2^2 = 2\sqrt{2}\chi_{T^2_{1,\mathbf{k}_2}} \quad v_3^2 = 2\sqrt{2}\chi_{T^2_{1,\mathbf{k}_3}} \quad v_4^2 = 2\sqrt{2}\chi_{T^2_{1,\mathbf{k}_4}}$$

We obtain three orthonormal functions supported on T^1, which are orthogonal to each other and orthogonal to $\mathbf{V}_0$. Let $T^i_{1,\mathbf{k}_s} = T^i_s$. Then

- $\psi_1^1 = \frac{\sqrt{2}}{\sqrt{3}}\,(3\chi_{T_1^1} - \chi_{T_2^1} - \chi_{T_3^1} - \chi_{T_1^2}) = \frac{1}{2\sqrt{3}}(3v_1^1 - v_2^1 - v_3^1 - v_1^2)$.
- $\psi_2^1 = \frac{\sqrt{3}}{6}\,(8\chi_{T_2^1} - 4\chi_{T_3^1} - 4\chi_{T_1^2}) = \frac{\sqrt{3}}{3\sqrt{2}}(2v_2^1 - v_3^1 - v_1^2)$.
- $\psi_3^1 = 2\,(\chi_{T_3^1} - \chi_{T_1^2}) = \frac{1}{\sqrt{2}}(v_3^1 - v_1^2)$.

Similarly, other three functions are obtained supported on T^2. The functions in this second set are orthogonal to the above one, because the supports are disjoint, and they are orthogonal to $\mathbf{V}_0$ by construction. So the set $\{\psi_s^1(A\cdot -\mathbf{k}),\ \psi_s^2(A\cdot -\mathbf{k}) : \mathbf{k}\in\Gamma,\ \ s = 1,2,3\}$ is a $bon(\mathbf{W}_0)$. Then $\{v^1,\ v^2,\ \psi_s^1(A\cdot -\mathbf{k}),\ \psi_s^2(A\cdot -\mathbf{k}) : \mathbf{k}\in\Gamma,\ \ s=1,2,3\}$ is a $bon(\mathbf{V}_1)$.

Yet a "traditional" way to construct an orthonormal basis of $\mathbf{W_0}$ is by using the self-similarity condition provided by the MRA. Since $\vec{\phi}\in\mathbf{V}_0\subset\mathbf{V}_1$ then the wavelets must be a linear combinations of shifts and dilations of the vector scaling function $\vec{\phi}$:

$$\psi^i = \sum_{\mathbf{k}} c^i_{\mathbf{k}}\vec{\phi}(Ax-\mathbf{k})$$

for some 2×2 matrices $c^i_{\mathbf{k}}$. For a particular choice of the entries of $c^i_{\mathbf{k}}$ we obtain the following set of functions which have 0-moments:

$$\begin{aligned}
\psi_1^1(A\cdot -\mathbf{k}_1) &= \phi^2_{1,\mathbf{k}_1} + \phi^1_{1,\mathbf{k}_1} - \phi^1_{1,\mathbf{k}_2} - \phi^1_{1,\mathbf{k}_3}\\
\psi_2^1(A\cdot -\mathbf{k}_2) &= \phi^2_{1,\mathbf{k}_1} + \phi^1_{1,\mathbf{k}_2} - \phi^1_{1,\mathbf{k}_1} - \phi^1_{1,\mathbf{k}_3}\\
\psi_3^1(A\cdot -\mathbf{k}_3) &= \phi^2_{1,\mathbf{k}_1} + \phi^1_{1,\mathbf{k}_3} - \phi^1_{1,\mathbf{k}_1} - \phi^1_{1,\mathbf{k}_2}\\
\psi_1^2(A\cdot -\mathbf{k}_4) &= \phi^1_{1,\mathbf{k}_4} + \phi^2_{1,\mathbf{k}_4} - \phi^2_{1,\mathbf{k}_3} - \phi^2_{1,\mathbf{k}_2}\\
\psi_2^2(A\cdot -\mathbf{k}_3)) &= \phi^1_{1,\mathbf{k}_4} + \phi^2_{1,\mathbf{k}_3} - \phi^2_{1,\mathbf{k}_4} - \phi^2_{1,\mathbf{k}_2}\\
\psi_3^2(A\cdot -\mathbf{k}_2) &= \phi^1_{1,\mathbf{k}_4} + \phi^2_{1,\mathbf{k}_2} - \phi^2_{1,\mathbf{k}_4} - \phi^2_{1,\mathbf{k}_3}
\end{aligned}$$

Again the set $\{\phi^i(A\cdot -\mathbf{k}),\ \psi_s^i(A\cdot -\mathbf{k}) : i = 1,2\ \mathbf{k}\in\Gamma \text{ and } s = 1,2,3\}$ is a $bon(\mathbf{V}_1)$.

The construction of bases for $\mathbf{W}_j$, $j\neq 0$, is now easily done, as a consequence of the following straightforward properties.

- $\mathbf{W}_j$ is similar to $\mathbf{W}_0$: $f\in\mathbf{W}_j$ if and only if $f(A^{-j}\cdot)\in\mathbf{W}_0$, for $j\in\mathbb{Z}$.
- $\{\psi_l^i(x-\mathbf{k}) : l = 1,2,3,\ i = 1,2 \text{ and } \mathbf{k}\in\Gamma\}$ is $bon(\mathbf{W}_0)$ if and only if $\{2^j\psi_l^i(A^jx-\mathbf{k}) : l = 1,2,3,\ \mathbf{k}\in\Gamma\}$ is $bon(\mathbf{W}_j)$ for each $j\in\mathbb{Z}$.

The set $\{2^j\psi_s^i(A^jx-\mathbf{k}) : i = 1,2,\ s = 1,2,3,\ j\in\mathbb{Z},\ \mathbf{k}\in\Gamma\}$ is an orthonormal basis of $L^2(\mathbb{R}^2)$ because of the MRAs structure.

Let us finally observe that if we keep the domains T^1, T^2 and $\Gamma = \mathbb{Z}^2$, but take $A = 3Id$, then $T^i = [\cup_{l=1}^6 T^i_{1,l}]\cup[\cup_{s=1}^3 T^j_{1,s}]$ with $i\neq j$. Now we need to construct eight functions ψ_i^s. The set $\{\psi_s^1(A\cdot -\mathbf{k}),\ \psi_s^2(A\cdot -\mathbf{k}) : \mathbf{k}\in\Gamma,\ \ s = 1,\cdots,8\}$ is a $bon(\mathbf{W}_0)$.

3 SECOND CASE: ANOTHER TRIANGULAR TILING OF $I\!R^2$

If we use the lattice $\Gamma = \frac{1}{2}\mathbb{Z}^2$ and the same dilation matrix A we end up with four triangles:

$T^1 = T^1_{0,0} : (0,0), (1,0), (\frac{1}{2}, \frac{1}{2})$, $\quad T^2 = T^2_{0,0} : (1,0), (1,1), (\frac{1}{2}, \frac{1}{2})$,

$T^3 = T^3_{0,0} :\ (1,1), (0,1), (\frac{1}{2}, \frac{1}{2})$, $\quad T^4 = T^4_{0,0} : (0,1), (0,0), (\frac{1}{2}, \frac{1}{2})$.

The Tiling Property is straighforward. To verify the Quasi-similarity Property we choose the following set $\mathcal{K}$ associated to Γ and A

$$\mathcal{K} = \Big\{\mathbf{k}_1 = (0,0), \mathbf{k}_2 = (1,0), \mathbf{k}_3 = (0,1), \mathbf{k}_4 = (1,1), \mathbf{k}_5 = (\tfrac{1}{2}, \tfrac{1}{2})\Big\}$$

$$\cup\Big\{\mathbf{s}_1 = (\tfrac{1}{2}, -\tfrac{1}{2}), \mathbf{s}_2 = (\tfrac{3}{2}, \tfrac{1}{2}), \mathbf{s}_3 = (\tfrac{1}{2}, \tfrac{3}{2}), \mathbf{s}_4 = (-\tfrac{1}{2}, \tfrac{1}{2})\Big\}.$$

Then each $T^i = T^i_{0,0}$ can be written as union of four triangles of the next level:

$$T^1_{0,0} = T^1_{1,\mathbf{k}_1} \cup T^1_{1,\mathbf{k}_2} \cup T^1_{1,\mathbf{k}_5} \cup T^3_{1,s_1}, \qquad T^2_{0,0} = T^2_{1,\mathbf{k}_2} \cup T^2_{1,\mathbf{k}_4} \cup T^2_{1,\mathbf{k}_5} \cup T^4_{1,\mathbf{s}_2},$$

$$T^3_{0,0} = T^3_{1,\mathbf{k}_3} \cup T^3_{1,\mathbf{k}_4} \cup T^3_{1,\mathbf{k}_5} \cup T^1_{1,s_3}, \qquad T^4_{0,0} = T^4_{1,\mathbf{k}_1} \cup T^4_{1,\mathbf{k}_3} \cup T^4_{1,\mathbf{k}_5} \cup T^2_{1,\mathbf{s}_4}.$$

Let $\phi^i = \sqrt{2}\ \chi_{_{T^i}}$ be the normalized characteristic function associated to the triangles T^i for $i = 1, \cdots, 4$. Then for $j \in \mathbb{Z}$

$\mathbf{V}_j = L^2$-closure of the subspace generated by $\{\phi^i_{j,\mathbf{k}} :\ i = 1, \cdots, 4,\ \mathbf{k} \in \Gamma\}$.

The sequence $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$ of L^2-closed subspaces defines a MRA of multiplicity $r = 4$ associated to the dilation matrix $A = 2I$ and lattice $\Gamma = \frac{1}{2}\mathbb{Z}^2$, the vector function $\vec{\phi} = (\phi^1, \phi^2, \phi^3, \phi^4)$ being the scaling vector for the MRAs.

Again the wavelet spaces $\mathbf{W}_j$ will be the orthogonal complement of $\mathbf{V}_j$ in $\mathbf{V}_{j+1}$. And because $\mathbf{W}_j$ is similar to $\mathbf{W}_0$, we only need to built a base for $\mathbf{W}_0$. To each T^i we associated three functions with 0-moments and orthogonal to each other. The set $\{\phi^i_{0,\mathbf{k}},\ \psi^i_{u,0,\mathbf{k}} : i = 1, \cdots, 4;\ u = 1, 2, 3;\ \mathbf{k} \in \Gamma\}$ is a $bon(W_0)$.

4 THIRD CASE: TILING OF $I\!R^3$ BY TETRAHEDRA

Let $\Gamma = \mathbb{Z}^3$ and $A = 2Id$. We use as basic domains six tetrahedra:

$T^1 : \mathbf{0},\ (1,0,1),\ (0,0,1)\ (1,1,1)$ $\quad T^2 : \mathbf{0},\ (1,0,1),\ (1,0,0),\ (1,1,1)$

$T^3 : \mathbf{0},\ (1,0,0),\ (1,1,0),\ (1,1,1)$ $\quad T^4 : \mathbf{0},\ (1,1,0),\ (0,1,0),\ (1,1,1)$

$T^5 : \mathbf{0},\ (0,1,0),\ (0,1,1),\ (1,1,1)$ $\quad T^6 : \mathbf{0},\ (0,1,1),\ (0,0,1),\ (1,1,1)$

As before we will denote $T^i = T^i_{0,0}$ and the $\Gamma-$translation as $T^i_{0,\mathbf{k}} = T^i_{0,0} + \mathbf{k}$ with $\mathbf{k} \in \Gamma$.

Since $\cup_{i=1}^6 T^i = [0,1)^3$ then it is clear that these tetrahedra satisfy the Tiling Property, i.e,

- $\cup_{\mathbf{k}\in\Gamma}[\cup_{i=1}^6 T^i_{0,\mathbf{k}}] = I\!R^3$.
- $T^i_{0,\mathbf{k}} \cap T^j_{0,\mathbf{l}} \simeq \emptyset$ for all $\mathbf{k} \neq \mathbf{l}$ and $i, j = 1, \cdots, 6$.

Here $\simeq$ means that the possible intersection has measure zero. But of course a strict disjoint partition of $I\!R^3$ can be done, like in $I\!R^2$.

To see the Quasi-similarity Property, we choose the following set of digits associated to Γ and A.

$\mathcal{K} = \{\mathbf{k}_1 = \mathbf{0},\ \mathbf{k}_2 = (1,0,0),\ \mathbf{k}_3 = (0,1,0),\ \mathbf{k}_4 = (0,0,1), \mathbf{k}_5 = (1,1,0),$

$\mathbf{k}_6 = (1,0,1),\ \mathbf{k}_7 = (0,1,1),\ \mathbf{k}_8 = (1,1,1)\}.$

Then we can write

$$T^1 = [T^1_{1,\mathbf{k}_1} \cup T^1_{1,\mathbf{k}_6} \cup T^1_{1,\mathbf{k}_4} \cup T^1_{1,\mathbf{k}_8}] \cup [T^5_{1,\mathbf{k}_6} \cup T^6_{1,\mathbf{k}_6}] \cup [T^2_{1,\mathbf{k}_4} \cup T^3_{1,\mathbf{k}_4}]$$

$$T^2 = [T^2_{1,\mathbf{k}_1} \cup T^2_{1,\mathbf{k}_6} \cup T^2_{1,\mathbf{k}_2} \cup T^2_{1,\mathbf{k}_8}] \cup [T^6_{1,\mathbf{k}_2} \cup T^1_{1,\mathbf{k}_2}] \cup [T^3_{1,\mathbf{k}_6} \cup T^4_{1,\mathbf{k}_6}]$$

$$T^3 = [T^3_{1,\mathbf{k}_1} \cup T^3_{1,\mathbf{k}_2} \cup T^3_{1,\mathbf{k}_5} \cup T^3_{1,\mathbf{k}_8}] \cup [T^1_{1,\mathbf{k}_5} \cup T^2_{1,\mathbf{k}_5}] \cup [T^4_{1,\mathbf{k}_2} \cup T^5_{1,\mathbf{k}_2}]$$

$$T^4 = [T^4_{1,\mathbf{k}_1} \cup T^4_{1,\mathbf{k}_3} \cup T^4_{1,\mathbf{k}_5} \cup T^4_{1,\mathbf{k}_8}] \cup [T^2_{1,\mathbf{k}_3} \cup T^3_{1,\mathbf{k}_3}] \cup [T^5_{1,\mathbf{k}_5} \cup T^6_{1,\mathbf{k}_5}]$$

$$T^5 = [T^5_{1,\mathbf{k}_1} \cup T^5_{1,\mathbf{k}_3} \cup T^5_{1,\mathbf{k}_7} \cup T^5_{1,\mathbf{k}_8}] \cup [T^3_{1,\mathbf{k}_7} \cup T^4_{1,\mathbf{k}_7}] \cup [T^6_{1,\mathbf{k}_3} \cup T^1_{1,\mathbf{k}_3}]$$

$$T^6 = [T^6_{1,\mathbf{k}_1} \cup T^6_{1,\mathbf{k}_4} \cup T^6_{1,\mathbf{k}_7} \cup T^6_{1,\mathbf{k}_8}] \cup [T^5_{1,\mathbf{k}_4} \cup T^4_{1,\mathbf{k}_4}] \cup [T^1_{1,\mathbf{k}_7} \cup T^2_{1,\mathbf{k}_7}].$$

Both, the definition of the MRAs and the construction of the wavelet space follow the pattern already described for $I\!R^2$.

The basic domains were obtained by using the three diagonal issuing from the vertex $(0,0,0)$, which divide each one of the three faces into two triangles. In each face we join the triangles vertices to the vertex $(1,1,1)$ obtaining two tetrahedra. For example, by using the diagonal between the vertices $(0,0,0)$ and $(1,0,1)$ we obtain T^1 and T^2. A similar construction can be obtained by using the diagonals that issue from any one of the other seven vertices of Q.

5 FINAL REMARKS

5.1 About the First Case

We first observe that if we keep $\Gamma = Z\!\!Z^2$ and use a non-dyadic dilation matrix A then we end up with triangular domains which are non-rectangular, but that still satisfy the two fundamental properties.

Secondly, by keeping as dilation matrix $A = 2Id$ we can extend the above construction to any number of subset as basic domains as long as they satisfy the Tiling and the Quasi-similarity properties. Each time we have to find out the appropriated nested tilings of the space. In [1] is proved the existence of such families of nested partitions on metric spaces with a very mild homogeneity property.

Thirdly, the domains T^1, T^2 were obtained by partitioning Q by the diagonal between the vertices $(1,0)$ and $(0,1)$. Everything works similarly for domains $\tilde{T}^1$, $\tilde{T}^2$ which are obtained by using the diagonal between the vertices $(0,0)$ and $(1,1)$.

5.2 About the Third Case

The scheme developed in the Third Case can be generalized to $I\!R^n$, where we will have $n!$ tetrahedra as basic domains, which can be obtained by using, for instance, the n diagonals issuing from $(0,\cdots,0)$. The Tiling Property is straighforward. As for the Quasi-similarity Property we can use the following set of digits associated to A and Γ:

$$\mathcal{K} = \{\mathbf{k} \in I\!R^n :\ \mathbf{k} = \sum_{i \in S} \vec{e}_i\}$$

where $S \subset \{1, 2, \cdots, n\}$ and $\vec{e}_i$ is a vector of the canonical basis of $\mathbb{R}^n$.

Again the construction of the functional and wavelet spaces follow the pattern described for $\mathbb{R}^2$.

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