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Analysis of the transpiration cooling of a thin porous plate in a hot laminar convective flow

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ABSTRACT

This paper deals with the asymptotic and numerical analysis for the steady-state transpiration cooling of a thin porous flat plate in a laminar hot convective flow, taking into account the longitudinal heat conduction through the plate. For very good conducting plates, a regular perturbation analysis has been done, obtaining a three-term asymptotic solution for the distribution of the temperature of the plate. Parallel we solved numerically the equations using a quasilinearization technique. The numerical results are in good agreement with the asymptotic solution close to the asymptotic limit studied.

RESUMEN

En este trabajo se estudia tanto asintótica como numéricamente del proceso de enfriamiento por transpiración de película de una placa plana porosa en un flujo convectivo forzado a alta temperatura, tomando en cuenta la difusión de calor longitudinal. Para placas muy conductoras, se realizó una solución asintótica para encontrar la distribución de temperatura en la placa. Paralelamente se encontró la solución numérica usando una ténica de cuasi-linealización. Hay buena concordancia entre los resultados numéricos y los analíticos cerca del límite estudiado.

INTRODUCTION

One of the most important problems in gas turbines development is the related with the increase of the efficiency by increasing the working temperature at the exit of the combustion chamber. However, this is limited by the blade material. Transpiration cooling in this case is used in order to protect any solid material in contact with very hot fluids. The injected cooling fluid travels through the porous material, being heated first by the wall and later by mixing with the hot fluid. The effect of wall transpiration on the heat transfer process, has been studied numerically in several works [1] - [5]. Brouwers in a recent paper [6] studied the heat and mass transfer between a permeable wall and a porous medium, including the effect of injection and suction on the process. He uses the so called film model, which is an approximation of the boundary layer flow. For an injection parameter of $B = 0.5$, he obtained a relatively good correlation with an accuracy of 87%.

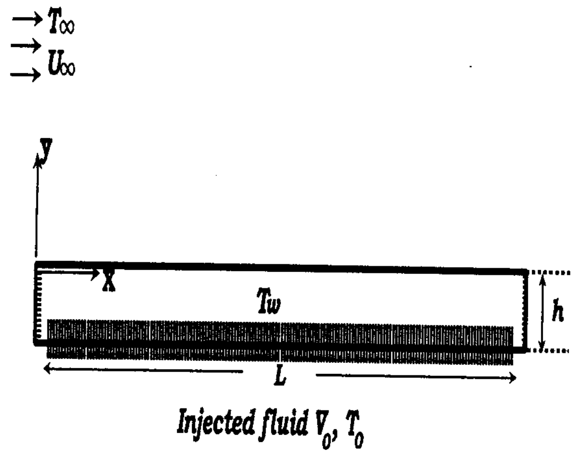

Figure 1: Schematics of the physical problem studied.

as compared with the boundary layer model. Eckert and Cho [7] obtained numerically the heat transfer characteristics of a porous wall in a turbulent boundary layer flow, using the $k-\varepsilon$ model. In none of these works, the longitudinal (parallel to the free stream) heat conduction through the porous material has been considered. The objective of the present work is to study both numerically and using asymptotic techniques the influence of the injection of a cooling fluid through a porous wall in contact with a laminar convective flow of a hot fluid. The longitudinal heat conduction through the wall is considered for the different appropriate regimes.

FORMULATION

The physical model analyzed is the following and showed in figure 1. A thin porous flat plate of length L , thickness $2h$ is placed parallel in a forced flow of a incompressible fluid with velocity U_∞ and temperature T_∞. A transpiration cooling flow is added through the porous plate, with a temperature T_0 and an injection velocity V_0. The thermal conductivity of the plate material makes it possible the heat conduction through the plate. Both edges of the plate are assumed for simplicity to be adiabatic. Introducing the following non-dimensional variables

$$\theta_w = \frac{(T_w - T_0)}{(T_\infty - T_0)}; \chi = \frac{x}{L}; z = \frac{y}{h} \tag{1}$$

$$\theta_g = \frac{(T_g - T_0)}{(T_\infty - T_0)}; \theta = \frac{(T - T_0)}{(T_\infty - T_0)}; \eta = \sqrt{\frac{U_\infty}{\nu x}} y; f = \frac{\psi}{\sqrt{U_\infty \nu x}} \tag{2}$$

the non-dimensional energy equations for the plate and the injected fluid are given by

$$\frac{\partial^2 \theta_w}{\partial \chi^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 \theta_w}{\partial z^2} - \frac{Pr\beta\gamma}{\alpha}(\theta_w - \theta_g) = 0 \text{ and } \gamma(\theta_w - \theta_g) = \frac{\partial \theta_g}{\partial z}. \tag{3}$$

where

$$\alpha = \frac{\lambda_w}{\lambda}\frac{h}{L}\frac{1}{Re_\infty^{1/2}}, \beta = \frac{V_0 Re_\infty^{1/2}}{U_\infty}, Re_\infty = \frac{U_\infty L}{\nu}, \varepsilon = \frac{h}{L}, Pr = \frac{\rho \nu c}{\lambda}.$$

Here, γ is the internal heat transfer parameter defined as $\gamma = Hh/(\rho c V_0)$, where H is the volumetric heat transfer coefficient between the plate and the injected fluid, with a local temperature T_g. For large values of γ, rapidly the injection fluid temperature reaches the temperature of the plate, that is $\theta_g = \theta_w$ at $z = 0$ for $\gamma \gg 1$. The governing equations for the hot fluid take the classical nondimensional form [8]

$$\frac{\partial^3 f}{\partial \eta^3} + \frac{f}{2}\frac{\partial^2 f}{\partial \eta^2} = \chi \left[\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \chi \partial \eta} - \frac{\partial f}{\partial \chi}\frac{\partial^2 f}{\partial \eta^2}\right] \tag{4}$$

$$\frac{1}{\mathrm{Pr}}\frac{\partial^2 \theta}{\partial \eta^2} + \frac{f}{2}\frac{\partial \theta}{\partial \eta} = \chi \left[\frac{\partial f}{\partial \eta}\frac{\partial \theta}{\partial \chi} - \frac{\partial f}{\partial \chi}\frac{\partial \theta}{\partial \eta}\right] \tag{5}$$

with the boundary conditions

$$f(0) + 2\sqrt{\chi}\beta(\chi) = \frac{\partial \theta_w}{\partial z} - \frac{\varepsilon^2}{\alpha}\left[\frac{1}{\sqrt{\chi}}\frac{\partial \theta}{\partial \eta} - \beta Pr(\theta_w - \theta_g)\right]$$

$$= \frac{\partial f}{\partial \eta} = \theta - \theta_w = 0 \text{ at } \eta = 0 \tag{6}$$

$$\frac{\partial f}{\partial \eta} - 1 = \theta - 1 = 0 \text{ at } \eta \to \infty \tag{7}$$

Here the injection function $\beta(\chi) = V_0\sqrt{Re_\infty}/U_\infty$ is assumed to be a function of the longitudinal coordinate χ. We introduce the normalized function $\tilde{\beta}(\chi) = \beta(\chi)/B$, such as $\int_0^1 \tilde{\beta}(\chi)d\chi = 1$. B is then the strength of the injection process.

THERMALLY THIN WALL REGIME

In this regime the non-dimensional transversal temperature variations in the plate are very small, of order ε^2/α. Integrating the energy equation across the solid and applying the boundary conditions we obtain

$$\alpha\frac{d^2\theta_w}{d\chi^2} + \frac{1}{\sqrt{\chi}}\left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} - \mathrm{Pr}\,\beta\theta_w = 0. \tag{8}$$

In this regime, the final equation does not depend on the value of γ. Therefore, it does not matter if the cooling fluid is heated by the wall or by the hot fluid. The result would be exactly the same.

Asymptotic Limit $\alpha >> 1$

For very large values of the parameter α compared with unity, the nondimensional temperature of the plate changes very little in the longitudinal direction of order α^{-1}. This limit is regular and is to be analyzed using α^{-1} as the small parameter of expansion. In this limit, the nondimensional temperature of the plate can be obtained using by the following asymptotic series

$$\theta_w(\alpha, \chi) = \sum_{j=0}^{\infty}\frac{1}{\alpha^j}\theta_{wj}(\chi). \tag{9}$$

Introducing the above relationship into the nondimensional governing eq. (0.8), we obtain the following set of equations

$$\frac{d^2\theta_{w0}}{d\chi^2} = 0, \tag{10}$$

$$\frac{d^2\theta_{wn}}{d\chi^2} = -\frac{1}{\sqrt{\chi}}\frac{\partial\theta_{n-1}}{\partial\eta}\bigg|_{\eta=0} + \Pr\beta\theta_{w(n-1)}) \text{ for } n \geq 1, \tag{11}$$

with the following adiabatic conditions at both edges

$$\frac{d\theta_{wn}}{d\chi} = 0 \text{ at } \chi = 0 \text{ and } 1 \text{ for all } n. \tag{12}$$

Solving equations (0.10) and (0.12), gives a constant value for θ_{0w}, which can be found after integrating the following higher order equation (0.11) with the corresponding adiabatic conditions at both edges. In this form, the solution for θ_{w0} is given by

$$\theta_{w0} = \frac{1}{1 + \frac{PrB}{2\overline{G_0}(m,B,Pr)}} \text{ with } \overline{G_0} = \frac{1}{2}\int_0^1 \frac{G_0 d\chi}{\sqrt{\chi}}. \tag{13}$$

Here $G_0(\chi : m, B, Pr)$ corresponds to the nondimensional temperature gradient at the surface of the wall obtained with the normalized conditions $\theta_0 = 0$ at $\eta = 0$ and $\theta_0 = 1$ as $\eta \to \infty$. In this case we represent the normalized injection function as $\widetilde{\beta}(\chi) = (1+m)\chi^m$. Any other function can be included without any difficulty. A first integration of Eq. (0.11) for $n = 1$, gives

$$\frac{d\theta_{w1}}{d\chi} = 2\overline{G_0}(1-\theta_{w0})\left[\chi^{m+1} - \frac{1}{2\overline{G_0}}\int_0 \chi\frac{G_0 d\chi}{\sqrt{\chi}}\right]. \tag{14}$$

For $m = -1/2$, we obtain that $\theta_{wn} = 0$ for all $n > 0$. For this specific injection function, the leading order solution is valid for all values of α. A second integration gives

$$\theta_{w1} = C_1 + 2\overline{G_0}(1-\theta_{w0})\left[\frac{\chi^{m+2}}{m+2} - \frac{1}{2\overline{G_0}}\int_0 \chi d\chi \int_0 \chi\frac{G_0 d\overline{\chi}}{\sqrt{\overline{\chi}}}\right] \tag{15}$$

where C_1 can be obtained by solving the higher order equation. For small values of B, from eq. (0.13) and assuming a solution of the form

$$f_0 = f_{00} + 2B\chi^{m+1/2}(1+m)(-1+g_1) + O(B^2)$$

$$\theta_0 = \theta_{00} + 2B\chi^{m+1/2}(m+1)\varphi_1 + O(B^2)$$

we obtain to the leading order

$$\frac{d^3 f_{00}}{d\eta^3} + \frac{f_{00}}{2}\frac{d^2 f_{00}}{d\eta^2} = 0 \text{ and } \frac{d^2\theta_{00}}{d\eta^2} + \frac{\Pr f_{00}}{2}\frac{d\theta_{00}}{d\eta} = 0, \tag{16}$$

with the well known solution for large Prandtl numbers

$$\theta_{00} = 0.7765\int_0^{(\frac{\Pr f''(0)}{4})^{1/3}\eta} \exp\left[\frac{-t^3}{3}\right] dt \text{ with } f''(0) = 0.332. \tag{17}$$

The first order equations take the form

$$\frac{d^3 g_1}{d\eta^3}+\frac{f_{00}}{2}\frac{d^2 g_1}{d\eta^2}-\left(m+\frac{1}{2}\right)\frac{df_{00}}{d\eta}\frac{dg_1}{d\eta}+(m+1)\frac{d^2 f_{00}}{d\eta^2}g_1=(m+1)\frac{d^2 f_{00}}{d\eta^2}, \tag{18}$$

$$\frac{1}{\Pr}\frac{d^2\varphi_1}{d\eta^2}+\frac{f_{00}}{2}\frac{d\varphi_1}{d\eta}-\left(m+\frac{1}{2}\right)\frac{df_{00}}{d\eta}\varphi_1+(m+1)\frac{d\theta_0}{d\eta}g_1=(m+1)\frac{d\theta_0}{d\eta}, \tag{19}$$

with the boundary conditions

$$g_1=\frac{dg_1}{d\eta}=\varphi_1=0 \text{ at } \eta=0 \text{ and } \frac{dg_1}{d\eta}=\varphi_1=0 \text{ for } \eta\to\infty. \tag{20}$$

The nondimensional temperature gradient then is given by

$$G_0=G_{00}+2B(m+1)\chi^{m+1/2}G_{01}, \tag{21}$$

where

$$G_{00}=\left.\frac{d\theta_{00}}{d\eta}\right|_{\eta=0}=0.3387\,\Pr^{1/3} \text{ and } G_{01}(m,Pr)=\left.\frac{d\varphi_1}{d\eta}\right|_{\eta=0} \tag{22}$$

The average value of the nondimensional temperature gradient is up to the first order terms

$$\overline{G_0}=G_{00}+BG_{01}+O(B^2) \tag{23}$$

By using eq. (0.13) and the previous result we obtain

$$\theta_{w0}=\left[1-\frac{\Pr B}{2G_{00}}+\frac{\Pr^2 B^2}{4G_{00}^2}\left(1+\frac{2G_{01}(m,Pr)}{Pr}\right)+O(B^3)\right]. \tag{24}$$

Fig. 2 shows the leading order solution θ_{w0} obtained numerically, as a function of the injection strength B, for a Prandtl number of unity and different values of m. The asymptotic solution, up to terms of order B, given by eq. (0.24), is also plotted. This approximation is enough to describe with acceptable accuracy the leading order solution. This function can be well correlated by

$$G_{01}\simeq -1.212-1.0487m+0.436m^2-1.1072(\Pr-1), \tag{25}$$

for values of $\Pr\sim 1$. Therefore up to the second order θ_{w1} is given by

$$\theta_{w1}=C_1+BPr\left[\frac{\chi^{m+2}}{m+2}-\frac{2}{3}\chi^{3/2}\right]+O(B^3). \tag{26}$$

After evaluating the constant C_1 by integrating the second order equation (0.11) we obtain finally up to the second order

$$\theta_w=\left\{1-\frac{\Pr B}{2G_{00}}+\frac{\Pr^2 B^2}{4G_{00}^2}\left(1+\frac{2G_{01}(m,Pr)}{Pr}\right)\right.$$
$$\left.+\frac{\Pr B}{\alpha 2G_{00}}\left[\frac{G_1(m+2)}{(m+2)(m+5/2)}-\frac{G_1(3/2)}{3}+2G_{00}\left(\frac{\chi^{m+2}}{m+2}-\frac{2}{3}\chi^{3/2}\right)\right]\right\}$$
$$+O(B^2/\alpha,B^3), \tag{27}$$

where G_1, obtained using the Lighthill approximation, is given by

$$G_1(n)=-\frac{4nG_{00}}{3}\int_0^1\frac{u^{4n/3-1}du}{(1-u)^{1/3}}.$$

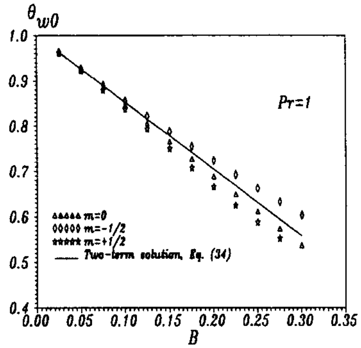

Figure 2: Leading order solution for the nondimensional temperature of the plate, θ_{w0}, obtained numerically, as a function of the injection strength B and three different values of m. The Prandtl number is Pr = 1. The two-term asymptotic solution given by eq. (0.24) is also plotted.

RESULTS AND CONCLUSIONS

The nondimensional governing equations (0.4) to (0.8) were solved numerically using a quasilinearization technique with a tridiagonal matrix solver using a mesh of 101 and 900 grid points in the longitudinal and transversal directions in the fluid phase, respectively. Fig. 3 shows the nondimensional temperature distribution for different values of the parameter α, for $m = 0$. The injection strength assumed for these calculation was $B = 0.1$ and the Prandtl number was $Pr = 1$. For large values of the parameter α, the large thermal conductivity of the wall does not permit large temperature gradients and the temperature distribution is almost flat. As the value of α decreases, the temperature at the leading edge increases, decreasing it at the trailing edge, thus producing important temperature gradients in the longitudinal direction. For large values of α, the temperature of the wall is decreases as the value of m increases, contrary of expected. However, as the value of α decreases, the maximum temperature (at the leading edge) increases by increasing the value of m. All this information can be obtained from the asymptotic solution, given by eq. (0.27). The nondimensional temperature at the leading edge, θ_{wl}, is after eq. (0.27)

$$\theta_{wl} \cong 1 + \frac{\Pr B}{2G_{00}} - 1 + \frac{\Pr B}{2G_{00}} \left[\left(1 + \frac{2G_{01}(m, Pr)}{Pr} \right) + \frac{1}{\alpha} \left[\frac{G_1(m+2)}{(m+2)(m+5/2)} - \frac{G_1(3/2)}{3} \right] \right]. \tag{28}$$

Here, all the parametric dependence is explicitly written. Thus

$$\frac{\partial \theta_{wl}}{\partial m} \cong \frac{\Pr B}{2G_{00}} \left[\frac{B}{G_{00}} \frac{dG_{01}}{dm} + \frac{1}{\alpha} \frac{d}{dm} \left(\frac{G_1(m+2)}{(m+2)(m+5/2)} \right) \right]. \tag{29}$$

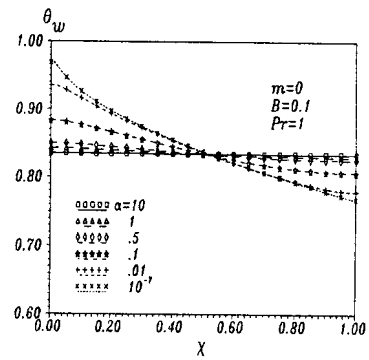

Figure 3: Nondimensional temperature of the plate as a function of the longitudinal coordinate for different values of the longitudinal heat conduction parameter α and $m = 0$. The injection strength is $B = 0.1$ and the Prandtl number is $\Pr = 1$.

Therefore

$$\frac{\partial \theta_{wl}}{\partial m} \cong \frac{\Pr B}{2G_{00}} \left[\frac{B}{G_{00}}(-1.0487 + .872m) \right.$$

$$\left. + \frac{4G_{00}}{3\alpha(m+5/2)} \int_0^1 \frac{u^{4m/3+5/3}du}{(1-u)^{1/3}} \left(\frac{4}{3} ln(1/u) + \frac{1}{(m+5/2)} \right) \right]. \qquad (30)$$

The first term in the right hand side of eq. (0.30) is negative, indicating that the temperature of the plate, to the leading order, decreases as the value of m increases. On the other hand, the second term, indicating the effect of α, is always positive, showing that the temperature at the leading edge increases as the value of m increases, for $m > -1/2$. For values of $m < -1/2$, the temperature distribution on the plate inverts, being very small at the leading edge but with the highest temperature at the trailing edge. In general, the numerical results are in good agreement with the asymptotic solution close to the asymptotic limits studied.

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