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SELECTION STRATEGIES IN ADAPTIVE REFINEMENT

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RESUMEN

Este trabajo trata sobre el problema de seleccionar elementos a refinar para la construcción de una nueva triangulación en un sistema de refinamiento adaptativo. El probema considerado es la resolución numérica de la ecuación de Poisson utilizando elementos finitos lineales e indicadores de error de tipo Babuška-Miller. Analizamos dos métodos de seleccionar triángulos en refinamiento adaptivo: la primer estrategia (ampliamente utilizada) es marcar aquellos elementos que tengan un indicador de error mayor que α veces el mayor de los indicadores, donde $0 \leq \alpha \leq 1$. Concluímos que este método es *robusto* en el siguiente sentido: si elegimos $\alpha \geq \alpha_0$ con α_0 pequeño, se recupera el orden de convergencia del problema regular con mallas cuasi-uniformes (medido con el número de elementos). Con este procedimiento también introducimos un *criterio de detención* para obtener el error final menor que una tolerancia prescripta. La segunda estrategia es marcar elementos que tengan un indicador mayor que un (*indicador admisible*. Este indicador admisible se define basado en el anterior criterio de detención. La relación entre el indicador elemental y el error admisible es utilizada para definir el nivel de refinamiento de cada elemento. Analizamos el comportamiento de ambas estrategias y las comparamos. Finalmente, discutimos algunos aspectos del proceso adaptivo como un todo.

ABSTRACT

This paper deals with the problem of selecting the elements to be refined for the construction of a new triangulation in an adaptive refinement system. The problem considered is the numerical solution of Poisson's equation using piecewise linear finite elements and local error indicators of Babuška-Miller-type. We analyze two ways of selecting triangles in adaptive refinement: the first strategy (widely used) is to mark elements that have an indicator greater than α times the largest of the indicators where $0 \leq \alpha \leq 1$. We conclude that this method is *robust* in the following sense: if we choose $\alpha \geq \alpha_0$ with α_0 small, the convergence order of the regular problem with quasiuniform meshes (measured with the number of elements) is recovered. In this procedure we also introduce a *stopping criterion* to obtain the final error measure smaller than a prescribed tolerance. The second strategy is to mark elements that have an indicator greater than an *admissible indicator*. This admissible indicator is defined based on the previous stopping criterion. The ratio between the elemental indicator and the admissible error is also used to define the level of refinement in each element. We analyze the behavior of both strategies and compare them. Finally, some remarks about the whole adaptive process are discussed.

INTRODUCTION

In Computational Mechanics it is usual to find the problem of increasing the accuracy of a solution without adding unnecessary degrees of freedom. Usually the overall accuracy of the numerical solution is degraded by a non-uniform distribution of the error, specially when the solution of the continuous problem has local singularities. The need for accurate solutions has made the use of adaptive procedures very attractive and necessary for large scale problems. It is therefore necessary to update the mesh to ensure that it becomes dense enough in the critical region while remaining reasonably coarse in the rest of the domain.

This procedure involves three steps: a) the evaluation of local error indicators (i.e.: estimates per element) and the error estimate, b) the selection of triangles that will be refined, and c)

the mesh refinement process based upon these indicators. Local a posteriori error estimators are the adequate tool to identify automatically these critical regions. They should use only given data and the numerical solution itself. Based on the information given by these estimators, it is possible to decide when the adaptive process must be stopped or, if this is not the case, where and how mesh refinement might be performed more efficiently. Several approaches have been introduced to define error estimators for different problems using the residual equation (see for example Noor and Babuška[1], Babuška and Miller[2], Babuška and Rheinboldt[3], Bank and Weisser[4] and Verfürth[5]).

The second step is the *selection strategy*, which consists of marking the triangles that will be refined. In this paper we deal with this step and show how the selection strategy influences the efficiency of the adaptive process. We analyze two ways of selecting triangles in adaptive refinement: the first strategy (widely used) is to mark elements that have an indicator greater than α times the largest of the indicators where $0 \leq \alpha \leq 1$. A *stopping criterion* is introduced to obtain the final error measure smaller than a prescribed tolerance. Based on this stopping criterion an *admissible error* can be defined. Moreover, the ratio between the elemental error indicator and the admissible error can be used to compute different levels of refinement in each element. This is the second selection strategy to be analyzed.

The main advantage of the second strategy is to account for the error distribution. Furthermore, the threshold to mark elements is based on global quantities, while in the first strategy it is defined through the indicator in one element.

An algorithm based on the subdivision of simplices, briefly described in the third section, is used in the mesh refinement process. This idea was successfully used by several authors (see for example Rivara[6]) and it is specially attractive for adaptive processes because it is possible to guarantee that elements will not degenerate.

The rest of the paper is organized as follows: in the following section we introduce the model problem and recall the finite element approximation. The error estimators and their equivalence with the error are introduced there. After that, we introduce the stopping criterion. The following two sections deal with the selection strategies including numerical results and comparative analysis. We also introduce an improvement of the second strategy to reduce the global computational cost. Finally, in the last section some concluding remarks are summarized.

MODEL PROBLEM

Let $\Omega \subset \mathbf{R}^2$ be a polygonal domain. The model problem is

$$\begin{cases} -\Delta u = f\,, & \text{in } \Omega \\ u = g_1\,, & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial n} = g_2\,, & \text{on } \Gamma_2 \end{cases} \tag{1}$$

where Γ_1 and Γ_2 are disjoint sets such that $\Gamma_1 \neq \emptyset$ and $\Gamma_1 \cup \Gamma_2 = \partial\Omega$. Standard notation for Sobolev spaces, norms and seminorms are used. For $\Gamma \subset \partial\Omega$ we set $H^1_\Gamma = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma\}$. Then, the solution of problem (1) satisfies

$$\int_\Omega \nabla u \nabla v = \int_\Omega fv + \int_{\Gamma_2} g_2 v\,, \quad \forall v \in H^1_{\Gamma_1} \tag{2}$$

Assume that we have a family $\{\mathcal{T}_j\}$ of triangulations of Ω such that any two triangles in $\mathcal{T}_j$ share at most a vertex or an edge and any $\mathcal{T}_j$ is consistent with the boundary data, i.e., a boundary side is contained in either Γ_1 or Γ_2. For any $\mathcal{T}_j$ we introduce the finite element space $V_j = \{v \in C^0(\Omega) : v|_T \in \mathcal{P}_1\,, \quad \forall T \in \mathcal{T}_j\}$ ($\mathcal{P}_1$ denotes the space of linear polynomials) and for $\Gamma \subset \partial\Omega$ let $V_{j,\Gamma} = V_j \cap H^1_\Gamma$.

For the sake of simplicity we assume that, g_1 is piecewise linear and g_2 is piecewise constant. These assumptions are not very restrictive. In fact, in the general case we may replace the data by appropriate interpolations and it is not difficult to see that the theorems below can be generalized assuming local regularity of the data (we refer to Babuška et al.[7], Verfürth[5] for details).

Then the finite element approximation to the solution of problem (1) is defined by $u_j \in V_j$ and,

$$\begin{cases} \int_\Omega \nabla u_j \nabla v = \int_\Omega f v + \int_{\Gamma_2} g_2 v & , \quad \forall v \in V_{j,\Gamma_1} \\ u_j = g_1 & , \quad \forall \ell \subset \Gamma_1 \end{cases} \tag{3}$$

Now, we introduce some notation that will be necessary in the definition of error estimators. Let E_I be the set of all interior edges and E_T the set of edges of T. For each interior edge ℓ we choose an arbitrary normal direction n and denote the two triangles sharing this edge T_{in} and T_{out} with $\mathbf{n}$ pointing out of T_{in}. For a boundary side ℓ we take n as the outward normal.

We set

$$\left[\!\left[\frac{\partial u_j}{\partial n}\right]\!\right]_\ell = \nabla(u_j|_{T_{\text{out}}})\mathbf{n} - \nabla(u_j|_{T_{\text{in}}})\mathbf{n} \quad , \qquad \forall \ell \in E_I$$

Note that this value is independent of the choice of n.

For a side ℓ define J_ℓ by

$$J_\ell = \begin{cases} \left[\!\left[\frac{\partial u_j}{\partial n}\right]\!\right]_\ell , & \text{if } \ell \in E_I \\ 2\left(g_2 - \frac{\partial u_j}{\partial n}\right)\Big|_\ell , & \text{if } \ell \subset \Gamma_2 \\ 0 , & \text{if } \ell \subset \Gamma_1 \end{cases}$$

For any triangle $T \in \mathcal{T}_j$, let η_T be the following indicator:

$$\eta_T = [\,|T|\,\|f\|^2_{0,T} + \frac{1}{2}\sum_{\ell \in E_T} |\ell|\,\|J_\ell\|^2_{0,\ell}]^{\frac{1}{2}} \tag{4}$$

where $|T|$ and $|\ell|$ are the area of T and the length of an edge ℓ , respectively. We define the global error estimator by

$$\eta = (\sum_T \eta_T^2)^{\frac{1}{2}}$$

Now, Theorem 1 can be stated. This is a standard result and a proof can be found in Babuška et al.[7].

Theorem 1 *There are two constants c_0,c_1 , which depend only on Ω and on the smallest angle in the triangulation $\mathcal{T}_j$, such that the estimates:*

$$c_0\,\eta \le |e_j|_1 \le c_1\,\eta \tag{5}$$

are valid.

We present the results of some numerical computations. Meshes $\{\mathcal{T}_j\}$ are generated in an adaptive way using η_T as an error indicator at the element T. Starting with a uniform triangulation $\mathcal{T}_0$, $\mathcal{T}_{j+1}$ is obtained from $\mathcal{T}_j$ refining the elements $T \in \mathcal{T}_j$ marked in the selection strategy.

The densification algorithm used is based on the idea proposed in Rivara[6]. Basically it has three steps: node addition for subdivision of the selected elements, node addition for the conforming process, and element subdivision. The algorithm may be summarized as follows:

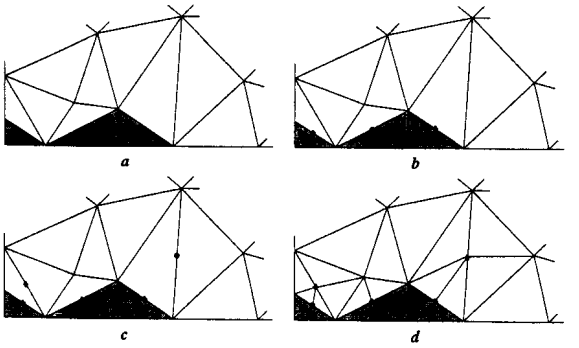

Figure 1: Refinement process sequence

(P)
- A node is added to the mid point of the longest edge of each triangle marked (see figure 1b).
- For every triangle in the mesh with nodes added to the mid point of an edge which is not the longest, add a node to the mid point of its longest edge. This step is repeated until every triangle with nodes added to its edges, has a node added to its longest edge. (see figure 1c)
- Finally, these triangles are bisected, first by their longest edges, and then by the other edges to which nodes were added (see figure 1d).

To use this algorithm in an adaptive environment, it is first necessary to guarantee that the new elements generated will not degenerate. In 2D the interior angle of any triangle generated by this procedure is not less than one half the smallest angle of the initial triangulation[6]. Then, if the initial mesh has good elements, the new meshes generated will also have good elements.

STOPPING CRITERION

In the rest of the paper the stopping criterion and the effectivity index will be used. The *stopping criterion* is defined by

$$\frac{\eta}{|u_j|_1} \leq \Theta$$

where Θ is another positive parameter introduced in the input and related to the final relative error. A standard measure of the quality of an estimator is the *effectivity index* defined by

$$eff = \frac{\eta}{|e|_1}$$

where $e = u - u_j$ is the error.

The relative error satisfies

$$\frac{|u - u_j|_1}{|u|_1} \leq \frac{\Theta}{eff}\left(\frac{|u_j|_1}{|u|_1}\right)$$

and using $|u_j|_1 \leq |u - u_j|_1 + |u|_1$ we get

$$\frac{|u - u_j|_1}{|u|_1} \leq \frac{\Theta}{eff - \Theta}$$

provided that $\Theta < eff$. Then, to obtain a relative error bounded by a given tolerance we need to estimate the effectivity index.

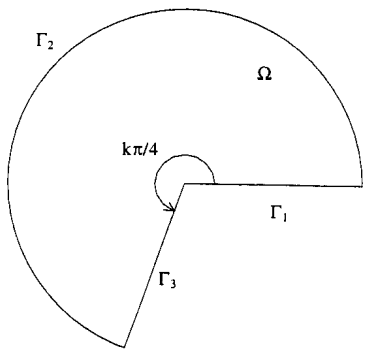

Figure 2: Domain for the test problems

We take a sequence of positive numbers (Θ_n) such that $0 < \Theta_{n+1} < \Theta_n < eff$ and $\Theta_n \to 0$ when $n \to \infty$. Starting from the initial mesh $\mathcal{T}_0$, we define U^1 as the solution that satisfies the stopping criterion for Θ_1. To define U^n with $n > 1$, we start from the final mesh used to obtain U^{n-1} and apply the adaptive method until the stopping criterion is satisfied for Θ_n. We have a sequence that verifies

$$\frac{|u - U^n|_1}{|u|_1} \leq \frac{\Theta_n}{eff - \Theta_n}$$

and then $U^n \to u$ in H_0^1 when $n \to \infty$.

Now, it is only necessary to show that the adaptive algorithm stops in a *finite number* of steps. Although there is no formal proof of this fact, the numerical examples (see following section) show strong evidence of this finite termination property.

FIRST SELECTION STRATEGY

The *first selection strategy* is defined by

$$\text{if} \quad \alpha\, \eta_{max} \leq \eta_T \quad \text{then} \quad \text{mark } T$$

where $\eta_{max} = \max\{\eta_T : T \in \mathcal{T}_j\}$ and $0 \leq \alpha \leq 1$.

If $\alpha = 0$ we will refine globally. In this case, in problems with singular solutions (i.e. solutions in $H_0^1 \setminus H^2(\Omega)$) the order of convergence with respect to the number of elements is less than the order of regular problems with quasiuniform meshes. The first step in our research is to analyze which α produces the same order than regular problems with quasiuniform meshes.

We consider the Laplace equation, with mixed boundary conditions, as test problems. We solve, for $k = 4, 6, 8$

$$\begin{cases} \Delta u = 0 & , \quad \text{in } \Omega \\ u = 0 & , \quad \text{on } \Gamma_1 \\ u = \sin\left(\frac{2\theta}{k}\right) & , \quad \text{on } \Gamma_2 \\ \frac{\partial u}{\partial n} = 0 & , \quad \text{on } \Gamma_3 \end{cases}$$

where $\Omega = \{(r, \theta) : 0 < r < 1, 0 < \theta < \frac{k\pi}{4}\}$, $\Gamma_1 = \{(r, \theta) : 0 < r < 1, \theta = 0\}$, $\Gamma_2 = \{(r, \theta) : r = 1, 0 < \theta < \frac{k\pi}{4}\}$ and, $\Gamma_3 = \{(r, \theta) : 0 < r < 1, \theta = \frac{k\pi}{4}\}$ (see figure 2).

The solution of this problem is $u(r, \theta) = r^{2/k} \sin\left(\frac{2\theta}{k}\right)$. The minimal angle in the initial meshes in all cases is equal to 38.8 degrees and (as a consequence of the algorithm of densification) the minimal angle in every mesh is greater than or equal to 19.4 degrees.

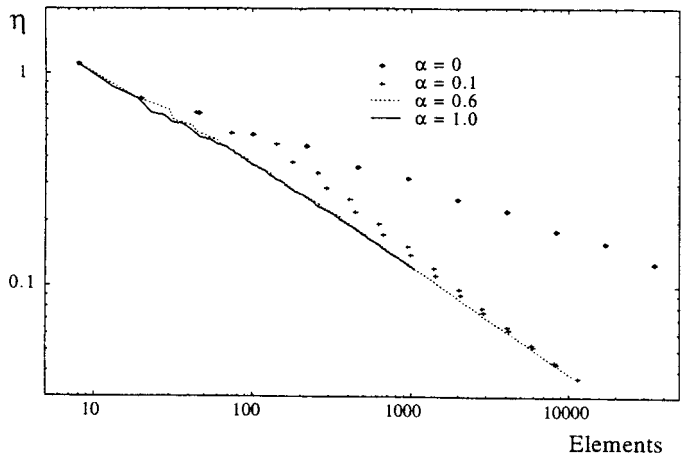

Figure 3: Convergence of the adaptive process using the first selection strategy with different values of α, $k = 4$.

Figures 3, 4 and 5 show the number of elements versus the global error estimator for $k = 4, 6$ and 8 respectively, and with several values of the parameter α. In these figures it can be seen that for α not too small we obtain $O(N^{-\frac{1}{2}})$. In fact, the adaptive algorithm based on the first selection strategy gives the same order of convergence as regular problems with quasiuniform meshes for all *practical* α, i.e., if we take α greater than, let's say 0.1, we get the order $O(N^{-\frac{1}{2}})$. The precise value of α for which the order becomes the optimal one can not be computed easily since that order is an asymptotic value and the size of the meshes grows dramatically for these values of α.

In Babuška et al.[7] it is shown that

$$c_0(\beta)\eta + o_1(h) \leq |e_j|_1 \leq c_1(\beta)\eta + |u - u_p| + o_2(h)$$

where u_p is a finite element solution of the problem using order p interpolants with $p \geq 2$ and β is the minimum angle in the mesh. The terms o_1 and o_2 with standard assumptions on f and g are of higher order.

Table I shows that $3 \leq \mathit{eff} \leq 4$ in all cases where *eff* is computed in the final mesh and gives numerical evidence that $c_0(\beta) \geq 3$.

α	0.1	0.3	0.5	0.7	0.9
$k = 4$	3.36	3.46	3.47	3.47	3.47
$k = 6$	3.35	3.23	3.24	3.26	3.26
$k = 8$	3.43	3.47	3.47	3.48	3.48

Table I : Effectivity index computed for the model problem

Therefore

$$\frac{|u - u_j|_1}{|u|_1} \leq \frac{\Theta}{c_0(\beta) - \Theta}$$

provided that $\Theta \leq c_0(\beta)$. Hence, if we want to have a relative error less than a tolerance τ we must take $\Theta \leq \tau(1+\tau)^{-1}c_0(\beta)$.

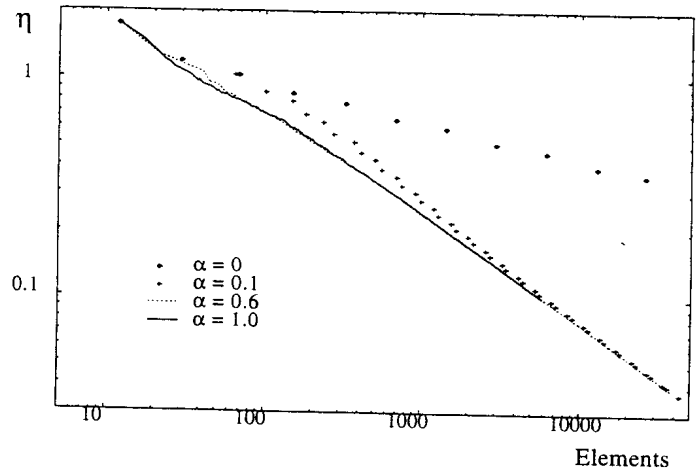

Figure 4: Convergence of the adaptive process using the first selection strategy with different values of α, $k = 6$.

SECOND SELECTION STRATEGY

Following similar ideas as those presented in Zienkiewicz and Zhu[8], the *second selection strategy* is defined by:

$$\text{if} \quad \eta_{adm} \leq \eta_T \quad \text{then} \quad \text{mark } T$$

Assuming that the error is equally distributed between elements, η_{adm}, the admissible error, is defined based on the stopping criterion:

$$\eta_{adm} = Nel^{-\frac{1}{2}} \ \Theta \ |u_j|_1$$

where Nel is the total number of elements.

Note that if $\eta_{adm} > \eta_T \ \forall T$ the stopping criterion is satisfied.

We define the *number N_T of subdivisions in the element T* by

$$\begin{aligned} N_T = 1 \quad &\text{if} \quad \eta_{adm} \leq \eta_T < 2\eta_{adm} \\ N_T = 2 \quad &\text{if} \quad 2\eta_{adm} \leq \eta_T < 4\eta_{adm} \\ N_T = 3 \quad &\text{if} \quad 4\eta_{adm} \leq \eta_T < 8\eta_{adm} \end{aligned}$$

In general, N_T verifies

$$2^{N_T-1}\eta_{adm} \leq \eta_T < 2^{N_T}\eta_{adm}$$

or, equivalently:

$$N_T = \left[\frac{\log(\eta_T/\eta_{adm})}{\log 2} + 1\right]$$

where $[x]$ is the greatest integer less than or equal to x.

Remark : In the general case of higher order elements, let us say p, in the previous definition we must replace 2 by $2^{\frac{p+1}{2}}$.

An upper bound N_M to limit the number of subdivisions can be established for marked elements $N_T \geq 1$.

The algorithm (P) is applied to every marked element, i.e. those for which $N_T > 0$. Because of the conforming algorithm, a triangle T may be subdivided in 2, 3 or 4 subtriangles T^* with

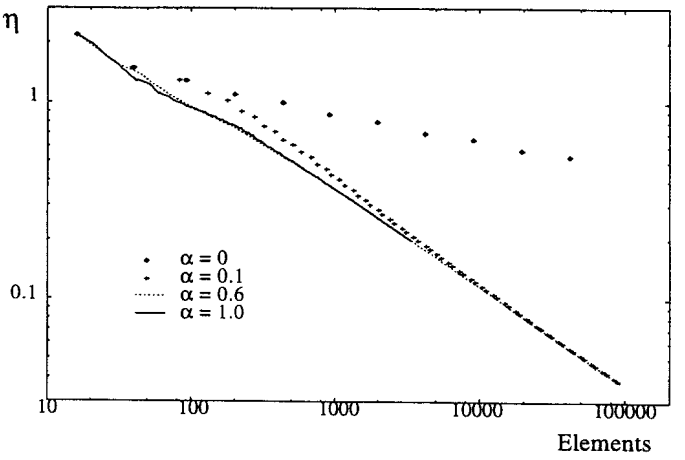

Figure 5: Convergence of the adaptive process using the first selection strategy with different values of α, $k = 8$.

areas one half or one quarter of the original triangle (see figure 1) . A new N_{T^*} is assigned to each subtriangle, based on the new area: $N_{T^*} = N_T - \log_2\left(\frac{|T|}{|T^*|}\right)$. This process is repeated until $N_{T^*} < 1 \quad \forall T^*$.

Now, we consider the second selection strategy, using the test problem with $k = 8$. Figure 6 shows the global error versus the number of elements when the above strategy for different N_M is used. A curve obtained using the first strategy with $\alpha = 0.6$ is also included.

Starting from the same initial mesh it can be seen that, except for the last steps, the first selection strategy gives much better meshes (with minimum number of elements for the same error). Additionally, the second selection strategy greatly reduces the number of adaptive steps required as it is shown in table II.

The second strategy can be improved to reduce the number of elements in the final mesh. With this aim, two positive parameters $\Theta_2 < \Theta_1$ are introduced. The idea is to apply the algorithm with $N_M = \infty$ twice: first use the algorithm with Θ_1 instead of Θ and then apply the algorithm with Θ_2 instead of Θ starting from the final mesh obtained with Θ_1. This algorithm will be referred to as *2-step algorithm*. The previous idea can be generalized for $n \geq 2$. In this case, it will be called *n-step algorithm* and it is defined introducing a new parameter M as follows:

$$M = \max\left\{\left[\frac{\log(\eta/\Theta\|u_j\|)}{\log 2} - 1\right], 0\right\}$$

and taking N_T as

$$N_T = \left[\frac{\log(\eta_T/\eta_{adm})}{\log 2} + 1\right] - M$$

Note that when $\eta/\Theta\|u_j\| < 2^2$ this algorithm reduces to the previous one. When $2^3 > \eta/\Theta\|u_j\| \geq 2^2$, N_T is computed as if the stopping criterion were $\eta/\|u_j\| < 2\Theta$ and so on. This procedure avoids the unnecessary degrees of freedom introduced in the first steps of the adaptive process when the second selection strategy is used. Figure 7 shows the behaviour of the 2-step and the n-step algorithms, compared to the first selection strategy, with $\alpha = 0.6$.

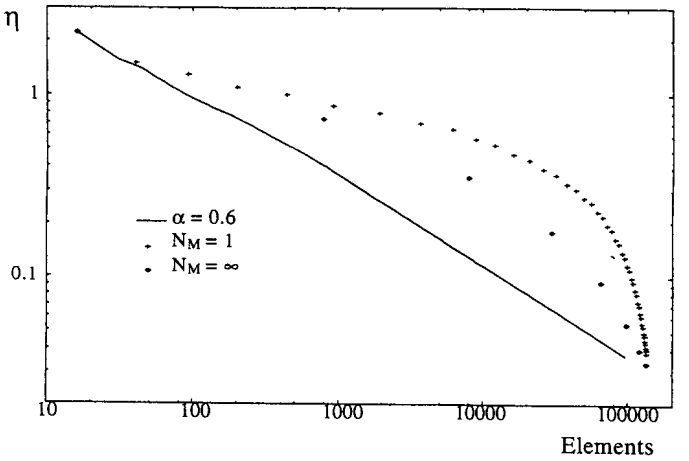

Figure 6: Convergence of the adaptive process for the test problem with $k = 8$. Second selection strategy with $N_M = 1, \text{inf}$. Also is shown the result for the first selection strategy with $\alpha = 0.6$.

Cases	Number of steps	Number of nodes in the final mesh
$\alpha = 0.1$	66	44714
$\alpha = 0.6$	87	48142
$\alpha = 0.9$	258	43825
$N_M = 1$	46	67262
$N_M = 3$	16	67854
$N_M = 6$	9	66057
$N_M = \infty$	8	66680
2-step	$7 + 3 = 10$	52013

Table II : Comparison between both strategies

CONCLUDING REMARKS

The method using the first selection strategy is *robust* in the following sense: if we choose $\alpha \geq \alpha_0$ with α_0 small, the convergence order is the same as that of the regular problem with quasiuniform meshes (measured with the number of elements).

An algorithm to get a relative error less than a prescribed tolerance is presented. This fact is desired in technological problems and it is not generally found in automatic systems.

To the authors' knowledge, there is no mathematical proof of the convergence of any adaptive procedure based on a posteriori error estimates for 2-D or 3-D. A convergence proof can be found in Babuška and Vogelius[9] for 1-D problems. It would be interesting to prove that the adaptive algorithm stops in a finite number of steps, as the numerical evidence shows. This would prove that the adaptive procedure is convergent in 2-D.

The results obtained in this paper are applicable to problems other than Poisson's. The method can be used in a large class of linear problems, as for example, linear elasticity or Stokes problem.

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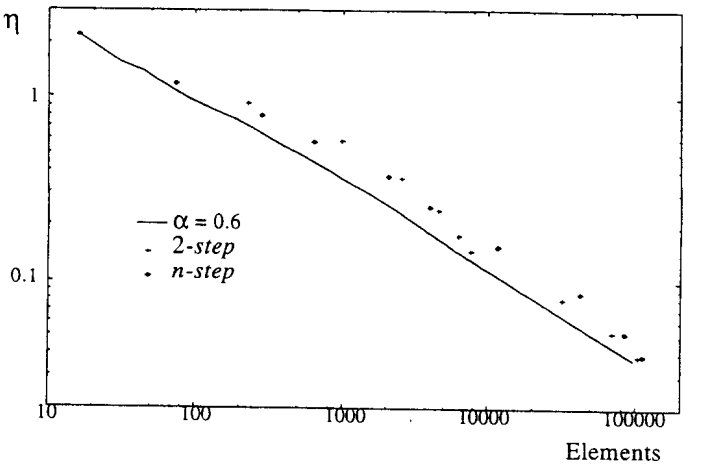

Figure 7: Convergence of the adaptive process for the test problem with $k = 8$. 2-step algorithm and n-step algorithm. Also is shown the result for the first selection strategy with $\alpha = 0.6$.

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