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A numerical method for solving the 2D direct problem in magnetotellurics

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Resumen

Presentamos un método numérico para resolver el problema directo bidimensional en magnetotelúrica. Las ecuaciones de Maxwell son tratadas como un sistema de ecuaciones diferenciales de primer orden, y se introducen condiciones de frontera absorbentes para minimizar efectos indeseados en los bordes. La soluciòn aproximada se obtiene utilizando un algoritmo de elementos finitos híbrido mixto de descomposición de dominio. Se presentan resultados obtenidos en una supercomputadora de arquitectura en paralelo IBM SP/2 de la Universidad de Purdue.

Abstract

We present a numerical method for solving the bidimensional forward problmem in magnetotellurics. Maxwell's equations are treated as a system of first order partial differential equations, and first order absorbing boundary conditions are introduced to minimize undesired border effects. The approximate solution is obtained by means of a mixed hybrid domain decomposed finite element procedure. Results obtained on a parallel supercomputer IBM SP/2 at Purdue University are shown.

Introduction

The magnetotelluric method consists in inferring the electrical conductivity distribution of the subsurface of the earth from measurements of natural electric and magnetic fields on its surface. This is a typical example of the so called inverse problems. When dealing with it, it is necessary to have at disposal numerical methods able to solve the forward problem (that is, to determine the induced electromagnetic fields in the subsurface given a electric conductivity distribution, when a plain electromagnetic wave impinges normally on the surface of the earth) as efficiently and accurately as possible.

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In this direction we present an algorithm to solve the direct 2D problem in magnetotellurics. Briefly stated, the problem consists in solving approximately Maxwell's equations for a given earth conductivity model.

The presented method, which deals with Maxwell's equations as a system of first order partial differential equations, is a mixed hybrid domain decomposed finite element procedure [1]. The chosen boundary conditions are first order absorbing ones, which allow for both a significant reduction in the size of the computational domain, and also for a reduction of the undesired effects generated by the artificial boundaries.

It is possible to find in the literature many algorithms devised to solve the forward 2D problem in magnetoellurics. See for example [2], [3], [4], [5], where global finite element procedures are designed to solve it. The method we introduce here is related to the ones described in [6], [7], [8],[9], [10], [11].

The domain decomposition technique naturally leads to the implementation of the procedure on parallel computers. Numerical results shown in this work were obtained on a IBM SP/2 parallel supercomputer at Purdue University.

In the next section we present the model and the differential problem to be solved, and afterwards we introduce the numerical method employed to solve it. Finally, we show some numerical results, and draw the conclusions.

The model

We want to solve the time-harmonic Maxwel's equations

$$\begin{aligned} &\text{(i)} && \nabla \times \mathbf{H} = (\sigma + iw\epsilon)\mathbf{E}, \\ &\text{(ii)} && \nabla \times \mathbf{E} = -iw\mu\mathbf{H}. \end{aligned} \tag{1}$$

where $\mathbf{E}$, $\mathbf{H}$ are electric and magnetic fields, ϵ is the electric permitivity, σ the electric conductivity, and μ the magnetic permeability of the medium. Together with these equations, the boundary conditions at the interface between two media of different physical properties must be considered. They are the continuity of the tangential and the normal components of the electric and magnetic fields, and the continuity of the normal component of the current density and magnetic flux.

The term $iw\epsilon\mathbf{E}$ (where w is the angular frequency) in the precedent equation represents displacement currents. It is in magnetotelluric sounding negligible compared to $\sigma\mathbf{E}$ (conduction currents). Therefore, displacement currents are discarded from now on.

Let us consider equations (1) in a two–dimensional domain Ω. Assume that Ω represents an horizontally-layered earth with an imbedded cylindrical inhomogeneity. The uppermost layer of Ω represents the air, with a very low (but positive) conductivity, and the other layers represent the subsurface, where the anomaly is located. We consider the z axis to be positive downwards, and the y axis to be the symmetry axis. According to the description of our domain, the electrical conductivity distribution is

$$\sigma(x,z) = \begin{cases} \sigma_p(z) & \text{in } \Omega_p \text{ (earth)} \\ \sigma_p(z) + \sigma_s(x,z) & \text{in } \Omega_s \text{ (inhomogeneity)} \end{cases} \tag{2}$$

Assuming that both sources and boundary conditions are also independent of y, it is well known [2],[5] that the electromagnetic response in Ω can be described by two uncoupled electromagnetic modes; the TE–mode involving field components (H_x, E_y, H_z) and the TM–mode involving field components (E_x, H_y, E_z). In what follows we will, for the sake of brevity, analyze in detail the latter and leave the former, arguing that its analysis shows no further difficulties.

For the TM–mode, eq. (1) yields:

$$\begin{aligned} &\text{(i)} && -\frac{\partial H_y}{\partial z} = \sigma E_x, \\ &\text{(ii)} && \frac{\partial H_y}{\partial x} = \sigma E_z \end{aligned} \tag{3}$$

and

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -iw\mu H_y. \tag{4}$$

Equations (3) and (4) have analytic solutions $\mathbf{E_p}(z) = (E_{xp}(z), 0, 0)$, and $\mathbf{H_p}(z) = (0, H_{yp}(z), 0)$ when an uniformly layered earth and an impinging wave of the form

$$\mathbf{E_0} = (E_{0x}, 0, 0) \qquad \mathbf{H_0} = (0, H_{0y}, 0) \tag{5}$$

are considered [12]. Following [13] we define residual electric $\mathbf{U}$ and magnetic $\mathbf{V}$ fields as

$$\begin{aligned} \mathbf{U} &= \mathbf{E} - \mathbf{E_p} = (U_x, 0, U_z), \\ \mathbf{V} &= \mathbf{H} - \mathbf{H_p} = (0, V_y, 0) = (0, v, 0), \end{aligned} \tag{6}$$

so that

$$\begin{aligned} &\text{(i)} && U_x = E_x - E_{xp}, \\ &\text{(ii)} && U_z = E_z, \\ &\text{(iii)} && v = H_y - H_{yp}. \end{aligned} \tag{7}$$

In order to simplify the notation, let us write for a scalar function $\varrho(x,z)$, curl $\varrho = \left(-\frac{\partial \varrho}{\partial z}, \frac{\partial \varrho}{\partial x}\right)$ and for a vector function $\mathbf{W} = (W_x(x,z), W_z(x,z))$ in the plane (x,z), define the scalar function curl $\mathbf{W} = \frac{\partial W_z}{\partial z} - \frac{\partial W_x}{\partial x}$. With the notation just introduced, the equations for the scattered fields can be written in the following way:

$$\text{curl } \mathbf{U} = -iw\mu v. \tag{8}$$

$$\sigma \mathbf{U} = \text{curl } v - (g, 0), \quad \text{with } g = (\sigma - \sigma_p) E_{xp}. \tag{9}$$

Some authors [5], [14], instead of working with these equations, derive the second order TM equation

$$-\nabla \cdot \left(\frac{1}{\sigma} \nabla v\right) + iw\mu v = -\frac{\partial}{\partial z}\left(\frac{\sigma_p}{\sigma}\right) E_{xp} - iw\mu \left(1 - \frac{\sigma_p}{\sigma}\right) H_{yp}. \tag{10}$$

for the potential v and from its solution derive $\mathbf{U}$. With this formulation, derivatives of the conductivity σ in the right-hand side of (10) are involved.

We choose to solve system (8) and (9) simultaneously, allowing for discontinuities in the conductivity σ, and avoiding the numerical calculation of the vector field $\mathbf{U}$ once the scalar v is known from (10).

Instead of using the normally applied Dirichlet boundary conditions, in order to minimize the effect of the artificial boundaries we will use the absorbing boundary condition [15]:

$$(1 - i) a \mathbf{U} \cdot \boldsymbol{\tau} + v = 0 \quad \text{on } \partial\Omega, \tag{11}$$

where $a = \left(\frac{\sigma}{2w\mu}\right)^{\frac{1}{2}}$. Here τ is the unit (counterclockwise) tangent vector and $\partial\Omega$ the boundary of the domain Ω.
In order to solve numerically system (9)-(11) we write it in *weak form*. In order to do so, we consider the space

$$H(\text{curl}\,,\Omega) = \{\Psi \in (L^2(\Omega))^2 : \text{curl}\,\Psi \in L^2(\Omega)\},$$

which is provided with the natural norm $\|\Psi\|_{H^c} = (\|\Psi\|_0^2 + \|\text{curl}\,\Psi\|_0^2)^{\frac{1}{2}}$.
Testing (9) against $\Psi \in H(\text{curl}\,,\Omega)$ and integrating by parts [16], using (11) we obtain

$$(\sigma \mathbf{U}, \Psi) - (v, \text{curl}\,\Psi) + \langle a(1-i)\mathbf{U}\cdot\tau, \Psi\cdot\tau\rangle = -((g,0),\Psi), \quad \Psi \in H(\text{curl}\,,\Omega). \tag{12}$$

Also, from (10) we get

$$(\text{curl}\,\mathbf{U}, \varphi) + (iw\mu v, \varphi) = 0, \quad \varphi \in L^2(\Omega). \tag{13}$$

As usual we have denoted $(\mathbf{v},\mathbf{w}) = \int_\Omega \mathbf{v}\cdot\overline{\mathbf{w}}\,d\mathbf{x}$, and $\langle \mathbf{v},\mathbf{w}\rangle = \int_{\partial\Omega} \mathbf{v}\cdot\overline{\mathbf{w}}\,dS$.
Thus, we give a mixed formulation for (9)-(11) as follows: Find $(\mathbf{U}, v) \in H(\text{curl}\,,\Omega)\times L^2(\Omega)$ such that

$$\begin{aligned} &(\sigma \mathbf{U}, \Psi) - (v, \text{curl}\,\Psi) + \langle a(1-i)\mathbf{U}\cdot\tau, \Psi\cdot\tau\rangle = -((g,0),\Psi), \quad \Psi \text{ in } H(\text{curl}\,,\Omega),\\ &(\text{curl}\,\mathbf{U}, \varphi) + (iw\mu v, \varphi) = 0, \quad \varphi \in L^2(\Omega). \end{aligned} \tag{14}$$

Numerical algorithm

Domain decomposition at the differential stage

Let us subdivide our original domain in a grid of non-overlapping rectangular subdomains Ω_{jk}, $j = 1,..,n_x; k = 1,..n_z$. Let $\partial\Omega_{jk}$ be the boundary of the subdomain Ω_{jk} and set $\partial\Omega_{jk} = \bigcup_{s=L,R,B,T}\Gamma_{jk}^s$, with Γ_{jk}^s $s = L,R,B,T$ being the left, right, bottom and top segments building the boundary $\partial\Omega_{jk}$ of the subdomain Ω_{jk}.

Being our goal to solve equations (14) in each subdomain Ω_{jk}, consistency conditions are to be imposed on all interior -artificial- boundaries Γ_{jk}^s (i.e., on all segments Γ_{jk}^s such that $\Gamma_{jk}^s \cap \partial\Omega = \emptyset$). The natural ones are the continuity of the tangential component of $\mathbf{U}_{jk}$ and the potential v_{jk} on Γ_{jk}^s. But instead of them, we will use an equivalent Robin transmission boundary condition of the form [17]

$$v_{jk} + \beta_{jk}^s \mathbf{U}_{jk}\cdot\tau_{jk} = v_{j^*k^*} - \beta_{jk}^s \mathbf{U}_{j^*k^*}\cdot\tau_{j^*k^*}, \quad \text{on } \Gamma_{jk}^s, \tag{15}$$

with $\beta_{jk}^s > 0$, and

$$\begin{aligned} &\{j^*k^*\} = \{j-1,k\} \text{ on } \Gamma_{jk}^L, \qquad &&\{j^*k^*\} = \{j+1,k\} \text{ on } \Gamma_{jk}^R,\\ &\{j^*k^*\} = \{j,k-1\} \text{ on } \Gamma_{jk}^B, \qquad &&\{j^*,k^*\} = \{j,k+1\} \text{ on } \Gamma_{jk}^T. \end{aligned} \tag{16}$$

We can state now the differential domain decomposition procedure as follows: For all pairs (j,k) find $(\mathbf{U}_{jk}, v_{jk})$ such that

$$\begin{aligned} (\sigma\mathbf{U}_{jk},\Psi)_{jk} - (v_{jk},\text{curl}\,\Psi)_{jk} - \sum_{s_i}\langle v_{jk},\Psi\cdot\tau_{jk}\rangle_{\Gamma_{jk}^s} + \langle a(1-i)\mathbf{U}_{jk}\cdot\tau_{jk},\Psi\cdot\tau_{jk}\rangle_{B_{jk}^a}&\\ = -((g,0),\Psi)_{jk}, \quad \Psi \in H(\text{curl}\,,\Omega_{jk}),&\\ (\text{curl}\,\mathbf{U}_{jk},\varphi)_{jk} + (iw\mu v_{jk},\varphi)_{jk} = 0, \qquad \varphi \in L^2(\Omega_{jk}).& \end{aligned} \tag{17}$$

Index s_i runs over all interior boundaries and $B_{jk}^a \cong \partial\Omega_{jk}\cap\partial\Omega$. We get the hybrid formulation of problem (17) when using eq. (15) to replace v_{jk} in the third term of the first equation above.

As we have already stated, our goal is to localize calculations on each subdomain Ω_{jk}. Taking into account that (15) and (16) involve adjacent subdomains, the following iterative procedure is suggested for (17): Choose $(\mathbf{U}_{jk}^0, v_{jk}^0)$ arbitrarily. Then compute $(\mathbf{U}_{jk}^{n+1}, v_{jk}^{n+1})$ as the solution of the equations:

$$\begin{aligned}(\sigma\mathbf{U}_{jk}^{n+1}, \Psi)_{jk} - (v_{jk}^{n+1}, \text{curl } \Psi)_{jk} + \sum_{s_i}\langle\beta_{jk}^s \mathbf{U}_{jk}^{n+1}\cdot\tau_{jk}, \Psi\cdot\tau_{jk}\rangle_{\Gamma_{jk}^s} \\ +\langle a(1-i)\mathbf{U}_{jk}^{n+1}\cdot\tau_{jk}, \Psi\cdot\tau_{jk}\rangle_{B_{jk}^a} \\ = -\sum_{s_i}\langle\beta_{jk}^s \mathbf{U}_{j^*k^*}^n\cdot\tau_{j^*k^*} - v_{j^*k^*}^n, \Psi\cdot\tau_{jk}\rangle_{\Gamma_{jk}^s} - ((g,0),\Psi)_{jk}, \quad \Psi\in H(\text{curl},\Omega_{jk}), \\ (\text{curl } \mathbf{U}_{jk}^{n+1}, \varphi)_{jk} + (iw\mu v_{jk}^{n+1}, \varphi)_{jk} = 0, \qquad \varphi\in L^2(\Omega_{jk}). \end{aligned} \tag{18}$$

We can now define the discrete version of the iterative procedure (18).

The finite element procedure

In order to simplify the description of the numerical procedure, we use the same partition (uniform rectangular cells) of the domain Ω for both the domain decomposition, and the finite element procedures. As we want to approximate simultaneously two different kind of functions $\mathbf{U}$ and v we need to define two different spaces. Let therefore

$$V^h = \{\mathbf{U}^h \in H(\text{curl},\Omega) : \mathbf{U}^h|_{\Omega_{jk}} \in P_{0,1}\times P_{1,0}\}, \quad W^h = \{v^h \in L^2(\Omega) : v^h|_{\Omega_{jk}} \in P_{0,0}\},$$

be the appropriated ones, and let $V_{jk}^h = V^h|_{\Omega_{jk}}, W_{jk}^h = W^h|_{\Omega_{jk}}$ be their restrictions to the domain Ω_{jk}. Here, $P_{s,t}$ denotes the polynomials of degree not greater than s in x and not greater than t in z.

Note that since functions in W^h are allowed to be discontinuous across the interior (artificial) boundaries, imposing the Robin transmission boundary conditions (15) on them would imply that the discrete approximation of v, $v^h \in W^h$ is a constant. We then introduce a set of Lagrange multipliers associated with the potentials $\lambda_{jk}^s \sim v_{jk}^h$ on the interior edges Γ_{jk}^s

$$\Lambda^h = \{\lambda^h : \lambda^h|_{\Gamma_{jk}^s} \in P_{m-1}(\Gamma_{jk}^s) = \Lambda_{jk}^{h,s}, \quad \Gamma_{jk}^s \cap \partial\Omega = \phi\}.$$

The consistency conditions (15) can be then rewritten as

$$\lambda_{jk}^s = \lambda_{j^*k^*}^s - \beta_{jk}^s(\mathbf{U}_{jk}\cdot\tau_{jk} + \mathbf{U}_{j^*k^*}\cdot\tau_{j^*k^*}), \quad \text{on } \Gamma_{jk}^s, \quad \Gamma_{jk}^s\cap\partial\Omega = \emptyset \tag{19}$$

where

$$\begin{aligned} s^* &= R \text{ for } s = L, & s^* &= L \text{ for } s = R, \\ s^* &= T \text{ for } s = B, & s^* &= B \text{ for } s = T. \end{aligned} \tag{20}$$

And finally, the hybridized mixed iterative finite element domain decomposition procedure is defined as follows (superscript h is not explicitly written):

1. Choose initial values $(\mathbf{U}_{jk}^0, v_{jk}^0, \lambda_{jk}^0)$ arbitrarily.

2. Compute $(\mathbf{U}_{jk}^{n+1}, v_{jk}^{n+1}, \lambda_{jk}^{n+1})$ as the solution of

$$(\sigma \mathbf{U}_{jk}^{n+1}, \Psi)_{jk} - (v_{jk}^{n+1}, \text{curl } \Psi)_{jk} + \sum_{s_i} \langle \beta_{jk}^{s} \mathbf{U}_{jk}^{n+1} \cdot \tau_{jk}, \Psi \cdot \tau_{jk} \rangle_{\Gamma_{jk}^{s}}$$
$$+ \langle a(1-i) \mathbf{U}_{jk}^{n+1} \cdot \tau_{jk}, \Psi \cdot \tau_{jk} \rangle_{B_{jk}^{a}}$$
$$= - \sum_{s_i} \langle \beta_{jk}^{s} \mathbf{U}_{j^*k^*}^{n} \cdot \tau_{jk} - \lambda_{j^*k^*}^{s^*,n}, \Psi \cdot \tau_{jk} \rangle_{\Gamma_{jk}^{s}} - ((g,0), \Psi)_{jk},$$

$$(\text{curl } \mathbf{U}_{jk}^{n+1}, \varphi)_{jk} + (iw\mu v_{jk}^{n+1}, \varphi)_{jk} = 0,$$

$$\lambda_{jk}^{s,n+1} = \lambda_{j^*k^*}^{s^*,n} - \beta_{jk}^{s} (\mathbf{U}_{jk}^{n+1} \cdot \tau_{jk} + \mathbf{U}_{j^*k^*}^{n} \cdot \tau_{j^*k^*}). \tag{21}$$

We can now point out some features of the proposed algorithm.

- The absorbing boundary condition introduced makes it unnecessary to consider big numerical domains.
- The algebraic problem associated with this algorithm is much easier to solve than that corresponding to a global finite element procedure, since the program reduces to the evaluations of algebraic expressions determining v_{jk}, U_{jk}^{s}, and λ_{jk}^{s}, for all s, j, k in each cell Ω_{jk} at the current iteration level.
- Because of the mixed treatment and the domain decomposition technique employed no global linear system appears, which means that simpler and faster calculations can be done, and that less data storage is needed.
- As in each step when the solution in the cell Ω_{jk} is built we employ information of the adjacent cells (through Robin's condition), it is possible to improve the efficiency of the algorithm using a *red - black* scheme i.e., the domain Ω is considered a chess-board and: a) the solution in the *red* cells is actualized using the solutions and Lagrange multipliers in adjacent cells , b) Lagrange multipliers are actualized in all *red* domains. c) we repeat steps a) and b) for the *black* cells.

Previous to the example, we can describe how the algorithm works on a parallel computer. The parallelization of the problem is done by assigning a certain number of processors in each direction of the numerical domain. The most efficient way to use them is to divide the problem into, as close as possible, an equal number of unknowns for which to solve on each processor [18]. Because each one needs to make calculations for a subset of the unknowns (a portion of the "chess-board" with the same number of red and black cells, in our case), and because the processors are making their calculations simultaneously, the time needed to reach the solution is reduced by a factor approximately equal to the number of processors involved. The ratio is not exactly the number of processors, because at each iteration information must be interchanged between 'adjacent' processors and this process, although very fast, is not instantaneous.

Data can be classified in two kinds, local and global. The former are e.g., conductivities of the cells, and the latter are variables that each processor needs to handle, as the frequency or position of the inhomogeneities. All processors read the same input data and, when needed, each transforms local into global data. As we have already said, data on the boundaries of regions corresponding to adjacent processors must be transferred among them. This is done in each iteration at the beginning and after actualizing the black cells.

Numerical example

We perform our calculation on the model proposed in [5], which is shown in Fig. 1. The rectangular anomaly, which has a conductivity $\sigma_2 = 0.5\,(\Omega m)^{-1}$ lays buried in an homogeneous background with conductivity $\sigma_1 = 0.01\,(\Omega m)^{-1}$. Because of the absorbing boundary condition,

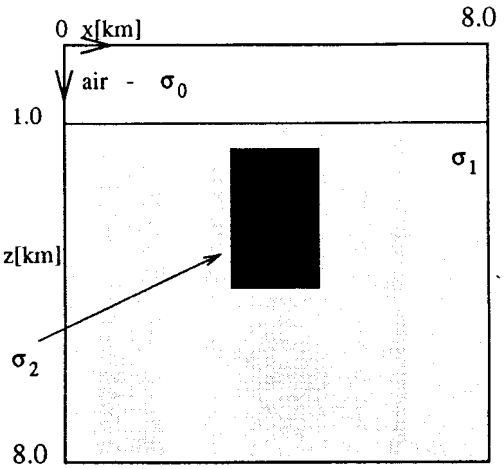

Figure 1: The model

we do not need to extend our computational domain far away from the anomaly and we take as computational domain a square, with side length of 8 km. A requirement of the presented algorithm is to consider a non-zero conductivity for the air region, which was chosen to be $\sigma_0 = 10^{-7}\,(\Omega\, m)^{-1}$ and with a height of 1 km.

In order to perform the calculations, the relative error required to stop the iterative process was chosen to be 10^{-5} (numerical experiments showed that beyond this number the results obtained did not displayed any observable change). The grid size was 64 × 64 cells, which yields a cell-size of 125 m. As we chose a frequency of 1 Hz for the example we show, this figure results less than one third of the skin-depth, which improves accuracy in the calculation [19]. Although fields **E**

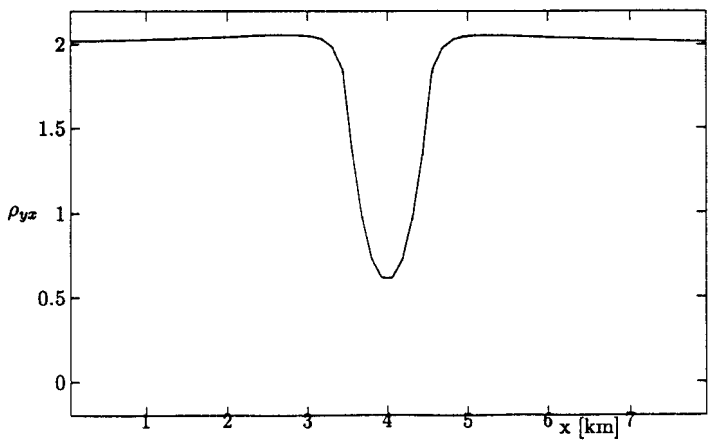

Figure 2: *TM* apparent resistivity at a frequency of 1 Hz, [Ohm-m]

and **H** are obtained, it is customary to display derived quantities

$$\Phi_{yx} \qquad \text{phase of } Z_{yx} = \frac{E_x}{H_y} \qquad \text{Impedance}$$

$$\rho_{yx} = \frac{Z_{yx}\overline{Z_{yx}}}{\mu\omega} \qquad \text{Apparent resistivity,}$$

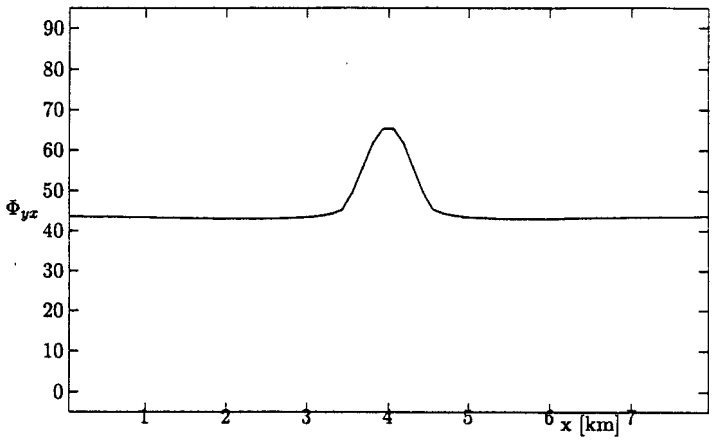

Figure 3: TM impedance phase, [Degrees]

because they are independent of the modulus of the impinging wave, usually unknown. Fig. 2 and Fig. 3 show results obtained for the apparent resistivity ρ_{yx} and impedance phase Φ_{yx} respectively, which are in very good agreement with the ones displayed in [5], calculated using a different procedure. Finally we show in Fig. 4 the performance of the algorithm on the already

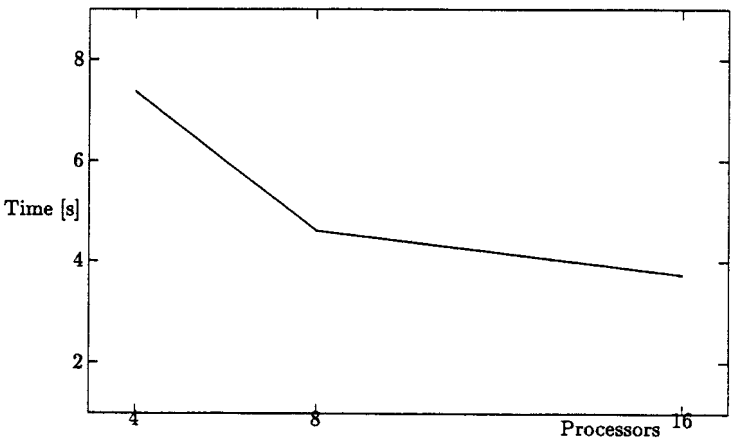

Figure 4: CPU time on the SP/2 parallel supercomputer, for the given example

mentioned parallel computer. The program was run three times, with four, eight and sixteen processors respectively. Parameters were the same as in the example mentioned above. The algorithm converged after 280 iterations.

Conclusions

We have presented a numeric algorithm to solve the forward 2D problem in magnetotellurics.

Instead of dealing with the second order problem, we treated Maxwell's equations as a system of first order partial differential equations. Absorbing boundary conditions were introduced, what allowed to consider smaller computational domains. The problem was solved by a mixed hybrid domain decomposed finite element procedure. The algorithm was successfully implemented on a parallel supercomputer, being the results obtained in accordance with already published calculations. It was observed that when the number of processors, and therefore the number of domains associated to them, gets closer to the grid-size the efficiency of the method diminishes, because of the time needed to interchange information among them. The results obtained encourage us to face the extension of the algorithm to deal with the forward 3D case, and also with the inverse problem both in two and three dimensions.

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