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AN EXPLICIT SCHEME FOR THE NUMERICAL SOLUTION OF THE SHALLOW WATER EQUATIONS

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RESUMEN

Este trabajo muestra la aplicación del método de elementos finitos para resolver problemas de flujos en aguas poco profundas, así como una comparación entre el uso del método 'lumped' SUPG y el método de masa consistente SUPG. Un esquema explícito ha sido desarrollado para la integración en la capa temporal. Diversos ejemplos basados en ecuaciones de aguas poco profundas unidimensionales ilustran la precisión y eficiencia alcanzada con tales métodos. Este trabajo forma parte de un proyecto cuyo fin es modelar problemas en aguas poco profundas incluyendo efectos de turbulencia usando cálculo paralelo.

ABSTRACT

This paper shows the application of a finite element method for solving shallow water flow problems as well as a comparison between the use of lumped mass SUPG and consistent mass SUPG method. An explicit scheme has been employed for the integration in the temporal layer. Several examples based on the one-dimensional shallow-water equations illustrate the accuracy and efficiency obtained with such methods. This work is the first stage of a big project oriented to parallel computing to solve turbulent shallow water equations.

1. INTRODUCTION

Applications of the shallow water equation include a wide variety of coastal phenomena such as drift and tidal current, pollutant dispersion, storm surge, tsunami wave propagation, drifts and transport. A great number of civil engineering projects in river hydraulics, coastal water and estuaries require predictive models of the flow. The trend is towards computational methods based on the one, or two dimensional shallow water equations. The use of a fine spatial grid is often required in practical applications thereby necessitating a very small integration time step if an explicit scheme is employed, having as a consequence, an unacceptable computing cost. Due to the current state of parallel computation, explicit methods have found a new interest because of their great adaptation to the parallel programming [1],[2],[3],[4],[5],[6].

In this paper an application of an explicit scheme in time using both Lumped SUPG mass matrix or Consistent SUPG mass matrix in combination with the use of a stabilized spatial discretization for solving shallow water flow problems has been examined[2],[7]. For simplicity we began adopting the lumped SUPG mass matrix scheme but our results show a very diffusive behavior in the description of a typical test problem like a solitary wave propagation along a one-dimensional channel with uniform bottom slope. On the other hand and thinking about to be consistent with the stabilized SUPG formulation we found that this scheme results to be unstable for the planar shallow water one-dimensional equation. These facts have been the main motivation to the introduction of a β-parameter in the mass matrix of the consistent SUPG method in order to recover part of its stability taking advantage of its reasonable accuracy. We have confirmed this hypotesis by numerical analysis arguments taking as a model equation the clasical unsteady one-dimensional advective equation. Moreover, several numerical examples based on the one-dimensional shallow-water equations illustrate the accuracy and efficiency obtained with such methods.

2. STABILITY ANALYSIS OF β-SUPG METHOD

In order to analize the stability of numerical schemes obtained by the application of the β-SUPG procedure in combination with an explicit scheme, let us consider one of the most representative equations for modeling transport phenomena, the convective, hyperbolic equation, written here as follows:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad 0 \le x \le L, \; t \ge 0$$

where u is the unknown function of (x, t) and a is the convection speed ($a > 0$). When linear elements are used, global matrices $\mathbf{M}$ and $\mathbf{K}$ will be obtained by assembling the element matrices. Matrix $\mathbf{M}^e$ may be diagonalized by using the row-sum lumping technique(see [8] for different choices of M^e arising from numerical integration). When this matrix is not diagonalized and the element matrices are assembled, a typical algorithmic equation for an internal node m (consistent mass SUPG method) may be written as:

$$\frac{1}{2}(\frac{1}{3} + \frac{\alpha}{2})(u_m^{n+1} - u_m^n) + \frac{4}{3}(u_m^{n+1} - u_m^n) + \frac{1}{2}(\frac{1}{3} - \frac{\alpha}{2})(u_m^{n+1} - u_m^n) =$$
$$\Delta t[\frac{\alpha a}{2h}(u_{m+1}^n - 2u_m^n + u_{m-1}^n) - \frac{a}{2h}(u_{m+1}^n - u_{m-1}^n)]$$

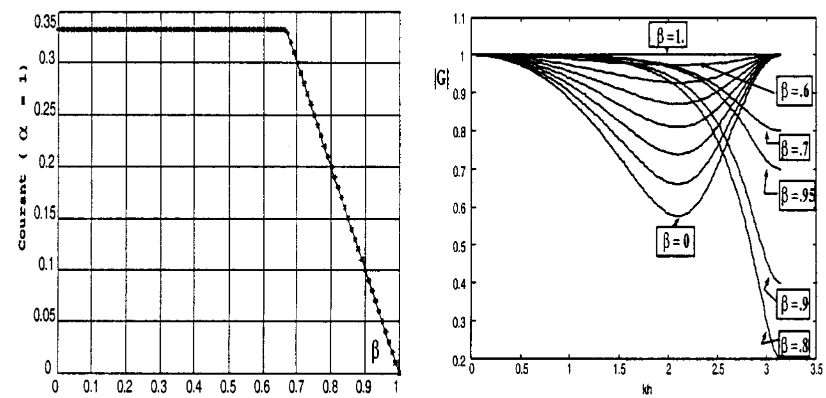
In the practice we have used $\alpha = 1$ and we have a compromise between the accuracy and stability of this procedure. One of the way to guarantee the stability property of this scheme is introducing a β parameter in the mass matrix corresponding to the temporal term. Based on this idea, using the β -SUPG discretization, we obtain an explicit scheme in the following form:

$$\frac{1}{2}(\frac{1}{3} + \beta\frac{\alpha}{2})(u_m^{n+1} - u_m^n) + \frac{4}{3}(u_m^{n+1} - u_m^n) + \frac{1}{2}(\frac{1}{3} - \beta\frac{\alpha}{2})(u_m^{n+1} - u_m^n) =$$
$$\Delta t[\frac{\alpha a}{2h}(u_{m+1}^n - 2u_m^n + u_{m-1}^n) - \frac{a}{2h}(u_{m+1}^n - u_{m-1}^n)]$$

Replacing in the above equation the following field $u_m^n = e^{i(kmh - \omega n \Delta t)}$, where i is the imaginary unit and k is the wave number in the x direction, we have an equation for the interior nodes in which the function G is the amplification factor

$$G = 1 + C\frac{\alpha(\cos(kh) - 1) - \mathrm{i}\sin(\mathrm{kh})}{\frac{\cos(kh)+2}{3} - \mathrm{i}\beta\frac{\alpha\sin(\mathrm{kh})}{2}}$$

where C is the Courant number. A scale of diffusivity has been obtained by means of the introduction of β parameter. By means of this scale the critical Courant number has been determined. In the Figure 1(a) the critical Courant values respect to the β parameter for $\alpha = 1$ are shown. For $\beta \geq .95$, we obtain a critical Courant value less than 0.05. We know that the "Consistent mass" SUPG method is obtained for $\beta = 1$ value, and their representative curve is not included in the family of allowed curves that characterize the β- SUPG method.



(a) optimal β-values

(b) Amplification Factor | G |

Figure 1: β-SUPG method.

Figure 1(b) shows the absolute values of amplification factor G. On this figure the stability property for several β values is represented. We can appreciate that β- SUPG method is unstable for $\beta = 1$ only.

3. NUMERICAL RESULTS

Several strategies of SUPG method have been applied to a number of shallow water flow problems, such that

One-dimensional flow along a channel of uniform width

Dam break problem

A solitary wave propagated along a channel with uniform bottom slope.

Water Flux over the shoal

Hydraulic jump a diverging rectangular channel

and others.

A solitary wave propagated along a one-dimensional channel with uniform bottom slope.

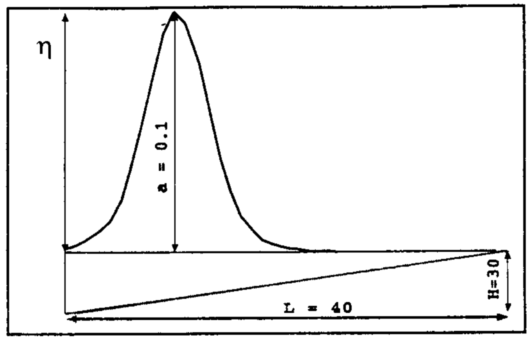


Figure 2: Coordinate system for shallow water equations.

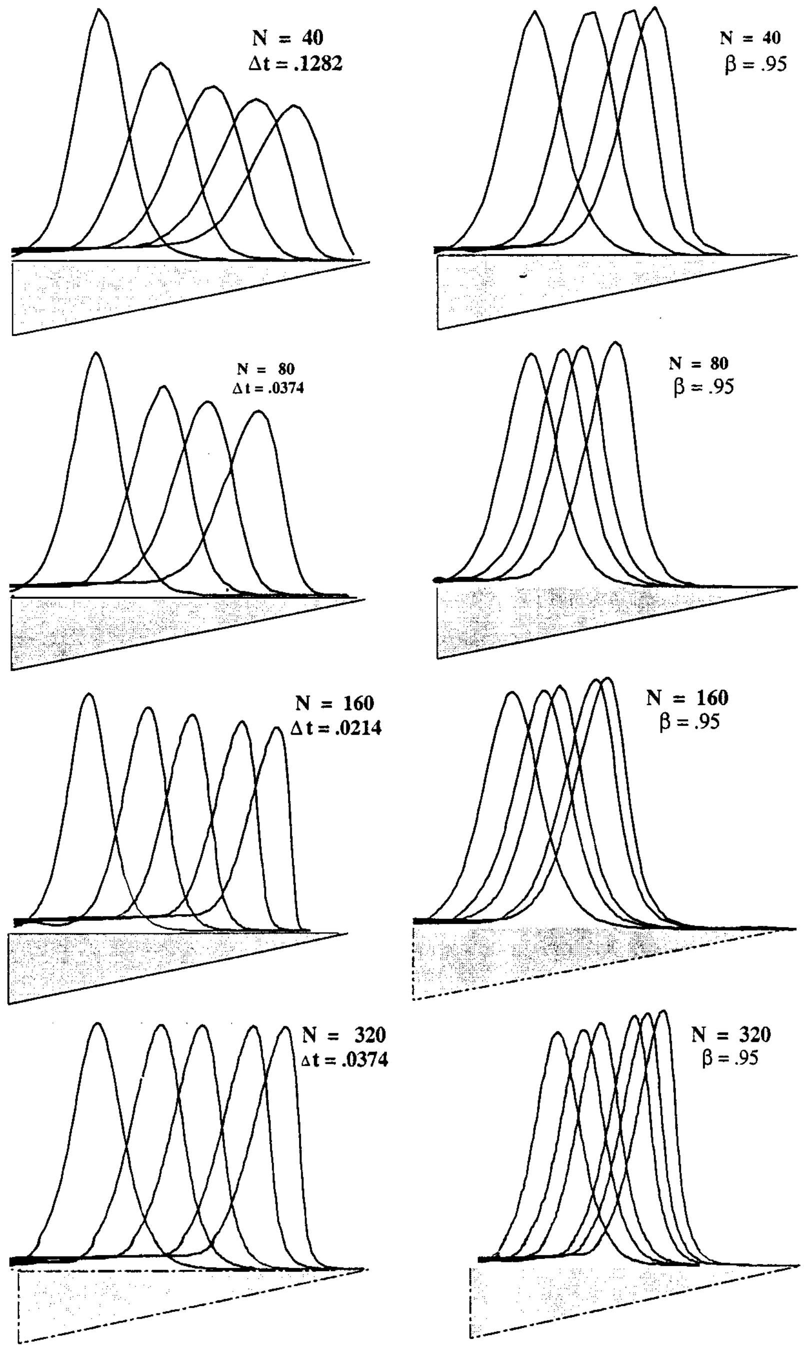
The following example is the analysis of a solitary wave propagated along a one-dimensional channel with uniform bottom slope. The initial configuration of the solitary wave and the variation of the depth is shown in Figure 2. The initial conditions are given by:

$$\eta = a\ \mathrm{sech}^2 \frac{1}{2}\sqrt{3a}\left(x - \frac{1}{\alpha}\right)$$
$$u = -(1 + a/2)\eta/(\alpha x + \eta)$$

where $a = 0.1, g = 1.0, \alpha = 1/30$. Figure 3 shows the computed results by "Lumped mass" SUPG method at left and β - "Consistent mass" SUPG method at right. For this problem the exact solution has a peak value of 1.2 times the initial peak value. This example allows to examine the numerical diffusivity property of several schemes[5],[6]. For the "Lumped mass" SUPG method the computed results seem to include a significant damping effect, which, as the number of subdivisions is increased(e.g. $N = 160$ or 320) the results seem to improve. It is very noticeable that the β - "Consistent mass" SUPG method produces better solutions even for relatively coarse grids (see figure 3(right) where the peak value increased up to 1.2 times the initial peak value).

4. CONCLUSIONS

In this paper a technique to solve shallow water equations system has been examined. The use of the SUPG method and an explicit scheme in the time discretization allows to describe several physical phenomena for inviscid flow. An analysis about the stability of this method for these equations system shows the convenience of using β-SUPG method in the solution of problems where others methods become very diffusive. A comparison between several methods has been developed. The numerical tests show that the numerical code describes adequately the physical phenomenon. The future work is oriented towards large scale simulation of a shallow water model with turbulent effects.



N = 40
Δt = .1282
N = 40
β = .95
N = 80
Δt = .0374
N = 80
β = .95
N = 160
Δt = .0214
N = 160
β = .95
N = 320
Δt = .0374
N = 320
β = .95

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