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A FINITE ELEMENT METHOD FOR THE APPROXIMATION OF WAVES IN FLUID SATURATED POROVISCOELASTIC MEDIA

Juan E. Santos[*]

[*]Facultad de Ciencias Astronómicas y Geofísicas,
Universidad Nacional de La Plata, CONICET
Paseo del Bosque s/n, 1900 - La Plata, Argentina
e-mail: santos@fcaglp.fcaglp.unlp.edu.ar
Also Department of Mathematics, Purdue University,
W. Lafayette, IN, 47907, USA
e-mail: santos@math.purdue.edu

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Abstract. *This work presents and analyzes a finite element procedure for the simulation of wave propagation in a porous solid saturated by a single-phase fluid. The equations of motion, formulated in the space-frequency domain, include dissipation due to viscous interaction between the fluid and solid phases and intrinsic anelasticity of the solid modeled using linear viscoelasticity. This formulation leads to the solution of a Helmholtz-type boundary value problem for each temporal frequency. For the spatial discretization, nonconforming finite element spaces are employed for the solid phase, while for the fluid phase the vector part of the Raviart-Thomas-Nedelec mixed finite element space is used. Optimal a priori error estimates for a standard Galerkin finite element procedure are derived.*

1 INTRODUCTION

The propagation of waves in a porous elastic solid saturated by a single–phase compressible viscous fluid was first analyzed by Biot in several classic papers,[1].[2] Biot assumed that the fluid may flow relative to the solid frame causing friction to arise. In the low frequency range, such flow is of laminar type and obeys Darcy's law for fluid flow in porous media. In the high frequency range, Biot pointed out that a frequency correction factor had to be introduced in the Darcy coefficient. Biot also predicted the existence of two compressional waves, which he denoted type I and type II compressional waves, and one shear wave. The three waves suffer attenuation and dispersion effects in the seismic to the ultrasonic range of frequencies. The type I and shear waves have a behavior similar to those in an elastic solid, with high phase velocities, low attenuation and very little dispersion. The type II wave behaves as a diffusion–type wave due to its low phase velocity and very high attenuation and dispersion. The experimental confirmation of Biot's theory was done by Plona, who reported the observation of the three waves in several papers.[3,4]

This article presents and analyzes a finite element method for the appoximate solution of Biot's equations of motion stated in the space-frequency domain, including solid matrix dissipation using a linear viscoelastic model and frequency dependent mass and viscous coupling coefficients.

The algorithm employs the nonconforming rectangular element defined in[5] to approximate the displacement vector in the solid phase. The dispersion analysis presented in[6] shows that employing this nonconforming element allows for a reduction in the number of points per wavelength necessary to reach a desired accuracy. The displacement in the fluid phase is approximated using the vector part of the Raviart-Thomas-Nedelec mixed finite element space of zero order, which is a conforming space.[7,8] Under minimal smoothnes requirements on the solution of Biot' s equations of motion, the error analysis yields optimal *apriori* error estimates both in $L^2(\Omega)$ and $H^1(\Omega)$, as well as an optimal error estimate in $L^2(\partial\Omega)$.

2 REVIEW OF BIOT'S THEORY

We consider a porous solid saturated by a single phase, compressible viscous and assume that the whole aggregate is isotropic. Let $u^{(1)} = (u_i^s)$ and $\tilde{u}^{(2)} = (\tilde{u}_i^{(2)})$, $i = 1, \cdots, d$ denote the averaged displacement vectors of the solid and phases, respectively, where d denotes the Eucliden dimension. Also let

$$u^{(2)} = \phi(\tilde{u}^{(2)} - u^{(1)}),$$

be the average relative fluid displacement per unit volume of bulk material, where ϕ denotes the effective porosity. Also set $u = (u^{(1)}, u^{(2)})$ and recall that

$$\xi = -\nabla \cdot u^{(2)},$$

represents the change in fluid content.

Let $\varepsilon_{ij}(u^{(1)})$ be the strain tensor of the solid. Also, let $\sigma_{ij},\ i, j = 1, \cdots, d$, and p_f denote the stress tensor of the bulk material and the fluid pressure, respectively. Following,[9] the stress-strain relations can be written in the form:

$$\sigma_{ij}(u) = 2\mu\,\varepsilon_{ij}(u^{(1)}) + \delta_{ij}(\lambda_c\,\nabla\cdot u^{(1)} - \alpha\,K_{av}\,\xi), \tag{1a}$$
$$p_f(u) = -\alpha\,K_{av}\,\nabla\cdot u^{(1)} + K_{av}\xi. \tag{1b}$$

The coefficient μ is equal to the shear modulus of the bulk material, considered to be equal to the shear modulus of the dry matrix. Also

$$\lambda_c = K_c - \frac{2}{d}\mu,$$

with K_c being the bulk modulus of the saturated material. Following[10,11] the coefficients in (1) can be obtained from the relations

$$\alpha = 1 - \frac{K_m}{K_s}, \quad K_{av} = \left[\frac{\alpha-\phi}{K_s} + \frac{\phi}{K_f}\right]^{-1} \quad K_c = K_m + \alpha^2 K_{av},$$

where K_s, K_m and K_f denote the bulk modulus of the solid grains composing the solid matrix, the dry matrix and the the saturant fluid, respectively. The coefficient α is known as the effective stress coefficient of the bulk material.

2.1 Modification of the elastic coefficients to introduce viscoelasticity

In order to introduce viscoelasticity we use the correspondence principle stated by M. Biot,[9] *i.e.,* we replace the real poroelastic coefficients in the constitutive relations by complex frequency dependent poroviscoelastic moduli satisfying the same relations as in the elastic case. In this work the linear viscoelastic model presented in[12] is used to make some of the moduli in (1) complex and frequency dependent. The set of poroviscoelastic moduli is computed using the following formula:

$$M(\omega) = \frac{M_{re}}{R_M(\omega) - iT_M(\omega)},$$

where the symbol 'M' represents any of the moduli in (1) and the coefficients M_{re} is the relaxed elastic modulus associated with M.[13]

The frequency dependent functions R_M and T_M, associated with a continuous spectrum of relaxation times, characterize the viscoelastic behavior and are given by[12]

$$R_M(\omega) = 1 - \frac{1}{\pi\widehat{Q}_M}\ln\frac{1+\omega^2 T_{1,M}^2}{1+\omega^2 T_{2,M}^2}, \qquad T_M(\omega) = \frac{2}{\pi\widehat{Q}_M}\tan^{-1}\frac{\omega(T_{1,M}-T_{2,M})}{1+\omega^2 T_{1,M}T_{2,M}}.$$

The quantity

$$\tan(\delta_M(\omega)) = \frac{\mathrm{Im}(M(\omega))}{\mathrm{Re}(M(\omega))} = \frac{1}{Q_M(\omega)}$$

is a measure of the viscoelastic behavior of the material. $Q_M(\omega)$ is called the quality factor.

The model parameters $\widehat{Q}_M$, $T_{1,M}$ and $T_{2,M}$ are taken such that the quality factors $Q_M(\omega)$ are approximately equal to the constant $\widehat{Q}_M$ in the range of frequencies where the equations are solved, which makes this model convenient for geophysical applications. Values of $\widehat{Q}_M$ range from $\widehat{Q}_M = 20$ for highly dissipative materials to about $\widehat{Q}_M = 1000$ for almost elastic ones.

2.2 The equations of motion

Let us consider a unit cube $\Omega \subset R^d$ of bulk material with boundary $\Gamma = \partial\Omega$. Let ρ_s and ρ_f denote the mass densities of the solid grains and the fluid and let

$$\rho_b = (1-\phi)\rho_s + \phi\rho_f$$

denote the mass density of the bulk material. Then Biot's equations of motion stated in the space-time domain are[1,10]

$$\rho_b \frac{\partial^2 u^{(1)}}{\partial t^2} + \rho_f \frac{\partial^2 u^{(2)}}{\partial t^2} - \nabla \cdot \sigma(u) = f^{(1)}, \tag{2a}$$

$$\rho_f \frac{\partial^2 u^{(1)}}{\partial t^2} + g \frac{\partial^2 u^{(2)}}{\partial t^2} + b \frac{\partial u^{(2)}}{\partial t} + \nabla p_f(u) = f^{(2)}. \tag{2b}$$

The mass coupling coefficient g represent the inertial effects associated with dynamic interactions between the solid and fluid phases, while the coefficient b include the viscous coupling effects between such phases. They are given by the relations

$$b = \frac{\eta}{k}, \qquad g = \frac{S\rho_f}{\phi}, \qquad S = \frac{1}{2}\left(1 + \frac{1}{\phi}\right), \tag{3}$$

where η is the fluid viscosity and k the absolute permeability. S is known as the structure or tortuosity factor. Above a certain critical frequency ω_c the coefficients b and g become frequency dependent.[2,14] This effect is associated with the departure of the flow from the laminar Poiseuille type at the pore scale, which occurs for angular frequencies greater than ω_c. The value of ω_c can be estimated by the formula

$$\omega_c = \frac{\eta\phi}{k\,\rho_f\,S}. \tag{4}$$

Let $u = (u^{(1)}, u^{(2)})$. When the frequency correction factor is included, the mass and viscous coupling coefficients g and b become

$$g(\omega) = \frac{\rho_f}{\phi}\left(S + \frac{F_i(\theta)}{\omega}\frac{\eta}{k}\frac{\phi}{\rho_f}\right), \quad b(\omega) = \frac{\eta}{k}F_r(\theta). \tag{5}$$

The function $F(\theta) = F_r(\theta) + iF_i(\theta)$ is the frequency correction factor proposed by Biot in[2] as a universal function representing frequency effects for arbitrary geometries. It has the expression

$$F(\theta) = \frac{1}{4} \quad \frac{\theta T(\theta)}{1 + \frac{2i}{\theta} T(\theta)}, \quad T(\theta) = \frac{ber'\theta + i\,bei'\theta}{ber\theta + i\,bei\theta} \tag{6}$$

with the argument θ being defined by

$$\theta = a_p \left(\omega \rho_f / \eta\right)^{1/2}. \tag{7}$$

In the formulas above, $ber\ z$ and $bei\ z$ are the real and imaginary parts of the Kelvin function of the first kind and zero order.

The pore–size parameter a_p in (7) has to be estimated from the data of the given formation. Following,[15] a_p can be computed using the relation

$$a_p = 2(A_0 k/\phi)^{1/2}, \tag{8}$$

where A_0 is the Kozeny–Carman constant[16] which in agreement with[15] may be taken to be 5.

Let the positive definite matrix $\mathcal{P}$ and the nonnegative matrix $\mathcal{B}$ be defined by

$$\mathcal{P} = \begin{pmatrix} \rho_b I & \rho_f I \\ \rho_f I & g(\omega) I \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0I & 0I \\ 0I & b(\omega) I \end{pmatrix},$$

where I denoted the identity matrix in $R^{d\times d}$. Next, let $\mathcal{L}(u)$ be the second order differential operator defined by

$$\mathcal{L}(u) = \left(\nabla \cdot \sigma(u), -\nabla p_f(u)\right)^t.$$

Then if $\omega = 2\pi f$ is the angular frequency, equations (2) stated in the space-frequency domain become,[1,2]

$$-\omega^2 \mathcal{P} u(x,\omega) + i\omega \mathcal{B} u(x,\omega) - \mathcal{L}(u(x,\omega)) = f(x,\omega), \quad x \in \Omega. \tag{9}$$

Let us denote by ν the unit outer normal on Γ. In the 2D case let χ be a unit tangent on Γ so that $\{\nu, \chi\}$ is an orthonormal system on Γ. In the 3D case let χ^1 and χ^2 be two unit tangents on Γ so that $\{\nu, \chi^1, \chi^2\}$ is an orthonormal system on Γ. Then, in the 2D case set

$$\mathcal{G}_\Gamma(u) = \Big(\sigma(u)\nu\cdot\nu, \sigma(u)\nu\cdot\chi, p_f(u)\Big)^t, \quad S_\Gamma(u) = \left(u^s\cdot\nu, u^s\cdot\chi, u^f\cdot\nu\right)^t, \tag{10}$$

and in the 3D case set

$$\begin{aligned} \mathcal{G}_\Gamma(u) &= \Big(\sigma(u)\nu\cdot\nu, \sigma(u)\nu\cdot\chi^1, \sigma(u)\nu\cdot\chi^2, p_f(u)\Big)^t, \\ S_\Gamma(u) &= \left(u^s\cdot\nu, u^s\cdot\chi^1, u^s\cdot\chi^2, u^f\cdot\nu\right)^t. \end{aligned} \tag{11}$$

Let us consider the solution of (9) with the following absorbing boundary condition

$$-\mathcal{G}_\Gamma(u(x,\omega)) = i\omega\mathcal{D}S_\Gamma(u(x,\omega)), \qquad x \in \Gamma. \tag{12}$$

The matrix $\mathcal{D}$ in (12) is positive definite. In the 2D case is given by the following relations, with the obvious extension to the 3D case: $\mathcal{D} = \mathcal{A}^{\frac{1}{2}}\mathcal{N}^{\frac{1}{2}}\mathcal{A}^{\frac{1}{2}}$, where $\mathcal{N} = \mathcal{A}^{-\frac{1}{2}}\mathcal{M}^{\frac{1}{2}}\mathcal{A}^{-\frac{1}{2}}$ and

$$\mathcal{A} = \begin{pmatrix} \rho_b & 0 & \rho_f \\ 0 & g - \frac{(\rho_f)^2}{g} & 0 \\ \rho_f & 0 & g \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} \lambda_c + 2\mu & 0 & \alpha\, K_{av} \\ 0 & \mu & 0 \\ \alpha\, K_{av} & 0 & K_{av} \end{pmatrix}.$$

3 A WEAK FORMULATION

For $X \subset \mathbb{R}^d$ with boundary ∂X, let $(\cdot,\cdot)_X$ and $\langle\cdot,\cdot\rangle_{\partial X}$ denote the complex $L^2(X)$ and $L^2(\partial X)$ inner products for scalar, vector, or matrix valued functions. Also, for $s \in \mathbb{R}$, $\|\cdot\|_{s,X}$ and $|\cdot|_{s,X}$ will denote the usual norm and seminorm for the Sobolev space $H^s(X)$. In addition, if $X = \Omega$ or $X = \Gamma$, the subscript X may be omitted such that $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ or $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\Gamma$. Also, set

$$H(\mathrm{div};\Omega) = \{v \in [L^2(\Omega)]^d : \nabla\cdot v \in L^2(\Omega)\}, \quad H^1(\mathrm{div};\Omega) = \{v \in [H^1(\Omega)]^d : \nabla\cdot v \in H^1(\Omega)\},$$

with the norms

$$\|v\|_{H(\mathrm{div};\Omega)} = \left[\|v\|_0^2 + \|\nabla\cdot v\|_0^2\right]^{1/2}; \quad \|v\|_{H^1(\mathrm{div};\Omega)} = \left[\|v\|_1^2 + \|\nabla\cdot v\|_1^2\right]^{1/2}.$$

It will be assumed that the solution of (9) with the boundary condition (12) exists and satisfies the regularity assumption

$$\|u^{(1)}\|_2 + \|u^{(2)}\|_{H^1(\mathrm{div};\Omega)} \le C(\omega)\|f\|_0, \tag{13}$$

where $f = (f^{(1)}, f^{(2)})$.

Let us introduce the space $\mathcal{V} = [H^1(\Omega)]^d \times H(\mathrm{div};\Omega)$. Then testing equation (9) with $v \in \mathcal{V}$, using integration by parts and the boundary condition (12) we conclude that the solution u of (9) and (12) satisfies the *weak form*:

$$-\omega^2(\mathcal{P}u,v) + i\omega(\mathcal{B}u,v) + \mathcal{A}(u,v) + i\omega\langle\mathcal{D}\,S_\Gamma(u), S_\Gamma(v)\rangle = (f,v), \qquad v = \left(v^{(1)}, v^{(2)}\right)^t \in \mathcal{V}, \tag{14}$$

where $\mathcal{A}(u,v)$ is the bilinear form defined as follows:

$$\mathcal{A}(u,v) = \sum_{l,m}\left(\sigma_{lm}(u), \varepsilon_{lm}(v^{(1)})\right) - \left(p_f(u), \nabla\cdot v^{(2)}\right), \; u,v \in \mathcal{V}. \tag{15}$$

Note that the bilinear form $\mathcal{A}(u,v)$ can be written in the form

$$\mathcal{A}(u,v) = (\mathbf{E}\,\tilde{\epsilon}(u), \tilde{\epsilon}(v)) = (\mathbf{E}_r\,\tilde{\epsilon}(u), \tilde{\epsilon}(v)) + i\,(\mathbf{E}_i\,\tilde{\epsilon}(u), \tilde{\epsilon}(v)), \; u,v \in \mathcal{V},$$

where $\mathbf{E} = \mathbf{E}_r + i\mathbf{E}_i$ is a complex matrix. Furthermore, we assume that the real part $\mathbf{E}_r$ is positive definite since in the elastic limit it is associated with the strain energy density. On the other hand, the imaginary part $\mathbf{E}_i$ is assumed to be positive definite because of the restriction imposed on our system by the First and Second Laws of Thermodynamics. In the 2D case the the matrix $\mathbf{E}$ and $\tilde{\epsilon}(u)$ are defined as follows, with the obvious extension to the 3D case:

$$\mathbf{E} = \begin{pmatrix} \lambda_c + 2\mu & \lambda_c & \alpha\, K_{av} & 0 \\ \lambda_c & \lambda_c + 2\mu & \alpha\, K_{av} & 0 \\ \alpha\, K_{av} & \alpha\, K_{av} & K_{av} & 0 \\ 0 & 0 & 0 & 2\mu \end{pmatrix}, \qquad \tilde{\varepsilon}(u) = \begin{pmatrix} \varepsilon_{11}(u^{(1)}) \\ \varepsilon_{22}(u^{(1)}) \\ \nabla \cdot u^{(2)} \\ \varepsilon_{12}(u^{(1)}) \end{pmatrix}.$$

Let us analyze the uniqueness of the solution of our differential model for the case of a unit square $\Omega = (0,1)^2$ in the (x_1, x_2)-plane to shorten the argument; the 3D case follows with the same argument. Then, set $f = 0$ and choose $v = u$ in (14). Taking the imaginary part in the resulting equation, we obtain

$$\omega\,(\mathcal{B}u, u) + (\mathbf{E}_i\, \tilde{\epsilon}(u), \tilde{\epsilon}(u)) + \omega\,\langle \mathcal{D}\, S_\Gamma(u), S_\Gamma(u)\rangle = 0.$$

Using that $\mathbf{E}_i$ and $\mathcal{D}$ are positive definite and $\mathcal{B}$ is nonnegative, we conclude that

$$u^{(2)} = 0, \quad u^{(1)} = 0, \quad u^{(2)} \cdot \nu = 0, \quad \Gamma. \tag{16}$$

Consider the part Γ_1 of the boundary Γ defined by $\Gamma_1 = \{x = (x_1, x_2) \in \Gamma : x_1 = 1, 0 < x_2 < 1\}$. Notice that (16) imply that

$$\frac{\partial u_1^{(1)}}{\partial x_2} = \frac{\partial u_2^{(1)}}{\partial x_2} = \quad \Gamma. \tag{17}$$

Next, thanks to (12) $\mathcal{G}_\Gamma(u) = 0$, which leads to the following relations on Γ_1

$$\sigma_{11}(u) \;=\; (\lambda_c + 2\mu)\,\frac{\partial u_1^{(1)}}{\partial x_1} + \alpha\, K_{av} \nabla \cdot u^{(2)} = 0, \tag{18}$$

$$\sigma_{12}(u) \;=\; \mu \frac{\partial u_2^{(1)}}{\partial x_1} = 0, \tag{19}$$

$$-p_f(u) \;=\; \alpha\, K_{av} \frac{\partial u_1^{(1)}}{\partial x_1} + K_{av} \nabla \cdot u^{(2)} = 0. \tag{20}$$

Now, since $\mu \neq 0$ and in any physically meaninful situation the determinant of the 2×2 linear system for $\dfrac{\partial u_1^{(1)}}{\partial x_1}$ and $\nabla \cdot u^{(2)}$ formed by equations (18) and (20) does not vanish, we conclude that

$$\frac{\partial u_2^{(1)}}{\partial x_1} = 0, \quad \frac{\partial u_1^{(1)}}{\partial x_1} = \nabla \cdot u^{(2)} = 0, \quad \Gamma_1. \tag{21}$$

The same argument applies for the validity of (17) and (21) in the rest of the boundary. Thus by the Cauchy-Kowalevsky theorem $u^{(1)} = 0$ in a neighborhood of any point on Γ where the coefficients are analytic and with the possible exception at the corners. Then the unique continuation principle[17] implies

$$u^{(1)} = 0, \quad \Omega. \tag{22}$$

Now from (16) and (22) we have uniqueness. The 3D case follows with the identical argument. We summarize the result in the following theorem.

Theorem 3.1. *Problem (9) with (12) has a unique solution for any $\omega \neq 0$.*

For the analysis that follows a similar result can be demonstrated for the adjoint problem to (9) and (12). Thus, the solution $\psi = \left(\psi^{(1)}, \psi^{(2)}\right)^t$ of

$$-\omega^2 \mathcal{P}\psi - i\omega\mathcal{B}\psi - \mathcal{L}^*(\psi) \;=\; F, \quad \Omega, \tag{23a}$$

$$\mathcal{G}^*_\Gamma(\psi) - i\omega\mathcal{D}S_\Gamma(\psi) \;=\; 0, \quad \Gamma, \tag{23b}$$

is unique and satisfies the regularity assumption

$$\|\psi^{(1)}\|_2 + \|\psi^{(2)}\|_{H^1(\mathrm{div};\Omega)} \leq C(\omega)\|F\|_0. \tag{24}$$

In (23a),

$$\mathcal{L}^*(\psi) = \left(\nabla\cdot\sigma^*(\psi), -\nabla p^*_f(\psi)\right)^t,$$

where $\sigma^*(\psi)$ and $p^*_f(\psi)$ are defined as in (1) but using the complex conjugates of the coefficients. Similarly, $\mathcal{G}^*_\Gamma(\psi)$ is defined as in (2) but using $\sigma^*(\psi)$ and $p^*_f(\psi)$ in those definitions. As before, existence for (23a)-(23b) will be assumed.

4 THE FINITE ELEMENT PROCEDURE

The numerical procedure will be defined and analyzed in two dimensions and for rectangular elements. With minor changes the arguments can be applied for the case of triangular elements and the three dimensional case. See[5] for the definition of the nonconforming spaces for triangles in the 2D case and the case of tetrahedrons or cubic elements in the 3D case.

Let $\mathcal{T}^h(\Omega)$ be a nonoverlapping partition of Ω into rectangles Q_j of diameter bounded by h such that $\overline{\Omega} = \cup_{j=1}^{J}\overline{Q}_j$. Denote by ξ_j and ξ_{jk} the midpoints of $\partial Q_j \cap \Gamma$ and $\partial Q_j \cap \partial Q_k$, respectively.

Let us denote by ν_{jk} the unit outer normal on $\partial Q_j \cap \partial Q_k$ from Q_j to Q_k and by ν_j the unit outer normal to ∂Q_j. Let χ_j and χ_{jk} be unit tangents on $\partial Q_j \cap \Gamma$ and $\partial Q_j \cap \partial Q_k$ so that $\{\nu_j, \chi_j\}$ and $\{\nu_{jk}, \chi_{jk}\}$ are orthonormal systems on $\partial Q_j \cap \Gamma$ and $\partial Q_j \cap \partial Q_k$, respectively.

To approximate each component of the solid displacement vector we employ the nonconforming finite element space as in,[5] while to approximate the fluid displacement vector we choose the vector part of the Raviart-Thomas-Nedelec space[7,8] of zero order. More specifically, set

$$\widehat{R} = [-1,1]^2, \qquad \widehat{\mathcal{N}C}(\widehat{R}) = \text{Span}\{1, \hat{x}_1, \hat{x}_2, \alpha(\hat{x}_1) - \alpha(\hat{x}_2)\}, \quad \alpha(\hat{x}_1) = \hat{x}_1^2 - \frac{5}{3}\hat{x}_1^4.$$

with the degrees of freedom being the values at the midpoint of each edge of $\widehat{R}$. Also, if $\psi^L(\hat{x}_1) = \frac{-1+\hat{x}_1}{2}$, $\psi^R(\hat{x}_1) = \frac{1+\hat{x}_1}{2}$, $\psi^B(\hat{x}_2) = \frac{-1+\hat{x}_2}{2}$, $\psi^T(\hat{x}_2) = \frac{1+\hat{x}_2}{2}$, we have that

$$\widehat{\mathcal{W}}(\widehat{R}) = \text{Span}\big\{(\psi^L(\hat{x}_1), 0)^t, (\psi^R(\hat{x}_1), 0)^t, (0, \psi^B(\hat{x}_2))^t, (0, \psi^T(\hat{x}_2))^t\big\}.$$

For each Q_j, let $F_{Q_j} : \widehat{R} \to Q_j$ be an invertible affine mapping such that $F_{Q_j}(\widehat{R}) = Q_j$, and define

$$\begin{aligned} \mathcal{N}C_j^h &= \{v = (v_1, v_2)^t : \; v_i = \widehat{v}_i \circ F_{Q_j}^{-1}, \; \widehat{v}_i \in \widehat{\mathcal{N}C}(\widehat{R}), \; i = 1,2\}, \\ \mathcal{W}_j^h &= \{w : \; w = \widehat{w} \circ F_{Q_j}^{-1}, \; \widehat{w} \in \widehat{\mathcal{W}}(\widehat{R})\}. \end{aligned}$$

Setting

$$\begin{aligned} &\mathcal{N}C^h = \{v \, : v_j = v|_{Q_j} \; \in \mathcal{N}C_j^h, \; v_j(\xi_{jk}) = v_k(\xi_{jk}) \; \forall (j,k)\}, \\ &\mathcal{W}^h = \{w \in H(\text{div};\Omega) : \; w_j = w|_{Q_j} \in \mathcal{W}_j^h\}, \end{aligned}$$

the global finite element space to approximate the solution u of (14) is defined by

$$\mathcal{V}^h = \mathcal{N}C^h \times \mathcal{W}^h.$$

In order to state the approximation properties of $\mathcal{V}^h$ let us introduce the space

$$\tilde{\Lambda}_s^h = \left\{ \tilde{\lambda}_s^h : \tilde{\lambda}_s^h|_{\partial Q_j \cap \partial Q_k} = \tilde{\lambda}_{s,jk}^h \in [P_0(\partial Q_j \cap \partial Q_k)]^2 \equiv \tilde{\Lambda}_{s,jk}^h, \quad \tilde{\lambda}_{s,jk}^h + \tilde{\lambda}_{s,kj}^h = 0 \right\},$$

where $P_0(\partial Q_j \cap \partial Q_k)$ denotes the constant functions defined on $\partial Q_j \cap \partial Q_k$. Also, define the projections $\Pi_h : [H^2(\Omega)]^2 \to \mathcal{NC}^h$ and $P_h : [H^2(\Omega)]^2 \times H^1(\text{div};\Omega) \to \tilde{\Lambda}_s^h$ by

$$(\varphi^{(1)} - \Pi_h \varphi^{(1)})(\xi) = 0, \quad \xi = \xi_{jk} \text{ or } \xi_j, \tag{25}$$

$$\langle \sigma(\psi_j)\nu - P_h(\psi_j), 1\rangle_B = 0, \quad B = \partial Q_j \cap \partial Q_k \;\text{ or }\; \partial Q_j \cap \Gamma, \tag{26}$$

for all $\varphi^{(1)} \in [H^2(\Omega)]^2$ and $\psi \in [H^2(\Omega)]^2 \times H^1(\text{div};\Omega)$. Then, standard approximation theory implies that, for all $\varphi = \left(\varphi^{(1)}, \varphi^{(2)}\right)^t \in [H^2(\Omega)]^2 \times H^1(\text{div};\Omega)$,

$$\begin{aligned} &\Bigg[\|\varphi - \Pi_h\varphi\|_0 + h\Big(\sum_j \|\varphi - \Pi_h\varphi\|_{1,Q_j}^2\Big)^{\frac{1}{2}} + h^2\Big(\sum_j \|\varphi - \Pi_h\varphi\|_{2,Q_j}^2\Big)^{\frac{1}{2}} \\ &\quad + h^{\frac{1}{2}}\Big(\sum_j |\varphi - \Pi_h\varphi|_{0,\partial Q_j}^2\Big)^{\frac{1}{2}} + h^{\frac{3}{2}}\Big(\sum_j |\sigma(\varphi_j)\nu_j - P_h\varphi_j|_{0,\partial Q_j}^2\Big)^{1/2}\Bigg] \\ &\leq Ch^2\left(\|\varphi^{(1)}\|_2 + \|\nabla\cdot\varphi^{(2)}\|_1\right). \end{aligned} \tag{27}$$

Next, let us define the projection $\mathbf{Q}_h$ associated with the displacement vector of the fluid phase as follows:

$$\mathbf{Q}_h : [H^1(\Omega)]^2 \to \mathcal{W}^h : \langle (\mathbf{Q}_h\varphi - \varphi)\cdot\nu, 1\rangle_B = 0,$$
$$B = \partial Q_j \cap \partial Q_k \text{ or } B = \partial Q_j \cap \Gamma.$$

Then, it follows from[7,8] that

$$\|\varphi - \mathbf{Q}_h\varphi\|_0 \leq Ch\|\varphi\|_1, \tag{28}$$
$$\|\varphi - \mathbf{Q}_h\varphi\|_{H(\mathrm{div};\Omega)} \leq Ch\left(\|\varphi\|_1 + \|\nabla\cdot\varphi\|_1\right). \tag{29}$$

Set

$$\mathcal{A}_h(u,v) = \sum_j \left[\sum_{l,m}\left(\sigma_{lm}(u), \varepsilon_{lm}(v^{(1)})\right)_{Q_j} - \left(p_f(u), \nabla\cdot v^{(2)}\right)_{Q_j}\right] \tag{30}$$

and

$$\Theta_h(u,v) = -\omega^2\left(\mathcal{P}u, v\right) + i\omega\left(\mathcal{B}u, v\right) + \mathcal{A}_h(u,v) + i\omega\left\langle \mathcal{D}\, S_\Gamma(u), S_\Gamma(v)\right\rangle.$$

Then the *global* finite element procedure is defined as follows: find $u^h = \left(u^{(1,h)}, u^{(2,h)}\right)^t \in \mathcal{V}^h$ such that

$$\Theta_h(u^h, v) = (f, v), \quad v = \left(v^{(1)}, v^{(2)}\right)^t \in \mathcal{V}^h. \tag{31}$$

Let us denote by $u_j^{(m,h)},\ j = 1, 2, ,$ the components of the vector $u^{(m,h)},\ m = 1, 2.$

Theorem 4.1. *Problem (31) has a unique solution for any* $\omega \neq 0$.

Proof. Set $f = 0$, choose $v = u^h$ in (31) and take the imaginary part in the resulting equation to obtain

$$\omega\left(\mathcal{B}u^h, u^h\right) + \sum_{Q_j}\left(\mathbf{E}_i\,\tilde{\epsilon}(u^h), \tilde{\epsilon}(u^h)\right)_{Q_j} + \omega\left\langle \mathcal{D}\, S_\Gamma(u^h), S_\Gamma(u^h)\right\rangle = 0. \tag{32}$$

Since each term in the left-hand side of (32) is nonnegative, in particular we have that $\left(\mathcal{B}u^h, u^h\right) = 0$, and the argument in the proof of Theorem 3.1 can be repeated to show that

$$u^{(2,h)} = 0, \quad \Omega. \tag{33}$$

To show that $u^{(1,h)} = 0$, take an element, say Q_1, among the four elements which intersect Γ at the vertices of Ω; two faces of Q_1 are contained in Γ. After a proper transformation, without loss of generality we can assume that $Q_1 = (-1,1)^2$ with the faces $\Gamma^R = \{(x_1,x_2) \in \Gamma : x_1 = 1\}$ and $\Gamma^T = \{(x_1,x_2) \in \Gamma : x_2 = 1\}$ contained in Γ. Set

$$u_1^{(1,h)} = a_1 + b_1x_1 + c_1x_2 + d_1(\alpha(x_1) - \alpha(x_2)).$$

Since the boundary term in (32) must vanish and the matrix $\mathcal{D}$ is positive definite, we conclude that $S_\Gamma(u^h) = 0$ and consequently $u^{(1,h)}(x_1, x_2)$ must vanish on $\Gamma^R \cup \Gamma^T$. In particular at the mid point of $\Gamma^R \cup \Gamma^T$ we have

$$u_1^{(1,h)}(1,0) = a_1 + b_1 - \frac{2}{3}d_1 = 0, \quad u_1^{(1,h)}(0,1) = a_1 + c_1 + \frac{2}{3}d_1 = 0, \tag{34a}$$
$$u_2^{(1,h)}(1,0) = a_2 + b_2 - \frac{2}{3}d_2 = 0, \quad u_2^{(1,h)}(0,1) = a_2 + c_2 + \frac{2}{3}d_2 = 0. \tag{34b}$$

Next, since the second term in the left-hand side of (32) is nonnegative and the matrix $\mathbf{E}_i$ is positive definite, for $(x_1, x_2) \in Q_1$ we must have

$$\varepsilon_{11}(u^{(1,h)}) = b_1 + 2d_1\left(x_1 - \frac{10}{3}x_1^3\right) = 0, \tag{35a}$$
$$\varepsilon_{22}(u^{(1,h)}) = c_2 - 2d_2\left(x_2 - \frac{10}{3}x_2^3\right) = 0, \tag{35b}$$
$$\varepsilon_{12}(u^{(1,h)}) = \frac{1}{2}\left[c_1 + b_2 - 2d_1\left(x_1 - \frac{10}{3}x_1^3\right) + 2d_2\left(x_2 - \frac{10}{3}x_2^3\right)\right] = 0. \tag{35c}$$

From (34) and (35) it follows that $u_1^{(1,h)}|_{Q_1} = u_2^{(1,h)}|_{Q_1} = 0$. Let us take an element Q_2 adjacent to Q_1 that intersects Γ and has a common face Γ_{12} with Q_1. Then $u_1^{(1,h)}$ and $u_2^{(1,h)}$ vanish at the mid points of Γ_2 and Γ_{12} and $\varepsilon_{11}(u^{(1,h)}), \varepsilon_{22}(u^{(1,h)})$ and $\varepsilon_{12}(u^{(1,h)})$ vanish identically on Q_2, so that repeating the above argument we verify that

$$u_1^{(1,h)}|_{Q_2} = u_2^{(1,h)}|_{Q_2} = 0. \tag{36}$$

Repeating the argument, one can show that (36) holds for all elements with a face contained in Γ. Next stripping out such boundary elements, take a boundary element with two faces common with the corner of stripped out domain and repeat the argument to show the validity of (36) for those elements. Then continue the process until the domain is exhausted. This completes the proof. □

5 A PRIORI ERROR ESTIMATES FOR THE GLOBAL PROCEDURE

In this section, we derive an error estimate between the solutions u and u^h defined by (14) and (31), respectively. The argument in this section is close to that given in[18] which uses a boot-strapping argument similar to[19] for nonconforming finite element methods for Helmholtz-type problems. Also, see[20] for such a boot-strapping argument for conforming finite element methods for the Helmholtz equation.

Set

$$\mathbf{Z_h} = \left(\Pi_h u^{(1)}, \mathbf{Q}_h u^{(2)}\right)^t, \quad \delta = u - u^h = \left(\delta^{(1)}, \delta^{(2)}\right), \quad \gamma = \mathbf{Z_h} u - u^h = \left(\gamma^{(1)}, \gamma^{(2)}\right).$$

Our first goal is to derive an estimate for $\|\gamma\|_0$, and for that purpose we will solve the adjoint problem (23) with γ as a source term. It is convenient to define the following broken norms and seminorms:

$$\|v\|_{s,h}^2 = \sum_j \|v\|_{s,Q_j}^2, \quad |v|_{s,h}^2 = \sum_j |v|_{s,Q_j}^2, \quad |v|_{s,h,\Gamma}^2 = \sum_j |v|_{s,\partial Q_j \cap \Gamma}^2.$$

First note that for $v = (v^{(1)}, v^{(2)})^t \in [L^2(\Omega)]^4$ such that $v^{(1)} \in [H^1(Q_j)]^2, v^{(2)} \in H(\text{div}; Q_j)$. Using integration by parts on each Q_j, we obtain

$$\begin{aligned}\Theta_h(u,v) &= \sum_j \left(-\omega^2 \mathcal{P} u + i\omega \mathcal{B} u - \mathcal{L}(u), v\right)_{Q_j} \\ &\quad + \sum_j \left\langle (\sigma(u)\nu, -p_f(u)\nu)^t, (v^{(1)}, v^{(2)})^t \right\rangle_{\partial Q_j \setminus \Gamma}. \end{aligned} \tag{37}$$

Thus from (31) and (37) we see that for $v \in \mathcal{V}^h$

$$\Theta_h(\delta, v) = \sum_j \left[\left\langle \sigma(u)\nu, v^{(1)} \right\rangle_{\partial Q_j \setminus \Gamma} - \left\langle p_f(u), v^{(2)} \cdot \nu \right\rangle_{\partial Q_j \setminus \Gamma} \right]. \tag{38}$$

Notice that the regularity assumption (13) implies that $p_f(u) \in H^{1/2}(\partial Q_j \cap \partial Q_k)$, which together with the fact that $v_j^{(2)} \cdot \nu_{jk} + v_k^{(2)} \cdot \nu_{kj} = 0$ in the sense of $H^{-1/2}(\partial Q_j \cap \partial Q_k)$, leads to

$$\sum_j \left\langle p_f(u), v^{(2)} \cdot \nu \right\rangle_{\partial Q_j \setminus \Gamma} = 0. \tag{39}$$

Hence, thanks to (39) and that $v^{(1)}$ is orthogonal to constants, (38) can be rewritten in the form

$$\Theta_h(\delta, v) = \sum_j \left\langle \sigma(u)\nu - P_h(u), v^{(1)} \right\rangle_{\partial Q_j \setminus \Gamma}, \quad v \in \mathcal{V}^h. \tag{40}$$

Let $\psi = \left(\psi^{(1)}, \psi^{(2)}\right)^t$ be the solution of the adjoint problem:

$$-\omega^2 \mathcal{P}\psi - i\omega \mathcal{B}\psi - \mathcal{L}^*(\psi) = \gamma, \quad \Omega, \tag{41a}$$

$$\mathcal{G}_\Gamma^*(\psi) - i\omega \mathcal{D} S_\Gamma(\psi) = 0, \quad \Gamma. \tag{41b}$$

According to (24), ψ satisfies the regularity assumption

$$\|\psi^{(1)}\|_2 + \|\psi^{(2)}\|_{H^1(\text{div},\Omega)} \le C(\omega) \|\gamma\|_0. \tag{42}$$

Using integration by parts on each Q_j and applying the boundary condition (41b), we get

$$\begin{aligned} -\left(\gamma, \mathcal{L}^*(\psi)\right) &= \mathcal{A}_h(\gamma, \psi) + i\omega \left\langle \mathcal{D}\, S_\Gamma(\gamma), S_\Gamma(\psi) \right\rangle \\ &\quad - \sum_j \left[\left\langle \gamma^{(1)}, \sigma^*(\psi)\nu \right\rangle_{\partial Q_j \setminus \Gamma} - \left\langle \gamma^{(2)} \cdot \nu, p_f^*(\psi) \right\rangle_{\partial Q_j \setminus \Gamma} \right]. \end{aligned} \tag{43}$$

Next, the argument used to show the validity of (39) can be applied to see that the last term in the right-hand side of (43) vanishes. Thus (43) implies that

$$\begin{aligned}\|\gamma\|_0^2 &= \big(\gamma, -\omega^2\mathcal{P}\psi - i\omega\mathcal{B}\psi - \mathcal{L}^*(\psi)\big) \\ &= \Theta_h(\gamma,\psi) - \sum_j \left\langle \gamma^{(1)}, \sigma^*(\psi)\nu \right\rangle_{\partial Q_j\setminus\Gamma}.\end{aligned} \tag{44}$$

Next, since $\sigma^*(\psi)\nu - P_h^*(\psi)$ has average value zero on $\partial Q_j \setminus \Gamma$, (here $P_h^*(\psi)$ is defined as in (26) replacing $\sigma(\psi)$ by $\sigma^*(\psi)$) we have that for any $q^{(1)} \in [P_0(Q_j)]^2$,

$$\left\langle q^{(1)}, \sigma^*(\psi)\nu - P_h^*(\psi) \right\rangle_{\partial Q_j\setminus\Gamma} = 0,$$

so that (44) can be stated in the form

$$\|\gamma\|_0^2 = \Theta_h(\gamma,\psi) - \sum_j \left\langle \gamma^{(1)} - q^{(1)}, \sigma^*(\psi)\nu - P_h^*(\psi) \right\rangle_{\partial Q_j\setminus\Gamma}. \tag{45}$$

Next use (40) to see that for $v \in \mathcal{V}^h$,

$$\begin{aligned}\Theta_h(\gamma,v) &= \Theta_h(\delta,v) - \Theta_h(u - \mathbf{Z_h}u, v) \\ &= \sum_j \left\langle \sigma(u)\nu - P_h(u), v^{(1)} \right\rangle_{\partial Q_j\setminus\Gamma} - \Theta_h(u - \mathbf{Z_h}u, v).\end{aligned} \tag{46}$$

Then use (46) in (45) to obtain

$$\begin{aligned}\|\gamma\|_0^2 &= \Theta_h(\gamma,\psi - v) - \Theta_h(u - \mathbf{Z_h}u, v) + \sum_j \left\langle \sigma(u)\nu - P_h(u), v^{(1)} \right\rangle_{\partial Q_j\setminus\Gamma} \\ &\quad - \sum_j \left\langle \gamma^{(1)} - q^{(1)}, \sigma^*(\psi)\nu - P_h^*(\psi) \right\rangle_{\partial Q_j\setminus\Gamma}.\end{aligned} \tag{47}$$

Next, since $\psi^{(1)} \in [H^2(\Omega)]^2$, (47) can be put in the equivalent form

$$\begin{aligned}\|\gamma\|_0^2 &= \Theta_h(\gamma,\psi - v) - \Theta_h(u - \mathbf{Z_h}u, v) \\ &\quad + \sum_j \left\langle \sigma(u)\nu - P_h(u), v^{(1)} - \psi^{(1)} \right\rangle_{\partial Q_j\setminus\Gamma} \\ &\quad - \sum_j \Big[\left\langle \gamma^{(1)} - q^{(1)}, \sigma^*(\psi)\nu - P_h^*(\psi) \right\rangle_{\partial Q_j\setminus\Gamma}.\end{aligned} \tag{48}$$

Let us bound each term in the right-hand side of (48). First, choose $v = \left(v^{(1)}, v^{(2)}\right)^t = \mathbf{Z_h}\psi \in \mathcal{V}^h$ such that

$$\|\psi^{(1)} - v^{(1)}\|_0 + h\|\psi^{(1)} - v^{(1)}\|_{1,h} + h^2\|v^{(1)}\|_{2,h} \le Ch^2\|\psi^{(1)}\|_2 \le Ch^2\|\gamma\|_0, \tag{49a}$$

$$\|\psi^{(2)} - v^{(2)}\|_0 \le Ch\|\psi^{(2)}\|_1 \le Ch\|\gamma\|_0, \tag{49b}$$

$$\left\|\nabla\cdot\left(\psi^{(2)} - v^{(2)}\right)\right\|_0 + h\left\|\nabla\cdot\left(\psi^{(2)} - v^{(2)}\right)\right\|_{1,h} \le Ch\|\nabla\cdot\psi^{(2)}\|_1 \le Ch\|\gamma\|_0. \tag{49c}$$

For the first term in the right-hand side of (48), using (49) we see that

$$\begin{aligned}|\Theta_h(\gamma,\psi-v)| &\le C(\omega)\Big[\|\gamma\|_0\|\psi-v\|_0+\|\gamma^{(1)}\|_{1,h}\|\psi^{(1)}-v^{(1)}\|_{1,h} \\ &\qquad +\|\nabla\cdot\gamma\|_0\|\nabla\cdot(\psi-v)\|_0+|\langle S_\Gamma(\gamma),S_\Gamma(\psi-v)\rangle|\Big] \\ &\le C(\omega)h\|\gamma\|_0\left[\|\gamma^{(1)}\|_{1,h}+\|\nabla\cdot\gamma^{(2)}\|_0+|\langle S_\Gamma(\gamma),S_\Gamma(\psi-v)\rangle|\right]. \end{aligned} \tag{50}$$

The boundary integral in the right-hand side of (50) can be bounded using (42) and the trace inequality as follows:

$$|\langle S_\Gamma(\gamma),S_\Gamma(\psi-v)\rangle| \le C\|\gamma\|_0 h^{3/2}\left[\|\gamma^{(1)}\|_{1,h}\right], \tag{51}$$

where we have used that

$$\sum_j \left\langle \left(\psi^{(2)}-\mathbf{Q}_h\psi^{(2)}\right)\cdot\nu,\gamma^{(2)}\cdot\nu\right\rangle_{\partial Q_j\setminus\Gamma}=0.$$

Hence, using (51) in (50), we get

$$|\Theta_h(\gamma,\psi-v)| \le C(\omega)h\|\gamma\|_0\left[\|\gamma^{(1)}\|_{1,h}+\|\nabla\cdot\gamma^{(2)}\|_0\right]. \tag{52}$$

By choosing $q_j^{(1)}=q^1|_{Q_j}$, to be the average value of $\gamma^{(1)}$ on Q_j and using the trace inequality, (27) and (42), the last term in (48) is bounded as follows:

$$\begin{aligned}&\left|\sum_j\left\langle\gamma^{(1)}-q^{(1)},\sigma^*(\psi)\nu-P_h^*(\psi)\right\rangle_{\partial Q_j\setminus\Gamma}\right| \\ &\le\left(\sum_j|\gamma^{(1)}-q^{(1)}|^2_{0,\partial Q_j\setminus\Gamma}\right)^{1/2}\left(\sum_j|\sigma^*(\psi)\nu-P_h^*(\psi)|^2_{0,\partial Q_j\setminus\Gamma}\right)^{1/2} \\ &\le\left(\sum_j h\|\gamma^{(1)}\|^2_{1,Q_j}\right)^{1/2} h^{1/2}\left(\|\psi^{(1)}\|_2+\|\nabla\cdot\psi^{(2)}\|_1\right) \\ &\le Ch\|\gamma\|_0\|\gamma^{(1)}\|_{1,h}. \end{aligned} \tag{53}$$

Next, using integration by parts in the $\mathcal{A}_h(u-\mathbf{Z_h}u,v)$-term and the boundary condition

(41b), the second term in the right-hand side of (48) can be written in the form

$$\begin{aligned}\Theta_h(u-\mathbf{Z_h}u,v) = &\sum_j \left(u-\mathbf{Z_h}u, -\omega^2\mathcal{P}v - i\omega\mathcal{B}v - \mathcal{L}^*(v)\right)_{Q_j}\\ &+\sum_j \langle S_\Gamma\left(u-\mathbf{Z_h}u\right), \mathcal{G}_\Gamma^*(v)\rangle_{\partial Q_j\setminus\Gamma} + \sum_j \langle S_\Gamma\left(u-\mathbf{Z_h}u\right), \mathcal{G}_\Gamma^*(v)\rangle_{\partial Q_j\cap\Gamma}\\ &+i\omega\langle \mathcal{D}S_\Gamma\left(u-\mathbf{Z_h}u\right), S_\Gamma(v)\rangle\\ = \quad &\sum_j \left(u-\mathbf{Z_h}u, -\omega^2\mathcal{P}v - i\omega\mathcal{B}v - \mathcal{L}^*(v)\right)_{Q_j}\\ &+\sum_j \langle S_\Gamma\left(u-\mathbf{Z_h}u\right), \mathcal{G}_\Gamma^*(v)-\mathcal{G}_\Gamma^*(\psi)\rangle_{\partial Q_j\cap\Gamma} + \sum_j \langle S_\Gamma\left(u-\mathbf{Z_h}u\right), \mathcal{G}_\Gamma^*(v)\rangle_{\partial Q_j\setminus\Gamma}\\ &+i\omega\langle \mathcal{D}S_\Gamma\left(u-\mathbf{Z_h}u\right), S_\Gamma(v-\psi)\rangle\\ \equiv\ & T_1+T_2+T_3+T_4. \end{aligned} \tag{54}$$

Let us bound each term in the right-hand side of (54). First, using (27), (28), and (49) we see that

$$|T_1| \le Ch\|\gamma\|_0\left(\|u^{(1)}\|_2 + \|u^{(2)}\|_1 + \|\nabla\cdot u^{(2)}\|_1\right).$$

For the T_2 term, applying the trace inequality, (27), (28), (24), and (49), one has

$$\begin{aligned}|T_2| \ \le\ &\sum_j |u^{(1)}-\Pi_h u^{(1)}|_{0,\partial Q_j\cap\Gamma}\,|(\sigma^*(v)-\sigma^*(\psi))\cdot\nu|_{0,\partial Q_j\cap\Gamma}\\ &\quad+\sum_j \left|\left(u^{(2)}-\mathbf{Q}_h u^{(2)}\right)\cdot\nu\right|_{-1/2,\partial Q_j\cap\Gamma}\left|p_f^*(v)-p_f^*(\psi)\right|_{1/2,\partial Q_j\cap\Gamma}\\ \le\ & C\|\gamma\|_0\left[h^2\left(\|u^{(1)}\|_2\right) + h\left(\|u^{(2)}\|_1 + \|\nabla\cdot u^{(2)}\|_1\right)\right].\end{aligned} \tag{55}$$

Next, we decompose T_3 as follows:

$$\begin{aligned}T_3 = &\sum_j \langle S_\Gamma\left(u-\mathbf{Z_h}u\right), \mathcal{G}_\Gamma^*(v)-\mathcal{G}_\Gamma^*(\psi)\rangle_{\partial Q_j\setminus\Gamma} + \sum_j \langle S_\Gamma\left(u-\mathbf{Z_h}u\right), \mathcal{G}_\Gamma^*(\psi)\rangle_{\partial Q_j\setminus\Gamma}\\ \equiv\ & T_{3,1}+T_{3,2}.\end{aligned} \tag{56}$$

Then, as in (55),

$$|T_{3,1}| \le C\|\gamma\|_0\left[h^2\left(\|u^{(1)}\|_2\right) + h\left(\|u^{(2)}\|_1 + \|\nabla\cdot u^{(2)}\|_1\right)\right].$$

The other term in (56) can be bounded by using again the fact that $\Pi_h u_j^{(1)} - \Pi_h u_k^{(1)}$ is orthogonal to constants

$$
\begin{aligned}
|T_{3,2}| \;\le\; & \Bigg| \sum_j \left\langle \left(u^{(1)} - \Pi_h u^{(1)}\right) \cdot \nu, \sigma^*(\psi)\nu \cdot \nu \right\rangle_{\partial Q_j \setminus \Gamma} \\
& + \left\langle \left(u^{(1)} - \Pi_h u^{(1)}\right) \cdot \chi, \sigma^*(\psi)\nu \cdot \chi \right\rangle_{\partial Q_j \setminus \Gamma} \\
& - \sum_j \left\langle \left(u^{(2)} - \mathbf{Q}_h u^{(2)}\right) \cdot \nu, p_f^*(\psi) \right\rangle_{\partial Q_j \setminus \Gamma} \Bigg| \\
\le\; & C h^2 \, \|\gamma\|_0 \|u^{(1)}\|_2,
\end{aligned}
$$

where we have used again the argument in (39) to cancel out the terms involving $u^{(2)}$ in the inequality above. Finally, in order to bound T_4, applying the trace inequality, (27), (28), and (49), we obtain

$$
\begin{aligned}
|T_4| \;\le\; & C \Bigg[\sum_j \left| u^{(1)} - \Pi_h u^{(1)} \right|_{0,\partial Q_j \cap \Gamma} \left| v^{(1)} - \psi^{(1)} \right|_{0,\partial Q_j \cap \Gamma} \\
& + \sum_j \left| \left(u^{(2)} - \mathbf{Q}_h u^{(2)}\right) \cdot \nu \right|_{0,\partial Q_j \cap \Gamma} \left| \left(v^{(2)} - \psi^{(2)}\right) \cdot \nu \right|_{0,\partial Q_j \cap \Gamma} \Bigg] \\
\le\; & \sum_j \|u^{(1)} - \Pi_h u^{(1)}\|_{0,Q_j}^{\frac{1}{2}} \|u^{(1)} - \Pi_h u^{(1)}\|_{1,Q_j}^{\frac{1}{2}} \|\psi^{(1)} - v^{(1)}\|_{0,Q_j}^{\frac{1}{2}} \|\psi^{(1)} - v^{(1)}\|_{1,Q_j}^{\frac{1}{2}} \\
& + \sum_j h^{\frac{1}{2}} |u^{(2)} \cdot \nu|_{\frac{1}{2},\partial Q_j \cap \Gamma} h^{\frac{1}{2}} |\psi^{(2)} \cdot \nu|_{\frac{1}{2},\partial Q_j \cap \Gamma} \\
\le\; & C\|\gamma\|_0 h^3 \|u^{(1)}\|_2 + Ch\|u^{(2)}\|_1 \|\psi^{(2)}\|_1 \\
\le\; & C\|\gamma\|_0 h^3 \|u^{(1)}\|_2 + Ch\|u^{(2)}\|_1.
\end{aligned}
$$

Collecting the estimates for T_1, T_2, T_3, and T_4, we conclude that

$$
|\Theta_h(u - \mathbf{Z_h} u, v)| \le C\|\gamma\|_0 \left[h^2 \|u^{(1)}\|_2 + h \left(\|u^{(2)}\|_1 + \|\nabla \cdot u^{(2)}\|_1 \right) \right]. \tag{57}
$$

Next, use the trace inequality, (27), and (52) to bound the third term in the right-hand side of (48) as follows:

$$
\begin{aligned}
& \left| \sum_j \left[\left\langle \sigma(u)\nu - P_h(u), v^{(1)} - \psi^{(1)} \right\rangle_{\partial Q_j \setminus \Gamma} \right| \\
& \le \left(\sum_j |\sigma(u)\nu - P_h(u)|_{0,\partial Q_j \setminus \Gamma}^2 \right)^{1/2} \left(\sum_j |v^{(1)} - \psi^{(1)}|_{0,\partial Q_j \setminus \Gamma}^2 \right)^{1/2} \\
& \le Ch^{1/2} \left(\|u^{(1)}\|_2 + \|\nabla \cdot u^{(2)}\|_1 \right) h^{3/2} \left(\|\psi^{(1)}\|_2 \right) \\
& \le Ch^2 \|\gamma\|_0 \left(\|u^{(1)}\|_2 + \|\nabla \cdot u^{(2)}\|_1 \right).
\end{aligned} \tag{58}
$$

Thus collecting the bounds in (52), (53), (57), and (58), we obtain

$$\|\gamma\|_0 \leq C(\omega)\left[h\left(\|\gamma^{(1)}\|_{1,h} + \|\nabla\cdot\gamma^{(2)}\|_0\right)\right.$$
$$\left. + h^2\left(\|u^{(1)}\|_2\right) + h\left(\|u^{(2)}\|_1 + \|\nabla\cdot u^{(2)}\|_1\right)\right]. \tag{59}$$

Using the triangle inequality, the last estimate (59), and the approximation properties of Π_h and $\mathbf{Q}_h$ in (27) and (28), we get

$$\begin{aligned}\|\delta\|_0 &\leq \|\gamma\|_0 + \|\mathbf{Z}_h u - u\|_0 \leq C(\omega)\left[h\left(\|\delta^{(1)}\|_{1,h} + \|\nabla\cdot\delta^{(2)}\|_0\right)\right.\\ &\quad + h\left(\|u^{(1)} - \Pi_h u^{(1)}\|_{1,h} + \|\nabla\cdot(u^{(2)} - \mathbf{Q}_h u^{(2)})\|_0\right)\\ &\quad \left. + h^2\|u^{(1)}\|_2 + h\left(\|u^{(2)}\|_1 + \|\nabla\cdot u^{(2)}\|_1\right)\right]\\ &\leq C(\omega)\left[h\left(\|\delta^{(1)}\|_{1,h} + \|\nabla\cdot\delta^{(2)}\|_0\right)\right. \qquad (60)\\ &\quad \left. + h^2\|u^{(1)}\|_2 + h\left(\|u^{(2)}\|_1 + \|\nabla\cdot u^{(2)}\|_1\right)\right].\end{aligned}$$

We next use a Gärding-type inequality to bound the δ-terms in (60) in terms of the u-terms in that inequality. First note that using Korn's second inequality[21,22] and that $\mathbf{E}_i$ is positive definite, we get

$$\begin{aligned}|\mathrm{Im}\left(\Theta_h(\delta,\delta)\right)| &= \omega\left(\mathcal{B}\delta,\delta\right) + \sum_j \left(\mathbf{E}_i\,\tilde{\epsilon}(\delta),\tilde{\epsilon}(\delta)\right)_{Q_j} + \omega\left\langle\mathcal{D}\,S_\Gamma(\delta),S_\Gamma(\delta)\right\rangle\\ &\geq C_1(\omega)\left[\|\delta^{(1)}\|_{1,h}^2 + \|\nabla\cdot\delta^{(2)}\|_0^2 + \left\langle S_\Gamma(\delta),S_\Gamma(\delta)\right\rangle\right] - C_2(\omega)\|\delta\|_0^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\|\delta^{(1)}\|_{1,h}^2 + \|\nabla\cdot\delta^{(2)}\|_0^2 + \left\langle S_\Gamma(\delta),S_\Gamma(\delta)\right\rangle\\ &\quad\leq C_3(\omega)\left|\Theta_h(\delta,\delta)\right| + C_2(\omega)\|\delta\|_0^2\\ &\quad\leq C_3(\omega)\left[\|\delta\|_0^2 + \left|\Theta_h(\delta,u-\mathbf{Z}_h u)\right| + \left|\Theta_h(\delta,\gamma)\right|\right]. \qquad (61)\end{aligned}$$

Since $\gamma \in \mathcal{V}^h$, the expression for $\Theta_h(\delta,\gamma)$ given in (40) can be replaced by using (61) so that

$$\begin{aligned}&\|\delta^{(1)}\|_{1,h}^2 + \|\nabla\cdot\delta^{(2)}\|_0^2 + \left\langle S_\Gamma(\delta),S_\Gamma(\delta)\right\rangle\\ &\quad\leq C_3(\omega)\Bigg[\|\delta\|_0^2 - \omega^2\left(\mathcal{P}\delta,u-\mathbf{Z}_h u\right) + i\omega\left(\mathcal{B}\delta,u-\mathbf{Z}_h u\right) + \mathcal{A}_h(\delta,u-\mathbf{Z}_h u) \qquad (62)\\ &\qquad + i\omega\left\langle\mathcal{D}\,S_\Gamma(\delta),S_\Gamma(u-\mathbf{Z}_h u)\right\rangle + \sum_j \left\langle\sigma(u)\nu - P_h(u),\gamma^{(1)}\right\rangle_{\partial Q_j\setminus\Gamma}\Bigg].\end{aligned}$$

Let us bound the last five terms in the right-hand side of (62). First, thanks to the approximation properties of Π_h and $\mathbf{Q}_h$ given in (27) and (28), it follows that

$$\left|-\omega^2\left(\mathcal{P}\delta,u-\mathbf{Z}_h u\right) + i\omega\left(\mathcal{B}\delta,u-\mathbf{Z}_h u\right)\right|$$
$$\leq C(\omega)\left[\|\delta\|_0^2 + h^4\|u^{(1)}\|_2^2 + h^2\|u^{(2)}\|_1^2\right]. \tag{63}$$

Again, due to (27) and (28),

$$
\begin{aligned}
|\mathcal{A}_h(\delta, u - \mathbf{Z}_h u)| &\leq C(\omega)\Big[\left(\|\delta^{(1)}\|_{1,h}\|u^{(1)} - \Pi_h u^{(1)}\|_{1,h}\right) \\
&\qquad + \|\nabla\cdot\delta^{(2)}\|_0 \|\nabla\cdot(u^{(2)} - \mathbf{Q}_h u^{(2)})\|_0 \Big] \\
&\leq \epsilon\left(\|\delta^{(1)}\|_{1,h}^2 + \|\nabla\cdot\delta^{(2)}\|_0^2\right) + C(\omega)h^2\left[\|u^{(1)}\|_2^2 + \|\nabla\cdot u^{(2)}\|_1^2\right].
\end{aligned} \tag{64}
$$

Next, using the trace inequality and approximation properties (27) and (28) again, we have

$$
\begin{aligned}
&|\omega\left\langle \mathcal{D}\, S_\Gamma(\delta), S_\Gamma(u - \mathbf{Z}_h u)\right\rangle| \\
&\quad\leq \epsilon\left\langle \mathcal{D} S_\Gamma(\delta), S_\Gamma(\delta)\right\rangle + C(\omega)\left\langle \mathcal{D} S_\Gamma(u - \mathbf{Z}_h u), S_\Gamma(u - \mathbf{Z}_h u)\right\rangle \\
&\quad\leq \epsilon\left\langle \mathcal{D} S_\Gamma(\delta), S_\Gamma(\delta)\right\rangle \\
&\qquad + C(\omega)\left[\sum_j |u^{(1)} - \Pi_h u^{(1)}|_{0,\partial Q_j\cap\Gamma}^2 + \sum_j |(u^{(2)} - \mathbf{Q}_h u^{(2)})\cdot\nu|_{0,\partial Q_j\cap\Gamma}^2\right] \\
&\quad\leq \epsilon\left\langle \mathcal{D} S_\Gamma(\delta), S_\Gamma(\delta)\right\rangle + C(\omega)\left[h^3\|u^{(1)}\|_2^2 + \sum_j h^2 |u^{(2)}\cdot\nu|_{1,\partial Q_j\cap\Gamma}^2\right] \\
&\quad\leq \epsilon\left\langle \mathcal{D} S_\Gamma(\delta), S_\Gamma(\delta)\right\rangle + C(\omega)\left[h^3\|u^{(1)}\|_2^2 + \sum_j h^2 \|u^{(2)}\|_{\frac{3}{2},Q_j}^2\right] \\
&\quad\leq \epsilon\left\langle \mathcal{D} S_\Gamma(\delta), S_\Gamma(\delta)\right\rangle + C(\omega)\left[h^3\|u^{(1)}\|_2^2 + h^2 \|u^{(2)}\|_{\frac{3}{2}}^2\right].
\end{aligned} \tag{65}
$$

Finally, owing to the orthogonality property of $\gamma^{(1)}$ to constants on $\partial Q_j \setminus \Gamma$, the trace inequality, and (27), it follows that

$$
\begin{aligned}
&\left|\sum_j \left\langle \sigma(u)\nu - P_h(u), \gamma^{(1)}\right\rangle_{\partial Q_j\setminus\Gamma}\right| = \left|\sum_j \left\langle \sigma(u)\nu - P_h(u), \gamma^{(1)} - q^{(1)}\right\rangle_{\partial Q_j\setminus\Gamma}\right| \\
&\quad\leq C\left(\sum_j |\sigma(u)\nu - P_h(u)|_{0,\partial Q_j\setminus\Gamma}^2\right)^{1/2}\left(\sum_j |\gamma^{(1)} - q^{(1)}|_{0,\partial Q_j\setminus\Gamma}^2\right)^{1/2} \\
&\quad\leq Ch^{1/2}\left(\|u^{(1)}\|_2 + \|\nabla\cdot u^{(2)}\|_1\right)\left(\sum_j h\|\gamma^{(1)}\|_{1,Q_j}\right)^{1/2} \\
&\quad\leq Ch\|\gamma^{(1)}\|_{1,h}\left(\|u^{(1)}\|_2 + \|\nabla\cdot u^{(2)}\|_1\right) \\
&\quad\leq Ch\left(\|\delta^{(1)}\|_{1,h} + \|u^{(1)} - \Pi_h u^{(1)}\|_{1,h}\right)\left(\|u^{(1)}\|_2 + \|\nabla\cdot u^{(2)}\|_1\right) \\
&\quad\leq \epsilon\|\delta^{(1)}\|_{1,h}^2 + Ch^2\left(\|u^{(1)}\|_2^2 + \|\nabla\cdot u^{(2)}\|_1^2\right).
\end{aligned} \tag{66}
$$

Hence using (63), (64), (65), and (66) in (62), we have the following estimate:

$$\begin{aligned} &\|\delta^{(1)}\|_{1,h} + \|\nabla \cdot \delta^{(2)}\|_0 + \langle S_\Gamma(\delta), S_\Gamma(\delta)\rangle^{\frac{1}{2}} \\ &\qquad \le C(\omega)\left[\|\delta\|_0 + h\left(\|u^{(1)}\|_2 + \|u^{(2)}\|_{\frac{3}{2}} + \|\nabla \cdot u^{(2)}\|_1\right)\right]. \end{aligned} \tag{67}$$

Next, use (67) in (60) to obtain

$$\|\delta\|_0 \le C(\omega)\left[h\|\delta\|_0 + h^2\left(\|u^{(1)}\|_2 + \|u^{(2)}\|_{\frac{3}{2}}\right) + h\left(\|u^{(2)}\|_1 + \|\nabla \cdot u^{(2)}\|_1\right)\right]. \tag{68}$$

Therefore, it follows from (68) that for sufficiently small $h > 0$ such that $0 < C(\omega)h < 1$,

$$\|\delta\|_0 \le C(\omega)\left[h^2\left(\|u^{(1)}\|_2 + \|u^{(2)}\|_{\frac{3}{2}}\right) + h\left(\|u^{(2)}\|_1 + \|\nabla \cdot u^{(2)}\|_1\right)\right]. \tag{69}$$

Finally using (69) in (67), we arrive at the following error estimate.

$$\begin{aligned} &\|\delta^{(1)}\|_{1,h} + \|\nabla \cdot \delta^{(2)}\|_0 + \langle S_\Gamma(\delta), S_\Gamma(\delta)\rangle \\ &\qquad \le C(\omega)h\left[\|u^{(1)}\|_2 + \|u^{(2)}\|_{\frac{3}{2}} + \|\nabla \cdot u^{(2)}\|_1\right]. \end{aligned}$$

We summarize the above in the following theorem:

Theorem 5.1. *Let $u \in \mathcal{V}$ and $u^h \in \mathcal{V}^h$ be the solutions of* (14) *and* (31)*, respectively. We then have the following energy-norm error estimate: for sufficiently small $h > 0$,*

$$\begin{aligned} &\|u^{(1)} - u^{(1,h)}\|_{1,h} + \|\nabla \cdot (u^{(2)} - u^{(2,h)})\|_0 \\ &\qquad + |u^{(1)} - u^{(1,h)}|_{0,\Gamma} + |(u^{(2)} - u^{(2,h)}) \cdot \nu|_{0,\Gamma} \\ &\qquad \le C(\omega)h\left[\|u^{(1)}\|_2 + \|u^{(2)}\|_{\frac{3}{2}} + \|\nabla \cdot u^{(2)}\|_1\right]. \end{aligned}$$

Also, we have the $[L^2(\Omega)]^6$-error estimate as follows: for sufficiently small $h > 0$,

$$\|u - u^h\|_0 \le C(\omega)\left[h^2\left(\|u^{(1)}\|_2 + \|u^{(2)}\|_{\frac{3}{2}}\right) + h\left(\|u^{(2)}\|_1 + \|\nabla \cdot u^{(2)}\|_1\right)\right].$$

REFERENCES

[1] M. A. Biot. Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low frequency range. *J. Acoust. Soc. Am.*, **28**, 168–171 (1956).

[2] M. A. Biot. Theory of propagation of elastic waves in a fluid-saturated porous solid. II. High frequency range. *J. Acoust. Soc. Am.*, **28**, 179–191 (1956).

[3] T. J. Plona. Observation of a second bulk compressional wave in a porous medium at ultrasonic frequencies. *Appl. Phys. Lett.*, **36**, 256–261 (1980).

[4] D. L. Johnson and T. J. Plona. Acoustic slow waves and the consolidation transition. *J. Acoust. Soc. Am.*, **72**, 556–565 (1982).

[5] J.Douglas Jr., J. E. Santos, D. Sheen, and X. Ye. Nonconforming Galerkin methods based on quadrilateral elements for second order elliptic problems. *RAIRO Math. Modeling and Numer. Analysis (M2AN)*, **33**, 747–770 (1999).
[6] F. I. Zyserman, P. M. Gauzellino, and J. E. Santos. Dispersion analysis of a non-conforming finite element method for the helmholtz and elastodynamic equations. *Int. J. Numer. Meth. Eng.*, **58**, 1381–1395 (2003).
[7] P. A. Raviart and J. M. Thomas. Mixed finite element method for 2^{nd} order elliptic problems. *Mathematical Aspects of the Finite Element Methods, Lecture Notes of Mathematics, vol. 606, Springer*, (1975).
[8] J. C. Nedelec. Mixed finite elements in r^3. *Numer. Math.*, **35**, 315–341 (1980).
[9] M. A. Biot. Mechanics of deformation and acoustic propagation in porous media. *J. Appl. Phys.*, **33**, 1482–1498 (1962).
[10] J. E. Santos, J. M. Corberó, C. L. Ravazzoli, and J. L. Hensley. Reflection and transmission coefficients in fluid-saturated porous media. *J. Acoust. Soc. Amer.*, **91**, 1911–1923 (1992).
[11] F. Gassmann. über die elastizität poröser medien" ("on the elasticity of porous media"). *Vierteljahrsschrift der Naturforschenden Gessellschaft in Zurich*, pages 1–23 (1951).
[12] H. P. Liu, D. L. Anderson, and H. Kanamori. Velocity dispersion due to anelasticity; implications for seismology and mantle composition. *Geophys. J. R. Astr. Soc.*, **l47**, 41–58 (1976).
[13] T. Bourbie, O. Coussy, and B. Zinszner. *Acoustics of Porous Media*. Editions Technip, Paris, (1987).
[14] J. M. Carcione. Wave fields in real media: Wave propagation in anisotropic, anelastic and porous media. *Handbook of Geophysical Exploration,Pergamon Press Inc.*, **31** (2001).
[15] J. M. Hovem and G. D. Ingram. Viscous attenuation of sound in saturated sand. *J. Acoust. Soc. Amer.*, **66**, 1807–1812 (1979).
[16] J. Bear. Dynamics of fluids in porous media. *Dover Publications, New York*, (1972).
[17] L. Nirenberg. Uniqueness in Cauchy problems for differential equations with constant leading coefficients. *Comm. Pure Appl. Math.*, **X**, 89–105 (1957).
[18] T. Ha, J. E. Santos, and D. Sheen. Nonconforming finite element methods for the simulation of waves in viscoelastic solids. *Computer Methods in Appl. Mech. and Eng.*, **191**, 5647–5670 (2002).
[19] J. Douglas Jr., J. E. Santos, and D. Sheen. Nonconforming galerkin methods for the helmholtz equation. *Numer. Methods for Partial Diff. Equations*, **17**, 475–494 (2001).
[20] J. Douglas Jr., J. E. Santos, D. Sheen, and L. Bennethum. Frequency domain treatment of one–dimensional scalar waves. *Math. Models Methods Appl. Sci.*, **3**, 171–194 (1993).
[21] G. Duvaut and J.-L. Lions. *Inequalities in Mechanics and Physics*. Springer–Verlag, Berlin Heidelberg, (1976).
[22] J. A. Nitsche. On Korn's second inequality. *RAIRO Anal. Numer.*, **15**, 237–248 (1981).