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FRACTAL TECHNICS TO MEASURE THE NUMERICAL INSTABILITY OF OPTIMIZATION METHODS

Andrés L. Granados M.
Departamento de Mecánica. Universidad Simón Bolívar
Valle de Sartenejas, Edo. Miranda, Apdo.89000, Caracas 108A, Venezuela.

RESUMEN

Todo proceso que puede ser definido en la forma de una ecuación iterativa del tipo $x=g(x)$, puede ser considerado como un sistema dinámico. La complejidad del sitema depende de cuan compleja es la función $y=g(x)$. Aún en los casos más simples, el comportamiento de tales sistemas dinámicos puede ser caóticos. En tales procesos se puede obtener un mapa coloreando los puntos iniciales con colores distintos, dependiendo del punto fijo al cual converga. Estos mapas son fractales si el sistema dinámico es caótico y su dimensión fractal puede representar una medida de cuan caótico (o inestable) es el sistema. En este contexto, se analizan varios métodos de optimización como Newton-Raphson, Método de la Secante, Cuasi-Newton y Método de Segundo Orden.

ABSTRACT

All process that can be defined in the form of an iterative algorithm of the form x=g(x), may be considered as a dynamical system. The complexity of the system depends on how complex is the function y=g(x). Even for the simplest cases, the behavior of such dynamical systems may be chaotic. In such processes it may be obtaines a map coloring the initial points with different colors, depending on the fixed point toward they converge. These maps are fractals if the system is chaotic and its fractal dimension may represent a measure of the chaotic quality (or instability) of the system. In this context , some optimization methods such as Newton-Raphson, Secant Method, Cuasi-Newton, and Second Order Methods are analized.

INTRODUCTION

In this paper we deal with iterative processes. All process that can be defined in the form of an iterative algorithm of the form

$$x^{k+1} = g(x^k) \tag{1}$$

may be considered as a dynamical system [1]. The complexity of the system depends on how complex is the function

$$y = g(x) \tag{2}$$

Even for the simplest cases the behavior of such dynamical systems may be apparently unpredictable. The sequences

$$x^0, x^1, x^2, \dots, x^s, x^{s+1}, \dots, x^\infty \tag{3}$$

in the aforementioned dynamical systems may transform itself into chaotic processes. The fixed-point methods (steepest descent, Newton-Raphson, Cuasi-Newton, second order methods, etc.), and other optimization methods are examples of these complex chaotic dynamical systems.

FRACTAL MAP

The solution of the equation

$$x = g(x) \tag{4}$$

is obtained when the iterative process find a fixed point $r=g(r)$. Suppose that the function is defined to be

$$g: \mathrm{R}^n \to \mathrm{R}^n \tag{5}$$

and every time the iterative process begin, a different initial point x^0 is selected. Imagine that every initial point

$$x^0 \in A \quad \text{with} \quad A \subseteq \mathrm{R}^n \tag{6}$$

is marked, for example with different colors, depending on the fixed point r to which the iterative process (1) converges. Thus, if the iterative process is deterministic, the subset of the space will be colored with different colors in different well-determined zones. Conversely, if the iterative process is chaotic, the mentioned subset will be drawn forming strange figures embedded in the n-dimensional space, where the colored zones are not well delimited. Such figures are known as *"Fractals"* [2]. If in these fractal figures one try to make a zoom to the boundary of a mono-colored zone, the same structure is found repeatedly within the fractal structure, up to the infinity. This property is known as *"Auto-Similarity"*.

FRACTAL DIMENSION

According to Mandelbrot [2], who coined the term *"Fractal"*, a subset A of the space is called fractal provided its Hausdorff dimension $Dh(A)$ is not an integer. Intuitively, $Dh(A)$ measures the growth of the number of sets of diameter e needed to cover A, when $e \to 0$. More precisely, if A is a subset of R^n, let $N(e)$ be the minimum number of n-dimensional balls of diameter e needed to cover A. Then, if $N(e)$ increases like $N(e)$ is proportional to e^{-Dh} when $e \to 0$., it can be said that A has a Hausdorff dimension Dh [3]. It is not difficult to show that the Hausdorff dimension can be obtained by

$$D_h = \lim_{e \to 0} \frac{\log[N(e)]}{\log(k/e)} \tag{7}$$

where k is the proportionality constant when

$$N \to k e^{-D_h} \qquad e \to 0 \tag{8}$$

For example, a monocolor rectangle with sides a and b can be covered with circles of diameter $e = a/n = b/m$, ordered in n lines by m rows, and the Hausdorff dimension is then obtained as

$$D_h = \lim_{n,m \to \infty} \frac{\log(nm)}{\log[k'\sqrt{ab}/\sqrt{(a/n)(b/m)}]} = 2 \tag{9}$$

where k' is a proportionality constant. For a square the result is trivial and it is exactly the same. For a fractal figure the result is not an integer as it was mentioned, but the used procedure may be the same.

Accordingly, the Hausdorff dimension represent a measure of how fractal is a figure embedded in R^n. Consequently, more fractal is the figure less stable is the dinamical system, and therefore less stable is the iterative process represented by that fractal map. In this paper the term *"fractal map"*, or simply *"fractal"*, will be used to name a complete fractal representation of a specific dynamical system. The term *"fractal figure"* or *"fractal region"* will be used to name the intricate configuration of colors and forms in the boundary of uniform extended color zones, which will be named "basins".

ANALYSIS OF ITERATIVE METHODS

Graphical figures are represented by this technics, for some iterative methods like Newton-Raphson, Secant Method, Cuasi-Newton and Second Order methods applied to some simple system. Then the Hausdorff dimension is numerically quantified in an approximate form. With special attention, the Second Order Method is studied exhaustively for the mentioned system, and, as a conclusion, this method is found to be the most stable.

The Problem

Let's now focus our attention to an specific problem in R^2. This problem consist of solving the following equation

$$f(z) = z^4 - 1 = 0 \tag{10}$$

which is equivalent to the next system of two equations

$$(x^2 - y^2)^2 - 4x^2y^2 - 1 = 0 \tag{11.a}$$

$$4xy(x^2 - y^2) = 0 \tag{11.b}$$

The equation (10) represent the equation of the quartic roots of 1 in the complex plane, where $z = x + \mathrm{i}\,y \in \mathbb{C}$. This equation, in the set of complex numbers, represents a system of two non-linear equations in the set of the real plane.

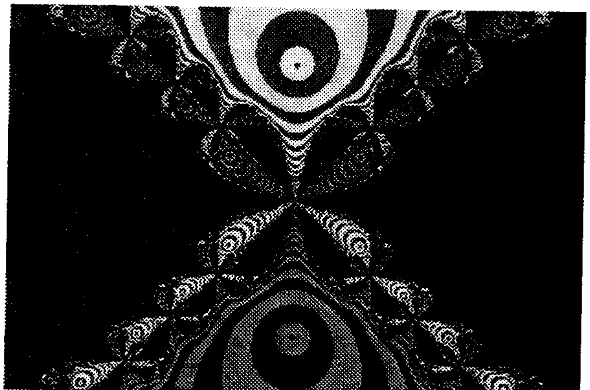

Fig. 1.a Newton-Raphson Method (K_{min}=3 , K_{med}=9.6 , K_{max}=83 , D_h=1.974688).

The Technics

The roots of the equation (10) are ±1 and ±i. The fractal figure is formed with combinations of different colors depending on which root the system converges, begining on the colored point with the initial value. Most of the fractal map were obtained with approximated 430x390 points, and the area covers the intervals from -1.5 to 1.5, both in x and y. The legend of each fractal map contains information about the minimum, averaged and maximum iteration number (K_{min},K_{med},K_{max}) and the Hausdorff fractal dimension (D_h) averaged for all the colors and for the subset of the plane shown. The Hausdorff dimensions here calculated are only aproximations due to the limited number of points used, and are numerically obtained by

$$D_h \approx 2\frac{\log[N(\text{COLOR})]}{\log[N_T(\text{COLOR})]} \tag{12}$$

where N(COLOR) is the number of points of color "COLOR" that are totally surrounded by points of the same color, and N_T (COLOR) is the total number of points of color "COLOR". Notice that the Hausdorff fractal dimension average for all the colors, named from now on simply *"fractal dimension"*, is aproximately 2, but a little smaller. If the map were a collection of colored polygons, then the fractal dimension would be exactly 2. While the fractal figure is smaller within the map, then the fractal dimension is closer to 2. While the fractal dimension is closer to 2, then the behavior of the system is more stable.

Fig. 1.b Newton-Raphson Method (Detail).

The chaotic process occurred in the systems described above is more or less stable depending on the location of the inicial point in the iterative process. When the iterative process begins in a point colored with a specific color, the consecutive iterations jump from a point of that specific color to another point with the same color, until a root in the same color basin is reached. This process is chaotic if the inicial point is located within the fractal figure and may be unstable if a jump is made outside the fractal map and falls in another point really far from the root. This process is absolutely stable if the inicial point is located in an attraction basin which contains a root.

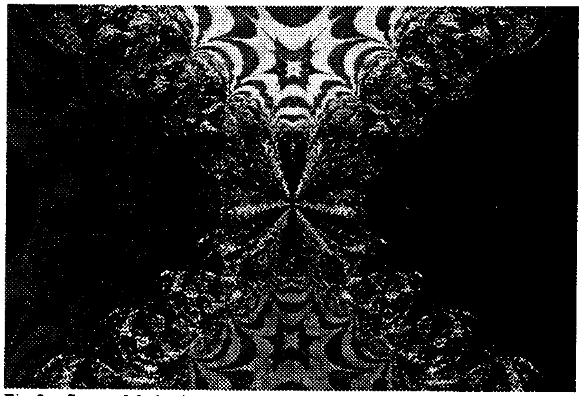

Fig.2.a Secant Method (K_{min}=3, K_{med}=12 , K_{max}=889 , D_h=1.943245).

Newton-Raphson Method

Figures 1 shows the fractal map of the Newton-Raphson method (Appeared early in [3]) for the solution of the equation (10), without any relaxation [4]. Additionally, figure 1.b shows a detail of the first quadrant of the fractal figure. There, it can be noted the recurrent property of autosimilarity, which is a special characteristic of the fractals. Notice that every colored basin surrounds a different root of the equations. In the middle of the map there is a rosette with eight petals. The intermitent shadowed and clear regions especify when the method passes from one iteration to another. The route of one iteration process may seem unpredictible when this begin in the fractal region. When the iteration process begin in a principal basin the route is directly toward the nearest root. Therefore, the stability of this method is good when the iteration process begin within a principal basin. Near the fractal region the number of iterations may increase substantially, and the method is said to be less stable. The maximum number of iteration may be 83 and the average value is almost 10. Clearly, every initial point finish in any of the mentioned roots, each one marked with a small circle in the principal basins.

Secant Method

The figure 2 shows the fractal map for the secant method. The algorithm formula for this method is similar to the Newton-Raphson method except for the calculation of Jacobian matrix, which is calculated with two consecutive iteration values in each direction [4]. Similar to the last section figure 2.b present a detail of the first quadrant of the fractal figure. Here the fractal figure is presented with diffusive regions in the boundaries of the basins, like a fractal dust with a different color for each grain. In the center, it appears a rosette with 24 petals. This mean that, in the center of the map, the root to converge in less predictible than for the Newton-Raphson method. The Secant method presents a stability similar to the Newton-Raphson method, mainly near the roots, but the diffusive region seem to reduce the fractal dimension and the stability of the method, and increase the number of iterations up to 889, which represent an approximate factor of 10. The average iteration number is lightly higher. The shadow and clear regions have cusp forms, which means that the convergence may have a lateral component, which retard the convergence. Like the Newton-Raphson method every the point converges to a root.

Fig. 2.b Secant Method (Detail).

Cuasi-Newton Method (Broyden Algorithm)

The Cuasi-Newton method here analyzed is that known as *"Broyden Algorithm"* [5]. This algorithm necessitates for starting few iterations with the Newton-Raphson method. The figures 3.a-d show the fractal maps obtained for this method using from one to ten Newton-Raphson starting iterations. The dark regions represent those initial points with no convergence. Thus, the color *black* marks the instability zones. This is one of the iterative methods which is not stable. Within the principal basins the convergence is similar to the Secant method with the shadow and clear cusped regions. Outside the principal basins the iterative process is completely unstable, except in the spots of uniform color fractal dust. These spots joint themselves while the starting iteration increase its number, until they touch the basins. In the center of the map, gradually grows a rosette with eigth petals. The cusped region and the central rosette, jointly with the fractal dust, combine the characteristics of the iterative processes for both the Newton Raphson and Secant methods. While starting iterations increase, the shadow and clear regions are bigger within the basin because the method necessitates less iteration to converge, since they approach the solution toward the roots. It is importatnt to note that after ten starting iterations (Fig.3.d) the Cuasi-Newton method continues being unstable in the small dark regions. Only in the basins is stable due to the help of the Newton-Raphson method.

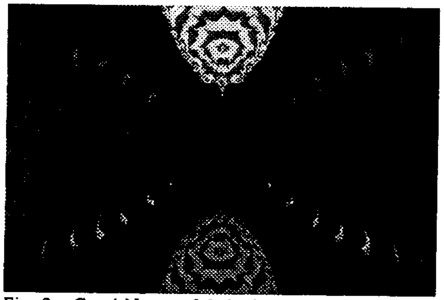

Fig. 3.a Cuasi-Newton Method, Broyden Algorithm, with one starting iteration using Newton-Raphson Method ($K_{min}=1$, $K_{med}=11.8$, $K_{max}=698626$, $D_h=1.811758$).

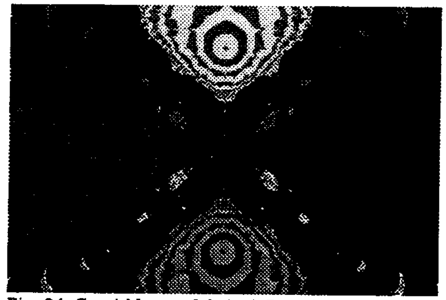

Fig. 3.b Cuasi-Newton Method, Broyden Algorithm, with 3 previous iteration using Newton-Raphson Method ($K_{min}=1$, $K_{med}=8.2$, $K_{max}=100000$, $D_h=1.909757$).

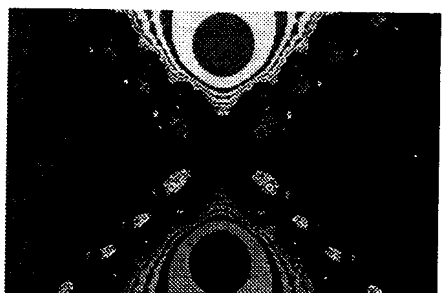

Fig. 3.c Cuasi-Newton Method, Broyden Algorithm, with 5 previous iteration using Newton-Raphson Method ($K_{min}=1$, $K_{med}=7$, $K_{max}=73527$, $D_h=1.946423$).

Second Order Method

The Second Order method is base on an expansion in Taylor Series up to the second order term. The Newton Raphson method is like, but it is developed using an expansion in Taylor Series up to the first order term. The Second Order method generates a quadratic system of equations, which may be solved using a secondary internal iteration procedure with the Newton-Raphson method. This procedure is performed in each principal iteration [6]. Figure 4 shows the influence of the number of internal iterations on the Second Order method. There are Three fractal maps with one, two and three internal iterations. The fractal dimension is a little closer to 2 when the internal iteration number increases. It should be noticed that the proximity of the fractal dimension to 2 and the stability of the systems are notable with the used procedure. This is appreciable, despite of the increment of the number of the averaged principal iterations, which is aproximately ten times higher than the standard Newton-Raphson

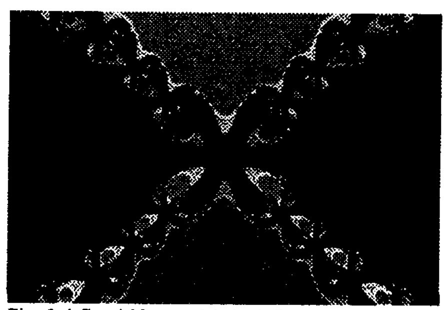

Fig. 3.d Cuasi-Newton Method, Broyden Algorithm, with 10 previous iteration using Newton-Raphson Method ($K_{min}=1$, $K_{med}=3$, $K_{max}=547$, $D_h=1.967462$).

method for the case of three internal iteration (Fig.4.c). While the iteration number is higher, the shadow and clear regions are smaller. Another feature presents in the Second Order method but not in the others is that, in both side of the fractal figure, there are small convergence hole that perturbes the convergence within the principal basins.

Fig. 4.a Second Order Method with 1 internal iteration ($K_{min}=12$, $K_{med}=24$, $K_{max}=199$, $D_h=1.984425$).

CONCLUSIONS

The conclusion of this analysis is that the Second Order methods offers more stability in the resolution of systems of non-linear equations than the classical methods, as Newton-Raphson method,and, most of the time, it is possible to find a root (or a minimum or a maximum, when used in optimization problems). However, the effort of calculation and the computational cost are greater. Even though, it is worthwhile for highly non-linear equations. Additionally, the Second Order methods increase, in average, the number of principal iterations. Nevertheless, the increment of the number of principal iteration is not so high, and this is compensated by stability reached by the algorithmic procedure. In summary, the higher stability is the main feature that make the second order methods attractive. On the other hand, the worst method is the Cuasi-Newton method because of its instability in the fractal region, and the high starting iteration necessary to be reasonably stable. In the middle there are the Newton-Raphson and the Secant methods, being the latter a little worse.

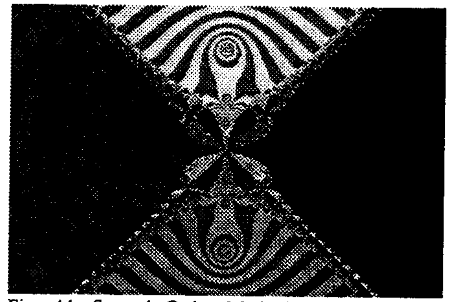

Fig. 4.b Second Order Method with 2 internal iteration ($K_{min}=25$, $K_{med}=48.6$, $K_{max}=300$, $D_h=1.988387$).

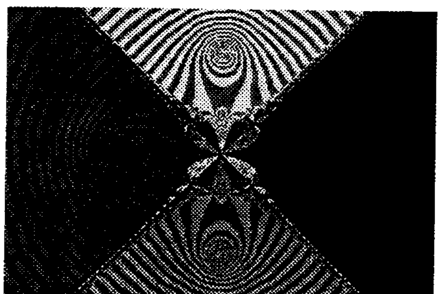

Fig. 4.c Second Order Method with 3 internal iteration ($K_{min}=47$, $K_{med}=94.3$, $K_{max}=469$, $D_h=1.990834$).

REFERENCES

[1] **Devaney, R. L.** *An Introduction to Chaotic Dynamical Systems.* Addison-Wesley, Reading, Massachusets, 1987.

[2] **Mandelbrot, B. B.,** *The Fractal Geometry of Nature,* Updated and Augmented Edition. W. H. Freeman and Company, New York, 1983.

[3] **Peitgen, H.-O.; Richter, P. H.,** *The Beauty of Fractals. Images of Complex Dynamical Systems.* Springer-Verlag, New York, 1986.

[4] **Ortega, J. M.; Rheinboldt, W. C.,** *Iterative Solution of NonLinear Equations in Several Variables*, Academic Press, New York, 1970.

[5] **Broyden, C. G.,** *A Class of Methods for Solving Non-Linear Simultaneous Equations*, Mathematics of Computation, Vol. 19, 1965, pags. 577-593.

[6] **Granados M., A. L.,** *Second Order Methods for Solving Non-Linear Equations*, INTEVEP, S. A. (Research Institute for Venezuelan Petroleum Industry), Tech. Rep. No.INT-EPPR/322-91-0002, Los Teques, Edo. Miranda, Jun. 1991, págs. 14-36.