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SENSITIVITY ANALYSIS FOR WATERHAMMER PROBLEMS IN PIPES

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ABSTRACT

The differential method was applied to the sensitivity analysis for waterhammer problems in hydraulic networks. Starting from the classical waterhammer equations in a single-phase liquid with friction (the direct problem) the state vector comprising the piezometric head and the velocity was defined.

Applying the differential method the adjoint operator, the adjoint equations with the general form of their boundary conditions, and the general form of the bilinear concomitant were calculated. The discretized adjoint equations and the corresponding boundary conditions were programmed and solved by using the so called method of characteristics.

As an example, a constant-level tank connected through a pipe to a valve discharging to atmosphere was considered. The bilinear concomitant was calculated for this particular case. The corresponding sensitivity coefficients due to the variation of different parameters by using both the differential method and the response surface generated by the computer code WHAT, solver of the direct problem, were also calculated. The results obtained with these methods show excellent agreement.

INTRODUCTION

The analysis of waterhammer transients plays an essential role in diverse areas such as hydroelectric projects, pumped-storage schemes, water supply systems, nuclear power plants, oil pipelines and industrial piping systems [1]. In nuclear power plants, various operating transients in the heat transport system can lead to significant pressure changes, which must be taken into account in the design for a safe operation [2].

Coupled to any transient calculation, there is a necessity of estimating the influence of different parameters on the obtained solution. This task, known as sensitivity analysis, can be performed by running repeatedly the computer code used in the calculations for different values of the parameters; in this way, a response surface is generated. However, this method requires time consuming calculations when many parameters are involved.

A different approach to perform the sensitivity analyses are the perturbation methods, which have been extensively used in reactor physics through the concept of the importance function [3,4,5]. The application of perturbation methods to the thermalhydraulics field has been first proposed by Oblow [6] by using the so called differential method; since then, it has been successfully extended [7]. The differential method has the following advantages:

i) The sensitivity analysis can be performed without choosing *a priori* any parameter.

ii) The calculations are faster and more efficient, since only one additional set of linear equations needs to be solved for a prescribed response.

The differential method is restricted to the linear behavior of the response surface in the vicinity of a specific design point, being this the main disadvantage.

The purpose of this paper is to outline the development of the sensitivity theory for a general waterhammer problem.

THEORY FOR A GENERAL WATERHAMMER PROBLEM

Direct equations. Consider the general one-dimensional waterhammer equations in a single-phase liquid with friction [8]:

$$\begin{aligned} m_1 &\equiv \frac{\partial H}{\partial t} + V \sin\theta + \frac{a^2}{g}\frac{\partial V}{\partial x} = 0 \\ m_2 &\equiv g\frac{\partial H}{\partial x} + \frac{\partial V}{\partial t} + \zeta\frac{V|V|}{2D} = 0 \end{aligned} \tag{1}$$

where H is the piezometric head, V the fluid velocity, a the wave propagation speed, D the pipe diameter, θ the pipe inclination angle, g the gravity acceleration, and ζ the Darcy friction factor (function of V).

The waterhammer equations can be written in a general way as

$$\vec{m}(\vec{f},\vec{p}) = \vec{0} \tag{2}$$

where $\vec{f} = [H,V]$ is the state vector and $\vec{p} = [p_1, p_2, \ldots, p_I]$ the parameter vector, which in turn depend on the generalized coordinate vector $\vec{r}$.

The corresponding boundary and initial conditions at the domain surface $\vec{r}_s$ can be written as

$$\vec{C}(\vec{f},\vec{p}) = \vec{0} \quad \text{at} \quad \vec{r} = \vec{r}_S \tag{3}$$

Consider now a response functional R given by the expression

$$R = \langle \vec{S}^{+} . \vec{f} \rangle \tag{4}$$

where $\vec{S}^+ \equiv [S_H^+, S_V^+]$ is an assigned vector weighting function, while the brackets represent integration over the whole domain Ω=[0,X]×[0,T]. In the following, we shall look for an expression for the change δR in the response functional in terms of the perturbations δp_i of the system parameters. In particular, expressions giving the sensitivity coefficients relevant to different parameters will be obtained.

Derived equations. Expanding (2) around a reference solution and considering the perturbations δp_i as independent, we obtain the general expression for the derived equations

$$\underline{\boldsymbol{H}}\,\vec{f}_{/i} = \vec{S}(i) \tag{5}$$

and boundary conditions

$$\vec{C}_{/i} + \frac{\bar{\partial}\vec{C}}{\partial\vec{f}}\,\vec{f}_{/i} = \vec{0} \quad \text{at} \quad \vec{r} = \vec{r}_S \tag{9}$$

where

$$\vec{f}_{/i} = \frac{\partial\vec{f}}{\partial p_i} = \left[H_{/i}, V_{/i}\right] \tag{6}$$

$$\vec{S}(i) = -\frac{\partial\vec{m}}{\partial p_i} = \left[S_H(i), S_V(i)\right] \tag{7}$$

$$\underline{\boldsymbol{H}} = \frac{\bar{\partial}\vec{m}}{\partial\vec{f}} \tag{8}$$

The symbol $\bar{\partial}$ denotes a Frechet derivative [9]. The final derived waterhammer equations are then

$$\begin{aligned} &\frac{\partial H_{/i}}{\partial t} + V_{/i}\sin\theta + \frac{a^2}{g}\frac{\partial V_{/i}}{\partial x} = S_H(i) \\ &g\frac{\partial H_{/i}}{\partial x} + \frac{\partial V_{/i}}{\partial t} + V_{/i}\frac{|V|}{D}\left(\zeta + \frac{V}{2}\frac{\partial\zeta}{\partial V}\right) = S_V(i) \end{aligned} \tag{10}$$

Expanding (4) to a first order, we obtain the change in the response functional (the sensitivity) as

$$\delta R = \sum_{i=1}^{I}\delta p_i\left(<\vec{S}^+_{/i}.\vec{f}> + <\vec{S}^+.\vec{f}_{/i}>\right) \tag{13}$$

Adjoint equations. Instead of solving (13) for each desired parameter to obtain the change in the response functional we can apply the concept of the adjoint function [10], defined by

$$<\vec{f}_{/i}.(\underline{\boldsymbol{H}}^*\vec{f}^*)> = <\vec{f}^*.(\underline{\boldsymbol{H}}\,\vec{f}_{/i})> + P(\vec{f}^*, \vec{f}_{/i}) \tag{14}$$

where $\vec{f}^*$ is the adjoint (importance) function, $\underline{\boldsymbol{H}}^*$ is the adjoint operator, and P the so called bilinear concomitant. The adjoint function is independent of the parameters. The linear adjoint system satisfies

$$\underline{\boldsymbol{H}}^*\vec{f}^* = \vec{S}^+ \tag{15}$$

with boundary conditions

$$\vec{C}^*(\vec{f}^*, \vec{p}) = \vec{0} \quad \text{at} \quad \vec{r} = \vec{r}_S \tag{16}$$

The particular form of the bilinear concomitant and the adjoint boundary conditions are determined for each problem, considering that:

i) The bilinear concomitant must not involve the derived functions, except when evaluated at the initial condition $t=0$.

ii) The boundary conditions for the adjoint equations must not involve the derived functions.

The change in the response functional can now be obtained as

$$\delta R = \sum_{i=1}^{L} \delta p_i \left[< \vec{S}^{+}_{/i} . \vec{f} > + < \vec{f}^{*} . \vec{S}(i) > + P \right] \tag{17}$$

where

$$P = -\int_0^X \left(H^* H_{/i} + V^* V_{/i} \right) \Big|_0^T dx - \int_0^T \left[H_{/i}\, g V^* + V_{/i} \frac{a^2}{g} H^* \right]_0^X dt \tag{19}$$

The final adjoint equations, linear in H^* and V^*, are

$$\begin{aligned} &-\frac{\partial H^*}{\partial t} - g\frac{\partial V^*}{\partial x} = S^{+}_{H} \\ &-\frac{a^2}{g}\frac{\partial H^*}{\partial x} + H^* \sin\theta - \frac{\partial V^*}{\partial t} + V^* \frac{|V|}{D}\left(\zeta + \frac{V}{2}\frac{\partial \zeta}{\partial V} \right) = S^{+}_{V} \end{aligned} \tag{20}$$

THE DISCRETE PROBLEM

Method of characteristics. It can be shown that the direct and adjoint equations are hyperbolic, so they can be transformed in ordinary differential equations along characteristic curves C^+ and C^-, respectively defined by $dx/dt=a$ and $dx/dt=-a$. For the direct equations we have [8]

$$\frac{DH}{Dt} + \frac{a}{g}\frac{DV}{Dt} + V\sin\theta + \frac{a\zeta V|V|}{2gD} = 0 \qquad ; along\ C^+ \tag{21}$$

$$\frac{DH}{Dt} - \frac{a}{g}\frac{DV}{Dt} + V\sin\theta - \frac{a\zeta V|V|}{2gD} = 0 \qquad ; along\ C^- \tag{22}$$

Correspondingly, for the adjoint equations on C^+ and C^- we have

$$\frac{DH^*}{Dt} + \frac{g}{a}\frac{DV^*}{Dt} - H^*\frac{g}{a}\sin\theta - V^*\frac{g}{a}\frac{|V|}{D}\left(\zeta + \frac{V}{2}\frac{\partial\zeta}{\partial V} \right) + S^{+}_{H} + \frac{g}{a}S^{+}_{V} = 0 \tag{23}$$

$$\frac{DH^*}{Dt} - \frac{g}{a}\frac{DV^*}{Dt} + H^*\frac{g}{a}\sin\theta + V^*\frac{g}{a}\frac{|V|}{D}\left(\zeta + \frac{V}{2}\frac{\partial\zeta}{\partial V} \right) + S^{+}_{H} - \frac{g}{a}S^{+}_{V} = 0 \tag{24}$$

Discretization. The characteristic curves are the same for the direct and adjoint equations and have a constant slope. Thus, it is possible to dis-cretize regularly the plane x-t into nodes and cycles, as in finite diffe-rences schemes, and perform the integration along the characteristics.

The direct equations have initial conditions, so the integration proceeds *forward* in time. For an inner node, the values of H and V after a time step are calculated by simultaneously solving finite difference approximations of the characteristic equations (21) and (22). For a node connected to a spatial boundary condition, it is necessary to solve simultaneously the appropriate characteristic equation and the corresponding boundary condition.

The strategy used in the integration of the adjoint equations is similar to the one detailed above. However, it will be shown later that the adjoint equations have *final* conditions; thus, the integration must proceed *backward* in time.

The direct problem was programmed in the computer code WHAT (Water Hammer Analysis in Tubes) [11]. With this code, any hydraulic network can be built by connecting different components (tanks, valves, pumps, tees, etc.) through pipes. The adjoint problem was programmed for different response functionals in the computer code ADWHAT, keeping the same philosophy.

APPLICATION EXAMPLE

Let us consider the problem of a single pipe connected at the end x=0 to a constant-level tank, while the end x=X is connected to a valve discharging to atmosphere

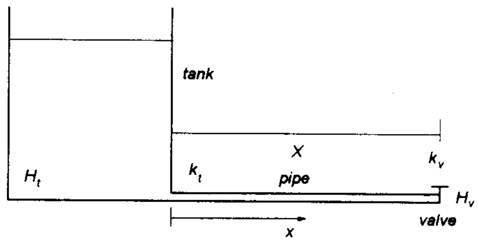

Figure 1. Hydraulic system

Direct equations. The direct boundary conditions for this case are

$$\begin{aligned}
&C_1 \equiv H + \frac{1}{2g}\left(V^2 + k_t V|V|\right) - H_t = 0 && at \quad x = 0 \\
&C_2 \equiv H - \frac{1}{2g} k_v V|V| - H_v = 0 && at \quad x = X \\
&C_3 \equiv H - \tilde{H} = 0 && at \quad t = 0 \\
&C_4 \equiv V - \tilde{V} = 0 && at \quad t = 0
\end{aligned} \tag{25}$$

were C_1 and C_2 represent the boundary conditions at the tank and at the valve, while C_3 and C_4 represent the initial conditions.

Derived equations. The derived boundary conditions result

$$\begin{aligned}
&H_{/i} + V_{/i}\frac{1}{g}\left(V + k_t|V|\right) + C_{1/i} = 0 && \text{at} \quad x = 0 \\
&H_{/i} - V_{/i}\frac{k_v}{g}|V| + C_{2/i} = 0 && \text{at} \quad x = X \\
&H_{/i} - \tilde{H}_{/i} = 0 && \text{at} \quad t = 0 \\
&V_{/i} - \tilde{V}_{/i} = 0 && \text{at} \quad t = 0
\end{aligned} \tag{28}$$

Adjoint equations and bilinear concomitant. The boundary conditions for the adjoint equations and the bilinear concomitant can be calculated by replacing (28) in (19) and considering the independence of the adjoint problem with respect to the derived functions. In this way, the boundary conditions for the adjoint equations result

$$\begin{aligned} C_1^* &\equiv \frac{a^2}{g}H^* - \left(V + k_t|V|\right)V^* = 0 && \text{at} \quad x = 0 \\ C_2^* &\equiv \frac{a^2}{g}H^* + k_v|V|V^* = 0 && \text{at} \quad x = X \\ C_3^* &\equiv H^* = 0 && \text{at} \quad t = T \\ C_4^* &\equiv V^* = 0 && \text{at} \quad t = T \end{aligned} \tag{29}$$

Note that the *initial* conditions in the direct problem have been transformed into *final* conditions in the adjoint problem.

Some response functionals. The solution of the adjoint equation involves the knowledge of the solution of the corresponding direct equations and the definition of the source term $\vec{S}^*$. The instantaneous values for the piezometric head and velocity can be chosen as response functionals by means of suitable distribution functions.

Instantaneous piezometric head: for the weighting function

$$\vec{S}^+ = \left[\delta(\bar{r} - \bar{r}_0), 0\right] \tag{31}$$

from equation (4), we get

$$R \equiv H(\bar{r}_0) = H(x_0, t_0) \tag{32}$$

Instantaneous velocity: let us consider the weighting function

$$\vec{S}^+ = \left[0, \delta(\bar{r} - \bar{r}_0)\right] \tag{33}$$

In this case, we get

$$R \equiv V(\bar{r}_0) = V(x_0, t_0) \tag{34}$$

Definition of the sensitivity problem. We consider the hydraulic system in a steady state for $t \leq 0$. For $t \geq 0$ the valve is operated in such a way that the friction coefficient changes linearly from k_{vi} to k_{vf} in a time interval τ, as shown in figure 2.

The following constants were chosen: H_f=13 m, H_v=10 m, X=30m, D =2.54 10^{-2} m, ε = 1. 10^{-4} m, k_{vi} = 0., k_{vf} = 1. 10^3, τ = 8.35 10^{-2} s, and k_t=0.5 if V>0 or k_t=1.0 if V<0. The fluid is water, with ν=1.0 10^{-6} m^2/s, a = 1437 m/s. The pipe was discretized into 11 nodes, resulting Δx = 3 m, $\Delta t = \Delta x/a$ = 2.087 10^{-3} s. We are interested in the sensitivity coefficients defined by equations (32) and (34) due to variations of different parameters.

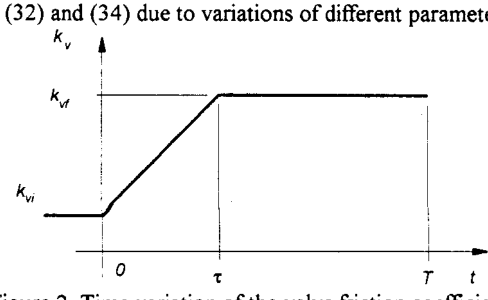

Figure 2. Time variation of the valve friction coefficient.

Some results. Once the direct problem for the given example was numerically solved with the WHAT code, six related adjoint problems were solved with de ADWHAT code. These six cases have different response functionals defined as:

a) R=$H(0,T)$, head at the tank outlet (node 1);
b) R=$H(X/2,T)$, head at the middle of the pipe (node 6);
c) R=$H(X,T)$, head at the valve inlet (node 11);
d) R=$V(0,T)$, velocity at the tank outlet (node 1);
e) R=$V(X/2,T)$, velocity at the middle of the pipe (node 6);
f) R=$V(X,T)$, velocity at the valve inlet (node 11).

As an example of the direct solution, the evolution of H and V during the transient at the three selected locations (nodes 1, 6 and 11) is shown in figures 3, 4 and 5.

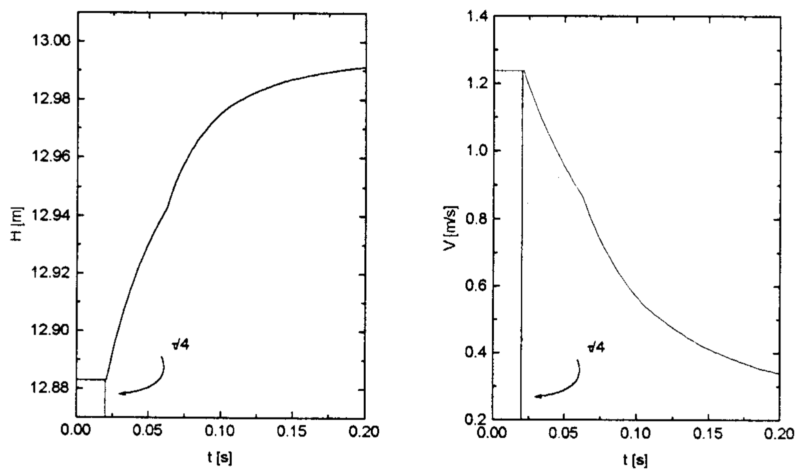

Figure 3. Evolution of H and V at the tank outlet (node 1)

For each one of the previously defined six adjoint cases, the sensitivity coefficients due to changes in the parameter k_{vf} and the closure time τ were calculated, by means of the computer code SANWHAT. Two different values of the reference time T (T=$\tau/2$ and T=20 τ) were chosen. Consequently, twenty-four different sensitivity coefficients have been evaluated. They are tabulated in Tables 1 and 2 along with the sensitivities evaluated by obtaining direct solutions with the selected parameters perturbed, using the WHAT code.

The closure time τ has been chosen equal to the period of the perturbation (4 X/a). For $T = \tau/2$ we get the maximum value of the piezometric head at the valve, while for $T = 20\ \tau$ a new steady state is achieved.

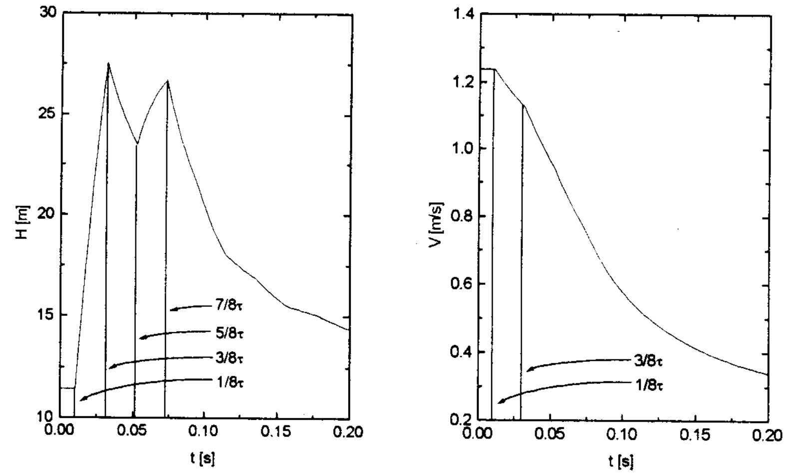

Figure 4. Evolution of H and V at the middle of the pipe (node 6)

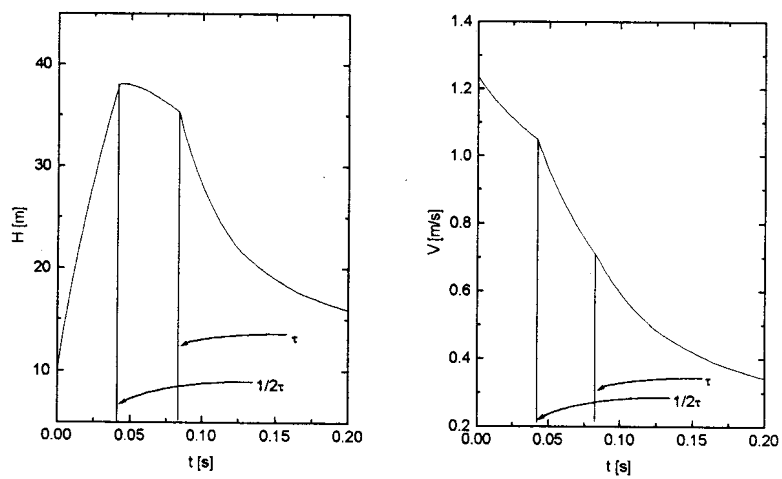

Figure 5. Evolution of H and V at the valve inlet (node 11)

Node	01		06		11	
	WHAT	*SANWHAT*	*WHAT*	*SANWHAT*	*WHAT*	*SANWHAT*
$\frac{\delta H}{\delta k_{vf}}[m]$	2.83 10^{-5}	2.83 10^{-5}	9.65 10^{-3}	9.64 10^{-3}	2.06 10^{-2}	2.06 10^{-2}
$\frac{\delta V}{\delta k_{vf}}[m/s]$	-1.81 10^{-4}	-1.81 10^{-4}	-1.71 10^{-4}	-1.71 10^{-4}	-1.38 10^{-4}	-1.38 10^{-4}
$\frac{\delta H}{\delta \tau}[m]$	-0.34	-0.33	-115.6	-115.5	-247.0	-247.1
$\frac{\delta V}{\delta \tau}[m/s]$	2.170	2.171	2.050	2.051	0.65	0.66

Table 1. Sensitivity coefficients for $T = \tau/2$

Node	01		06		11	
	WHAT	*SANWHAT*	*WHAT*	*SANWHAT*	*WHAT*	*SANWHAT*
$\frac{\delta H}{\delta k_{vf}}[m]$	4.11 10^{-6}	4.14 10^{-6}	6.15 10^{-5}	5.27 10^{-5}	1.19 10^{-4}	1.01 10^{-4}
$\frac{\delta V}{\delta k_{vf}}[m/s]$	-1.13 10^{-4}	-1.14 10^{-4}	-1.13 10^{-4}	-1.14 10^{-4}	-1.13 10^{-4}	-1.14 10^{-4}
$\frac{\delta H}{\delta \tau}[m]$	-6.0 10^{-8}	-7.1 10^{-8}	-8.2 10^{-5}	-8.4 10^{-5}	-8.2 10^{-5}	-8.7 10^{-5}
$\frac{\delta V}{\delta \tau}[m/s]$	1.8 10^{-6}	2.0 10^{-6}	1.8 10^{-6}	2.0 10^{-6}	3.3 10^{-6}	3.6 10^{-6}

Table 2. Sensitivity coefficients for $T = 20\ \tau$

From tables 1 and 2 it can be observed that the agreement between the results obtained from the codes WHAT and SANWHAT is excellent for short observation times. For the steady state, there are slight discrepancies due primarily to the treatment of the friction term in the direct equations when the method of characteristics is applied.

In figure 6, the adjoint solutions H^* and V^* for case (a) at some selected cycles is shown. This figure is only intended to show the kind of traveling delta-like waves that are typical of the adjoint waterhammer problems.

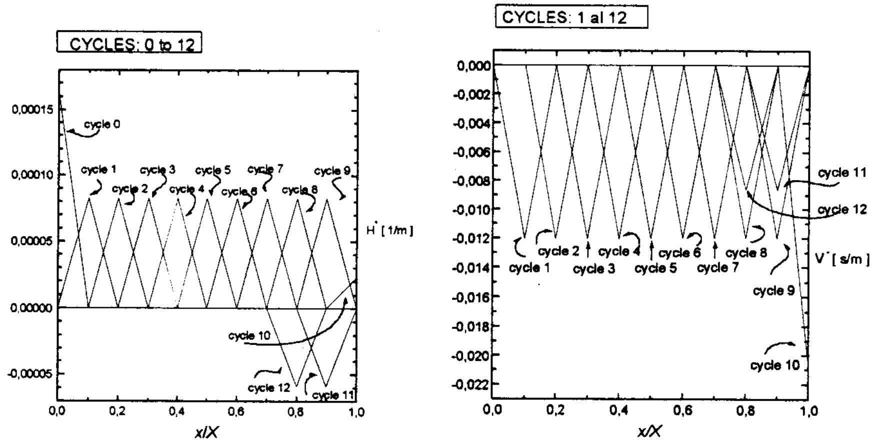

Figure 6. Adjoint solutions $H^*(x,t)$ and $V^*(x,t)$ at some selected times, corresponding to cycles 0 to 12 (cycle 0 refers to the final time T).

CONCLUSIONS

The development of the sensitivity theory by the differential method for a general waterhammer problem was outlined. The adjoint equations and the general form of the bilinear concomitant were obtained. The methodology was applied to a simple problem, showing excellent agreement between the sensitivity coefficients calculated with the differential method and the ones obtained via the solution of many perturbed direct problems.

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