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# Space-Frequency Domain Approximation of Waves in Dispersive Media

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Abstract. A finite element iterative domain decomposition algorithm is used to simulate the propagation of waves in bounded viscoacoustic media with absorbing boundary conditions at the artificial boundaries. For each frequency, the space-frequency iterative domain decomposition formulation leads to the solution of a collection of non-coercive elliptic problems with Robin type boundary conditions being employed to transmit information between subdomains. Numerical examples showing the implementation of the procedure are also presented.

**1. Introduction.** The attenuation and dispersion of waves travelling in rocks and other solid materials has been observed both in field measurements and laboratory experiments. The analysis of the attenuation and dispersion phenomena is important because yields information about rock properties such as porosity and saturation levels. The mathematical models developped to represent this phenomena are usually formulated in the space-frequency domain As examples of materials presenting this behaviour we can mention sedimentary rocks and fluid-saturated porous solids. The former is usually modeled as a linear viscoelastic material with frequency-dependent coefficients in the constitutive relations [21,15,16]. In the latter, dispersion and its related attenuation was first described by Biot [1, 2, 3, 19, 20] by using frequency dependent coefficients in the constitutive relations and in the mass and viscous coupling coefficients associated with the relative flow between fluid and solid. The inclusion of the dispersion terms in the dynamic equations leads to a non-coercive elliptic system of differential equations in the space-frequency domain which in general does not have a corresponding closed integro-differential form in the space-time domain. Even in the linear viscoelastic case, in which such integro-differential description exists, the addition of the associated absorbing boundary conditions at the artificial boundaries of the model generates non-linear terms in frequency which do not have an equivalent closed differential form in the space-time domain. Consequently, it seems to be natural to define and analyze numerical algorithms to find approximate solutions to such mathematical models in the space-frequency domain rather than in the the space-time domain. Later an approximation to the inverse Fourier transform can be used to obtain the desired space-time solution. This approach has been used by the author and some of his colleagues to treat wave propagation phenomena in viscoacoustic and viscoelastic bounded media; see [10,9, 17,7,11]. In this work we will consider a model problem describing the propagation of pressure waves in a two-dimensional bounded dispersive medium. For a given frequency, an iterative finite element domain decomposition procedure is used to solve approximately the equations of motion in the space-frequency domain. Once the solution is known in the range of frequencies of interest, (defined in terms of the source

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amplitude spectrum), an approximation to the inverse Fourier transform was employed to recover the space-time solution. For some other domain decomposition procedures defined to solve elliptic partial differential equations we refer to [5,6,8,12,13,14].

The organization of this paper is as follows. In §2 we present the differential model and state an existence and uniqueness result. Also we present an equivalent hybrid domain decomposition formulation of the original problem and the associated space discretization using a finite element procedure. Finally in §3 we show the results of experimental calculations.

**2. The Differential Model and the Iterative Domain Decomposition Formulation.** Let $\Omega = (0,1)^2$ and $\Gamma = \partial\Omega$. We will consider the following problem. Find $\widehat{u}(x,\omega)$ such that

$$(2.1)\qquad \begin{aligned} &\text{i)}\quad A(x,\omega)\widehat{u}(x,\omega) - \nabla\cdot\left(\frac{1}{\rho}(x)\nabla\widehat{u}(x,\omega)\right) = \widehat{f}(x,\omega), && x\in\Omega,\quad \omega\in R,\\ &\text{ii)}\quad \frac{\partial\widehat{u}(x,\omega)}{\partial\nu} + i\omega\alpha(x,\omega)\widehat{u}(x,\omega) = 0, && x\in\Gamma,\quad \omega\in R. \end{aligned}$$

In (2.1), $\widehat{u}(x,\omega)$ represents the Fourier transform of the pressure $u(x,t)$, $\rho(x)$ is the density, and $A(x,\omega) = -\omega^2/K(x,\omega)$ where

$$K(x,\omega) = K_r(x,\omega) + iK_i(x,\omega) = \frac{KR(x)}{\beta(x,\omega) - i\gamma(x,\omega)}$$

is the complex bulk modulus of the viscoacoustic material. The real and imaginary parts of $K(x,\omega)$ are related to the quality factor $Q(x,\omega)$ by the equation

$$(2.2)\qquad \frac{K_i}{K_r} = \frac{1}{Q(x,\omega)} = \frac{\gamma(x,\omega)}{\beta(x,\omega)},$$

where the coefficients $\beta(x,\omega)$ and $\gamma(x,\omega)$, which are associated with a continuous distribution of relaxation times, are given by (see [15,16])

$$(2.3)\qquad \beta(x,\omega) = 1 - \frac{\widetilde{C}(x)}{2}\ln\frac{1+\omega^2\tau_1^2}{1+\omega^2\tau_2^2}, \qquad \gamma(x,\omega) = \widetilde{C}(x)\tan^{-1}\frac{\omega(\tau_1-\tau_2)}{1+\omega^2\tau_1\tau_2};$$

$\tau_1$ and $\tau_2$ are given angular frequencies such that the quality factor $Q(x,\omega)$ is approximately equal to a constant $Q_m(x)$ in the range $\tau_1^{-1} \le \omega \le \tau_2^{-1}$. The constant $\widetilde{C}(x)$ satisfies the relation

$$(2.4)\qquad \widetilde{C}(x) = 2/(\pi Q_m(x)).$$

Equation (2.1.ii) is a first order absorbing boundary condition. Its derivation can be found in [16]. The complex coefficient $\alpha(x,\omega)$ in (2.1.ii) can be written as

$$\alpha(x,\omega) = M(x,\omega) - iN(x,\omega),$$

with $M(x,\omega)$ and $N(x,\omega)$ being given by

$$M(x,\omega) = C_r(2(C_r^4 + C_i^4))^{-1/2}\left[1 + (1 + (C_i/C_r)^4)^{1/2}\right]^{1/2},$$
$$N(x,\omega) = \frac{C_i^2}{C_r}(2(C_r^4 + C_i^4))^{-1/2}\left[1 + (1 + (C_i/C_r)^4)^{1/2}\right]^{-1/2},$$

and

$$C_r^2(x,\omega) = K_r(x,\omega)/\rho(x), \qquad C_i^2(x,\omega) = K_i(x,\omega)/\rho(x).$$

The argument given in [9] can be employed to show that there exists a unique solution $\widehat{u}(x,\omega)$ of (2.1).

Next, we formulate a domain decomposition procedure to find the solution $\widehat{u}(x,\omega)$ of (2.1). Let $\tau^{N_{x_1},N_{x_2}}$ be a nonoverlapping partition of $\Omega$ into rectangles $\Omega_{jk}$, $j = 1,\dots,N_{x_1}$, $k = 1,\dots,N_{x_2}$, and set

$$\partial\Omega_{jk} = \bigcup_{s=L,R,T,B} \Gamma_{jk}^s,$$

where $\Gamma_{jk}^L, \Gamma_{jk}^R, \Gamma_{jk}^B$ and $\Gamma_{jk}^T$ denote the left, right, top and bottom boundaries of the rectangle $\Omega_{jk}$, respectively.

Let us introduce the Lagrange multipliers

$$\lambda_{jk}^s = -\frac{1}{\rho}\frac{\partial u_{jk}}{\partial \nu_{jk}} \text{ on } \Gamma_{jk}^s, \quad s = L,R,T,B, \tag{2.5}$$

for all interior boundaries $\Gamma_{jk}^s$ such that $\Gamma_{jk}^s \cap \Gamma = \phi$. Then, a hybrid domain decomposition formulation of (2.1) can be stated as follows: for all pairs $\{j,k\}$, find $(\widehat{u}_{jk}, \lambda_{jk}) \in H^1(\Omega_{jk}) \times H^{-1/2}(\partial\Omega_{jk})$ such that

$$(A\widehat{u}_{jk},\varphi)_{\Omega_{jk}} + \left(\frac{1}{\rho}\nabla\widehat{u}_{jk},\nabla\varphi\right)_{\Omega_{jk}} + \langle i\omega\frac{\alpha}{\rho}\widehat{u}_{jk},\varphi\rangle_\Gamma + \sum_{s=L,R,T,B}\langle\lambda_{jk}^s,\varphi\rangle_{\Gamma_{jk}^s} = (f,\varphi)_{\Omega_{jk}}, \quad \varphi \in H^1(\Omega_{jk}), \tag{2.6}$$

$$-\lambda_{jk}^s + i\beta_{jk}^s\widehat{u}_{jk} = \lambda_{j^*k^*}^{s^*} + i\beta_{jk}^s\widehat{u}_{j^*k^*}, \text{ on } \Gamma_{jk}^s, \quad s = L,R,T,B, \tag{2.7}$$

where

$$s^* = R, \ \text{for } s = L, \qquad s^* = L, \ \text{for } s = R, \qquad s^* = T, \ \text{for } s = B, \qquad s^* = B, \ \text{for } s = T. \tag{2.8}$$

and

$$\text{(2.9)}\qquad \begin{array}{ll} \{j^*,k^*\}=\{j-1,k\} \ \text{ on } \Gamma^L_{jk}, & \{j^*,k^*\}=\{j+1,k\} \ \text{ on } \Gamma^R_{jk}, \\ \{j^*,k^*\}=\{j,k-1\} \ \text{ on } \Gamma^B_{jk}, & \{j^*,k^*\}=\{j,k+1\} \ \text{ on } \Gamma^T_{jk}. \end{array}$$

Equation (2.7) is a Robin boundary condition, with $\beta^s_{jk}=\beta^{s^*}_{j^*k^*}$ being a complex constant such that $\mathrm{Re}(\beta^s_{jk})>0$ to insure existence and uniqueness of the solution of the local problems in each $\Omega_{jk}$ [9, 16].

It can be shown is easy that (2.6)–(2.9) is equivalent to (2.1). Also, (2.6)–(2.9) leads naturally to the following iterative finite element procedure. To simplify, we treat only the case in which the partition of $\Omega$ defining the finite element space coincides with $\tau^{N_{x_1},N_{x_2}}$. Let

$$V^h=\{\varphi\in C^0(\overline{\Omega}):\varphi|_{\Omega_{jk}}\in P_{1,1}(\Omega_{jk}),\quad 1\le j\le N_{x_1},\quad 1\le k\le N_{x_2}\},$$

where $P_{1,1}(\Omega_{jk})$ denotes bilinear functions on $\Omega_{jk}$, and let

$$\text{(2.10)}\qquad V^h_{jk}=V^h|_{\Omega_{jk}},\qquad W^h_{jk}=\sum_{s=L,R,T,B}P_1(\Gamma^s_{jk}),$$

with $P_1(\Gamma^s_{jk})$ denoting linear functions on $\Gamma^s_{jk}$.

The iterative finite element domain decomposition procedure is defined as a follows. Choose $\left(u^{h,0}_{jk},(\lambda^{h,s,0}_{jk})_{s=L,R,T,B}\right)\in V^h_{jk}\times W^h_{jk}$ arbitrarily. Then, compute $\left(\widehat{u}^{h,n+1}_{jk},(\lambda^{h,s,n+1}_{jk})_{s=L,R,T,B}\right.$ . $V^h_{jk}\times W^h_{jk}$ as the solution of the equations

$$\text{(2.11)}\qquad \begin{aligned} &(A\widehat{u}^{h,n+1}_{jk},\varphi)_{\Omega_{jk}}+\left(\frac{1}{\rho}\nabla\widehat{u}^{h,n+1}_{jk},\nabla\varphi\right)_{\Omega_{jk}}+\left\langle i\omega\frac{\alpha}{\rho}\widehat{u}^{h,n+1}_{jk},\varphi\right\rangle_{\Gamma} \\ &\quad+\sum_{s=L,R,T,B}\langle\lambda^{h,s,n+1}_{jk},\varphi\rangle_{\Gamma^s_{jk}}=(f,\varphi)_{\Omega_{jk}},\quad \varphi\in V^h_{jk}, \end{aligned}$$

$$\text{(2.12)}\qquad \lambda^{h,s,n+1}_{jk}=i\beta^s_{jk}\widehat{u}^{h,n+1}_{jk}-[\lambda^{h,s^*,n}_{j^*k^*}+i\beta^s_{jk}\widehat{u}^{h,n}_{j^*k^*}],\quad \text{on } \Gamma^s_{jk},\quad s=L,R,T,B.$$

**3. Numerical Examples.** The domain $\Omega$ was taken to be a square of side length 1 m with the partition $\tau^{N_{x_1},N_{x_2}}$ consisting of squares of side length $h=1/N_{x_1}$. The source function $\widehat{f}(x,\omega)$ was the Fourier transform of the function [18]

$$\text{(3.1)}\qquad f(x,t)=-2\xi(t-t_0)e^{-\xi(t-t_0)^2}\delta(x_1-x_{1s})\delta(x_2-x_{2s}),\quad t\ge 0,$$

with $\xi=8f_0^2$, $t_0=.8/f_0$, $f_0$ being the main source frequency; $\widehat{f}(x,\omega)$ was filtered linearly between $\omega_*=40$ kHz and $\omega^*=50$ kHz. The iterative domain decomposition procedure

(2.11)–(2.12) was employed to compute the appzoximate solution $\widehat{u}^h_{jk}$ at a finite number of frequencies between zero and $\omega^*$, and the time domain solution was obtained using an approximation to the inverse Fourier transform. The constants $Q_m$, $\tau_1$, and $\tau_2$ in (2.3)–(2.4) were chosen to be 100, .1591 $10^6$ msec, and $10^3$ msec, respectively, so that $Q(\omega) \approx Q_m$ in the range $[f_1, f_2] = [10^{-6}$ kHz, $10^3$ kHz$]$.

Figure 1 shows a snapshot of the real part of $\widehat{u}^h_{jk}(x, \omega)$ at 10 kHz for a homogeneous medium with $N_{x_1} = N_{x_2} = 101$ and with the point source located at the center of $\Omega$. The medium was chosen to have density $\rho = 1$ gr/cm$^3$ and relaxed bulk modulus $KR = 10^{10}$ dynes/cm$^2$ [4], so that the reference wave speed at zero frequency is $v_e(x) = \sqrt{KR/\rho} = 1$ km/s. The solution shown in Figure 1 was compared with another solution to a standard finite element discretization of (2.1) computed as described in [9]. A perfect match between the two solutions was observed.

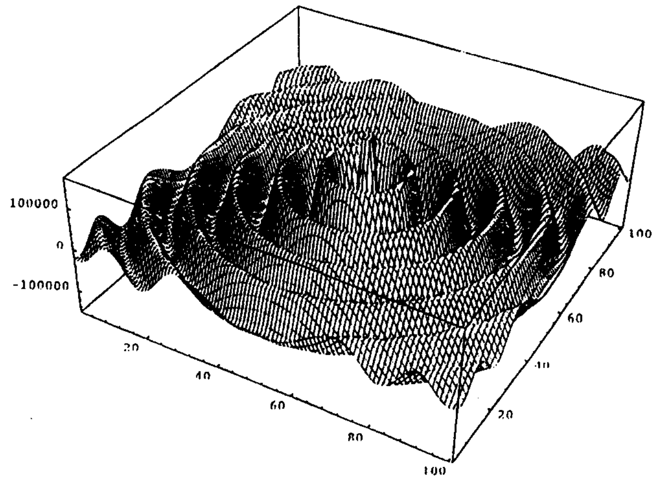


Figure 1. Real part of $\widehat{u}^h_{jk}$ at 10 kHz. Homogeneous medium

The next two experiments were performed to test the behavior of (2.11)-(2.12) for inhomogeneous media. First, we chose a layered model consisting of a layer $\{(x_1, x_2) : .4 \le x_2 \le .8\}$ having a reference velocity $v_e = 2$ km/s and with the reference velocity $v_e$ being 1 km/s in the remainder of $\Omega$. The source was located at $(x_{1_s}, x_{2_s}) = (.5, .01)$; $N_{x_1} = N_{x_2} = 101$. Figure 2 shows a snapshot of the solution at the frequency 10 kHz. The change in character of the solution across the change in the material properties is clearly observable.

Finally, Figures 3-4 show time domain snapshots of the solution at .4 and .6 msec for a corner model consisting of a region $\{(x_1, x_2) : .66 \le x_i \le 1\}$ with a reference velocity $v_e = 4$ km/s. In the remainder of $\Omega$, the reference velocity $v_e$ was 2 km/s. The source, with a main frequency $f_0$ of 20 kHz, was located at $x_{1_s} = x_{2_s} = .1$; $N_{x_1} = N_{x_2} = 120$.

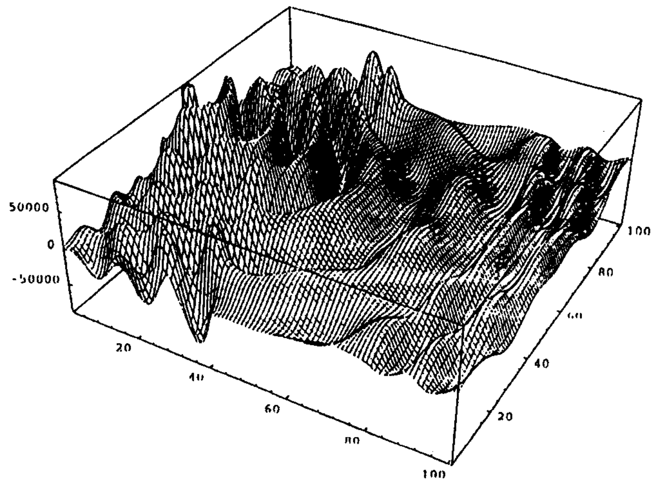


Figure 2. Real part of $\widehat{u}^h_{jk}$ at 10 kHz. Layered model

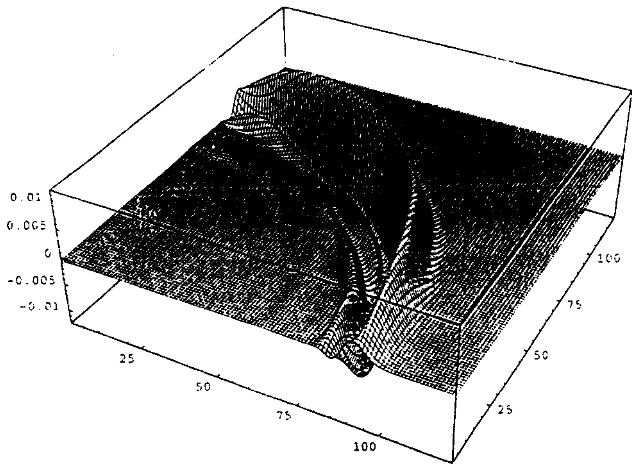


Figure 3. Snapshot of the space-time solution at .4 msec

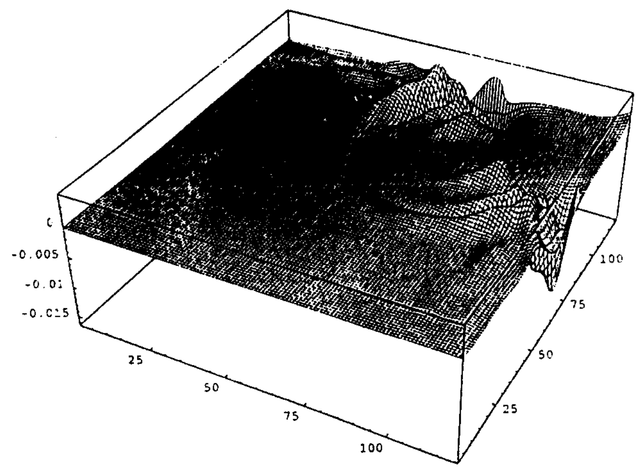


Figure 4. Snapshot of the space-time solution at .6 msec

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