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CONTACT SHAPE OPTIMIZATION

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ABSTRACT

A finite element approach for shape optimization in 2D frictionless contact problem for two different cost functions is presented in this work. The goal is to find an appropriate shape for the contact boundary, performing an almost constant contact-stress distribution. The whole formulation, including mathematical model for the unilateral problem, sensitivity analysis and geometry definition is treated in a continuous form, independently of the discretization in finite elements. Shape optimization is performed by direct modification of geometry through B-Spline curves and an automatic mesh generator is used at each new configuration to provide the finite element input data. Augmented-Lagrangian techniques (to solve the contact problem) and an interior-point mathematical-programming algorithm (for shape optimization) are used to obtain numerical results.

1. INTRODUCTION

Mechanical problems are, in general, strongly dependent on the domain $\Omega \subset R^n$ in which they are defined. This dependence is the cause of one of the main research branchs in mechanics: shape optimization and shape sensitivity analysis.

Due to its relevance, optimal shape design has received special attention in the last years, when real applications were possible due to development of powerful mathematical programming algorithms, efficient numerical methods (such as Finite Elements) and their integration with solid modeling, visualization of engineering data, automatic mesh generation and adaptivity.

Following above considerations, our aim in this paper is to present a finite element approach for an optimal shape design for the Signorini problem using two different cost functions: ***Total Potential Energy*** (Benedict and Taylor (1981)) and ***Reciprocal Energy*** (Haslinger and Klarbring (1993)).

The presentation is organized as follows: In section 2 the contact problem is introduced. The optimal shape design problem is formulated in section 3 where the shape of the contact boundary, defined mathematically by B-Splines curves, is taken as design variable. Sensitivity analysis is performed in terms of distributed parameters, approach also known as "speed method" (Zolésio (1981)) (section 4). In order to solve this mathematical model (section 5), standard finite elements are used to approximate the contact problem and sensitivity expressions. The finite element data for contact and optimization solvers are equiped by an automatic mesh generator (Fancello et al. (1991)) (section 6). Finally, the behavior of the contact stress distribution associated to each cost function is presented in section 7.

2. CONTACT PROBLEM

Consider a bounded region Ω in $\mathbf{R}^2$ with the boundary $\Gamma = \Gamma_c \cup \Gamma_f \cup \Gamma_u$ occupied by an elastic homogeneous body $\mathcal{B}$ submitted to surface tractions f over Γ_f and body forces b over Ω. Displacements u take a prescribed value in Γ_u (equal to zero for simplicity) and the unilateral contact between $\mathcal{B}$ and a fixed surface takes place over Γ_c (Γ_a is the boundary of a rigid foundation $\mathcal{F}$).

Let $\mathbf{T}$ be the stress-tensor field, obtained by the derivative of a function W with respect to the symmetric gradient of displacements:

$$\mathbf{T}(u) = \frac{\partial W}{\partial (\nabla u)^s}, \quad W(u) = \frac{1}{2} C(\nabla u)^s \cdot (\nabla u)^s, \tag{1}$$

where C is the fourth-order elastic tensor satisfying the usual assumptions of symmetry and strong ellipticity. Given a local orthonormal system (τ, n) at each point $x \in \Gamma_c$ (tangential and outward unit normal vectors respectively), we call $\sigma_n = \mathbf{T}n \cdot n$ the normal component of the reaction force defined on Γ_c. Let s, $s(x) \geq 0$ for any $x \in \Gamma_c$, be the initial gap between Γ_c and Γ_a in the normal direction n.

We define

$$\mathcal{V}(\Omega) = \{v \in (H^1(\Omega))^2 \ : v = 0 \text{ on } \Gamma_u\},$$

$$K(\Omega) = \{v \in \mathcal{V}(\Omega) : g(v) \equiv v \cdot n - s \leq 0 \text{ on } \Gamma_c\},$$

where $K(\Omega)$ is the convex set of admissible displacements, i.e. compatible with the kinematical constraints over Γ_c and Γ_u.
Given the bilinear form $a_\Omega(\cdot,\cdot)$ and the linear form $l_\Omega(\cdot)$,

$$a_\Omega(u,v) = \int_\Omega \mathrm{T}(u) \cdot \nabla v^s \, d\Omega, \qquad l_\Omega(v) = \int_\Omega b \cdot v \, d\Omega + \int_{\Gamma_f} f \cdot v \, d\Gamma, \tag{2}$$

the solution of the Signorini problem without friction is given by the following minimization problem:
Find $u \in K(\Omega)$ such that

$$u = \arg \inf_{v \in K(\Omega)} J_{(\Omega)}(v), \qquad J_{(\Omega)}(v) = \frac{1}{2} a_\Omega(v,v) - l_\Omega(v), \tag{3}$$

or, equivalently,

$$u = \arg \inf_{v \in \mathcal{V}(\Omega)} \mathcal{L}_{(\Omega)}(v), \qquad \mathcal{L}_{(\Omega)}(v) = J_{(\Omega)}(v) + I_{K(\Omega)}(v), \tag{4}$$

where $I_{K(\Omega)}$ is the indicator function of the convex set $K(\Omega)$.
Conditions for existence and uniqueness of solution of this abstract problem are thoroughly analyzed in Panagiotopoulos (1985), Kikuchi and Oden (1988), Hlaváček et al. (1982).

3. OPTIMAL SHAPE DESIGN

In shape optimization problems, the goal is to find an element Ω belonging to an *admissible set* $\mathcal{O}$ such that minimizes (locally) a cost function $\psi(\Omega) = \mathcal{J}(\Omega, u(\Omega))$, with $u(\Omega)$ being the solution of a state problem (in our case, the solution of the frictionless contact problem (3) defined in domain Ω).
We look for a uniform pressure distribution along the contact boundary, taking the boundary shape as a design variable. This problem was thoroughly analyzed in Haslinger and Neittaanmäki (1988) where it is shown that finding an appropriate cost function is not an easy task. One of the difficulties is the lack of differentiability of the state problem (3) with respect to changes of the shape. Although in most linear problems the mapping $\Omega_t \to u_t$ is smooth, in contact problems is, generally, only Lipschitz continuous (Sokolowski (1987)), and the material derivative $\dot{u}$ can be computed as a directional derivative. In order to overcome this inconvenience we use two cost functions where no differentiation of the state relation is necessary to calculate their sensitivity. The first cost function used in this paper (proposed originally by Benedict and Taylor (1981)) is the *Total Potential Energy* evaluated at the equilibrium state:

$$\psi_1(\Omega, u(\Omega)) = \int_\Omega (W(u(\Omega)) - b \cdot u(\Omega)) \, d\Omega - \int_{\Gamma_f} f \cdot u(\Omega) \, d\Gamma \,. \tag{5}$$

In Haslinger (1991), for a scalar Signorini-Dirichlet problem and in Klarbring and Haslinger (1993), for the Signorini contact problem, it was proved that the minimization of this functional in certain admissible set $\mathcal{O}$ and under some assumptions leads to a uniform distribution of the flux on the contact boundary.
The second one, proposed and analyzed in Haslinger and Klarbring (1993), is given by:

$$\psi_2(\Omega, u(\Omega)) = \frac{1}{2} \int_\Omega \mathrm{T}(z(\Omega)) \cdot (\nabla z(\Omega))^s \, d\Omega \,, \tag{6}$$

where $z(\Omega) \in \mathcal{V}_M(\Omega)$ is the solution of the following variational problem:

$$a_\Omega(z(\Omega), v) = \int_\Gamma (\mathrm{T}(u(\Omega))n - z_d) \cdot v \, d\Gamma \quad \forall v \in \mathcal{V}_M(\Omega) \,, \tag{7}$$

$$\mathcal{V}_M(\Omega) = \{v \in (H^1(\Omega))^2 : v = 0 \in \Gamma/\overline{M}\}$$

with $M \subset \Gamma$ being an open subset of Γ such that $\Gamma_c \cap M \neq \emptyset$ and $\Gamma_u \cap M = \emptyset$.
We use a particular case of this functional, namely

$$\begin{aligned} &z_d = f \text{ on } \Gamma_f \text{ and } z_d = 0 \text{ on } \Gamma_c \\ &\Gamma/\overline{M} = \Gamma_u, \text{ i.e., } M = \Gamma_c \cup \Gamma_f \Rightarrow \mathcal{V}_M(\Omega) = \mathcal{V}(\Omega)\,. \end{aligned} \tag{8}$$

Under these hypotheses, $z(\Omega) \in \mathcal{V}(\Omega)$ is now the solution of the variational problem

$$a_\Omega(z(\Omega), v) = \int_{\Gamma_c} \sigma_n(u(\Omega))(v \cdot n)\, d\Gamma \quad \forall v \in \mathcal{V}(\Omega)\,, \tag{9}$$

and cost function ψ_2 represents the *Reciprocal Energy* as shown in Haslinger and Klarbring (1993).
During the shape optimization procedure, changes of shape are performed by variations of the contact boundary Γ_c (keeping Γ_a fixed) through a finite number of control parameters (design variables) χ_i. This implies that the initial gap s in the definition of $K(\Omega)$, depends on the control parameters χ_i, i.e. $s = s(\chi_i)$. Let $\chi \in R^m$ be the vector of design variables χ_i. Thus, an optimal shape design for contact problems can be stated as follows:
Determine $\chi \in R^m$ such that

$$\begin{aligned} &\Omega = \Omega(\chi) \in \mathcal{O} \\ &\psi_i(\Omega(\chi), u(\Omega(\chi))) \leq \psi_i(\Omega(\hat\chi), u(\Omega(\hat\chi))) \qquad \forall \Omega(\hat\chi) \in \mathcal{O},\ i = 1,2 \end{aligned} \tag{10}$$

where

$$\mathcal{O} = \{\Omega(\chi) : \underline{\chi} \leq \chi \leq \overline{\chi}\,;\ \text{ meas } (\Omega(\chi)) = \beta_1 > 0\,;\ \|\frac{\partial n}{\partial \tau}\| \leq \beta_2\}$$

and $u(\Omega(\chi))$ is the solution of (3) at the domain $\Omega(\chi)$. The symbol $\mathcal{O}$ stands for the set of admissible domains, where $\underline{\chi}, \overline{\chi} \in \mathbf{R}^n$, $\beta_1, \beta_2 > 0$ are given.
Three constraints are used to define $\mathcal{O}$; the first one ensures boundedness and domain non-degeneracy, the second preserves area and the third avoids spurious oscillations of the contact boundary Γ_c. We should remark that, for a good choice in the control parameters limits, this last constraint can be avoided in the numerical treatment of the problem.

4. SENSITIVITY ANALYSIS

The objective of shape sensitivity analysis is the evaluation of how and how much a system changes its behavior due to changes in shape of the body $\mathcal{B}$. This change in shape can be simulated by a motion from an original configuration to a "deformed" one through a known mapping:

$$x_t = p(x,t) = x + tV(x), \tag{11}$$

$$\Omega_t = p(\Omega, t) = \Omega + tV(\Omega)\,. \tag{12}$$

where $V \in (H^{1,\infty}(\Omega))^2$ is the direction of the domain variation. This means that, for a given direction V, the variable Ω is uniquely controlled by the parameter $t \in [0, \delta)$, $\delta > 0$.
This treatment, also known as "speed method", has been introduced by Céa (1981), Zolesio (1981), and widely discussed in Haug et al. (1986).
Let Ω_t be an element of the set $\mathcal{O}$, dependent on the parameter t. Let us call $\Omega = \Omega_t|_{t=0}$, u and u_t the solutions ($u(\Omega)$ and $u_t(\Omega_t)$) of the variational problem (3) defined in the domain Ω and Ω_t respectively.
With these assumptions we want to compute the material derivative of the objective function ψ at the configuration Ω:

$$\dot\psi = \dot\psi(u; V) = \lim_{t \to 0+} \frac{\psi(\Omega_t, u_t) - \psi(\Omega, u)}{t}. \tag{13}$$

Next, the dot above a quantity stands for the material derivative of this quantity evaluated at $t = 0$.

Using this method, the expressions for the sensitivity of the two cost functions take the following forms (Fancello et. al., 1994):

$$\begin{aligned}\dot{\psi}_1 = & \int_\Omega (-\mathbf{T}\cdot(\nabla u \nabla V)^s + W \,\mathrm{div} V)\, d\Omega \\ & - \int_\Omega (\dot{b}\cdot u + b\cdot u \,\mathrm{div}V)\, d\Omega - \int_{\Gamma_f} (\dot{f}\cdot u + f\cdot u\, \mathrm{div}_\Gamma V)\, d\Gamma \\ & + \int_{\Gamma_c} \sigma_n (V\cdot n)\, d\Gamma .\end{aligned} \tag{14}$$

$$\begin{aligned}\dot{\psi}_2 = & \int_\Omega C(\nabla z \nabla V)^s \cdot (\nabla z)^s\, d\Omega - \frac{1}{2}\int_\Omega C(\nabla z)^s\cdot(\nabla z)^s \mathrm{div} V\, d\Omega \\ & + \int_{\Gamma_c} \sigma_n(u)(V\cdot n)\, d\Gamma - \int_\Omega C(\nabla u \nabla V)^s\cdot(\nabla z)^s\, d\Omega \\ & - \int_\Omega C(\nabla u)^s\cdot(\nabla z\nabla V)^s\, d\Omega + \int_\Omega C(\nabla u)^s \cdot (\nabla z)^s \,\mathrm{div}V\, d\Omega \\ & - \int_\Omega (\dot{b}\cdot z + b\cdot z \mathrm{div} V)\, d\Omega - \int_{\Gamma_f} (\dot{f}\cdot z + f\cdot z \mathrm{div}_\Gamma V)\, d\Gamma .\end{aligned} \tag{15}$$

It must be enforced that $(u, \sigma_n(u))$ is the solution of Signorini problem (3), $z = z(u)$ the solution of the adjoint equation (9). Once again, we see that ψ_2 is once continuously differentiable as is has already mentioned in Haslinger and Klarbring (1993).
The derivation of the constant volume constraint meas $(\Omega) = \beta_1$ takes an analogous form:

$$\text{meas }(\Omega) = \int_\Omega d\Omega, \qquad (\text{ meas }(\Omega))^{\cdot} = \int_\Gamma V\cdot n\, d\Gamma = \int_\Omega \mathrm{div} V\, d\Omega . \tag{16}$$

5. NUMERICAL APPROXIMATION

Up to this point the state problem, cost function, constraints, derivatives (sensitivity analysis) and even geometry of the problem are independent of the discretization and can be considered as belonging to the same abstract level. Following this insight, our aim in shape optimization is to control the geometry of $\mathcal{B}$ and not specific nodes of a particular mesh. Thus, geometry is defined by topological entities where boundaries are described by oriented straight lines, arches and B-Spline curves through a set of "control points". After this, an automatic mesh generator provides an appropiate mesh, the set of boundary conditions and load system.
As the boundary Γ_c is defined by a B-Spline curve, any variation of the shape is performed by changing the position of its control points. This defines a continuous velocity V, necessary for computing the material derivatives (14), (15) and (16).

5.1 Mesh Generation.

In the automatic mesh generator (Fancello et al. (1991)), domain Ω is defined by key-points, lines, arches and B-Spline curves. It is an unstructured frontal mesh generator (triangular elements with 3 or 6 nodes) where size and distribution of the elements are controlled by the nodal values of a background finite element grid generally formed by few elements. This technique, originally proposed in Peraire et al. (1987) has the advantage that it provides an easy way to deal with adaptive generation. After an analysis and an error estimation, the first mesh generated is now used as background grid having in its nodes the information of the new size of elements to be generated. We used linear triangular elements for the numerical examples of this paper.

5.2 B-Splines curves.

The coordinates of a point in a B-Spline curve are given by the expression

$$x(\xi) = \sum_{k=1}^{2}\sum_{i=1}^{n} X_i^k \Phi_i^k(\xi), \quad \Phi_i^1 = \begin{pmatrix}\varphi_i \\ 0\end{pmatrix}, \quad \Phi_i^2 = \begin{pmatrix} 0 \\ \varphi_i\end{pmatrix}, \tag{17}$$

where n is the number of control points, X_i^k their coordinates, Φ_i^k the blending functions and ξ a curvilinear coordinate. Each control point influences only a portion of the curve proportional to the degree of the polynomial functions φ (quadratic in this paper).
The gradient of the coordinates with respect to the control-points coordinates X_i^k are the blending functions themselves, providing a natural way to define the velocity field V on the boundary described by the B-Spline curve:

$$V_i^k(x(\xi)) = [\nabla_{X_i^k} x(\xi)] = \Phi_i^k(\xi). \tag{18}$$

In many situations it is useful to define, at a control point i, a preferential direction η in which the variation of the boundary takes place. In this case, the degree of freedom of this control point is reduced, providing only one design variable instead of two. The velocity V takes the form

$$V_i^\eta(x(\xi)) = \sum_{k=1}^{2} V_i^k(x(\xi))\eta^k. \tag{19}$$

Introducing in equation (14) any regular vector field V such that $V|_{\Gamma_c} = V_i^k(x(\xi))$ or $V|_{\Gamma_c} = V_i^\eta(x(\xi))$, we obtain the material derivative of ψ_1 due to a variation of the coordinate X_i^k or to a variation of the coordinates X_i in the direction η.
From this considerations it follows that if all control-points coordinates X_i^k are design variables then $m = 2n$. Otherwise, if all control points have preferential directions η we have that $m = n$, being m is the number of components of the vector χ of design variables.

5.3 F.E. approximation of the contact problem.

Denoting by I a the set of q indices i such that $x_i \in \Gamma_c$ is a nodal point, the convex set K can be approximated by:

$$K_h = \{v_h \in V_h : g_i = g(v_h(x_i)) \leq 0, \quad i \in I\}.$$

We also call R_+^q the q-dimensional vector space, elements of which have non-negative components.
The finite dimensional counterpart of (3) can be written as:

$$u_h = \arg \inf_{v_h \in K_h} J_h(v_h), \tag{20}$$

A possible way to solve the primal problem is to use penalty techniques. In this case, the indicator function is modeled by a regular penalty function P_ϵ, $\epsilon > 0$ obtained from an augmented Lagrangian formulation (Bertsekas, 1982):

$$u_h = \arg \inf_{v_h \in V_h} \{J_h(v_h) + P_\epsilon(v_h)\}, \tag{21}$$

$$0 \leq \lambda_h \leq C,$$

$$P_\epsilon(v_h, \lambda_h) = \frac{\epsilon}{2} \sum_{i \in I} \{ [\max(0\,;\, \lambda_{hi} + \frac{1}{\epsilon} g(v_h(x_i)))]^2 - \lambda_{hi}^2 \}. \tag{22}$$

5.4 F.E. approximation of the shape optimization problem.

Using the F.E. approximation described in Section 5.3., we write the discrete expressions of cost functions and their derivatives. In the case of the first cost function, (5), (14) and (16), are given by:

$$\psi_{1h} = \sum_{e=1}^{nel} \int_{\Omega^e} (W_h^e - b \cdot u_h^e)\, d\Omega - \sum_{e=1}^{nel} \int_{\Gamma_f^e} f \cdot u_h^e \, d\Gamma, \tag{23}$$

$$\begin{aligned} \dot{\psi}_{1h} = & \sum_{e=1}^{nel} \int_{\Omega^e} (-\mathbf{T}_h^e \cdot (\nabla u_h^e \nabla V_h^e)^s + W_h^e \operatorname{div} V_h^e)\, d\Omega \\ & - \sum_{e=1}^{nel} \int_{\Omega^e} (\dot{b} \cdot u_h^e + b \cdot u_h^e \operatorname{div} V_h^e)\, d\Omega \\ & - \sum_{e=1}^{nel} \int_{\Gamma_f^e} (\dot{f} \cdot u_h^e + f \cdot u_h^e \operatorname{div}_\Gamma V_h^e)\, d\Gamma + \sum_{e=1}^{nel} \int_{\Gamma_c^e} \sigma_{nh}(V_h^e \cdot n_h)\, d\Gamma, \end{aligned} \tag{24}$$

where $u_h \in K_h$ solves (20) on Ω_h.
For the second one we recall that z_h is the solution of the approximated linear variational problem

$$
\begin{aligned}
& z_h \in V_h \\
& a_{\Omega_h}(z_h, v_h) = \int_{\Gamma_{ch}} \sigma_{nh}(u_h)(v_h \cdot n_h)\, d\Gamma \quad \forall\, v_h \in V_h,
\end{aligned}
$$

and u_h the solution of the approximated Signorini problem. Thus,

$$
\psi_{2h} = \sum_{e=1}^{nel} \int_{\Omega^e} \frac{1}{2} C(\nabla z_h^e)^s \cdot (\nabla z_h^e)^s \, d\Omega. \tag{25}
$$

$$
\begin{aligned}
\dot{\psi}_{2h} = \; & \sum_{e=1}^{nel} \int_{\Omega^e} C(\nabla z_h^e \nabla V_h^e)^s \cdot (\nabla z_h^e)^s \, d\Omega - \frac{1}{2} \sum_{e=1}^{nel} \int_{\Omega^e} C(\nabla z_h^e)^s \cdot (\nabla z_h^e)^s \operatorname{div} V_h^e \, d\Omega \\
& + \sum_{e=1}^{nel} \int_{\Gamma_c^e} \sigma_{nh}^e(u_h^e)(V_h^e \cdot n_h^e)\, d\Gamma - \sum_{e=1}^{nel} \int_{\Omega^e} C(\nabla u_h^e \nabla V_h^e)^s \cdot (\nabla z_h^e)^s \, d\Omega \\
& - \sum_{e=1}^{nel} \int_{\Omega^e} C(\nabla u_h^e)^s \cdot (\nabla z_h^e \nabla V_h^e)^s \, d\Omega + \sum_{e=1}^{nel} \int_{\Omega^e} C(\nabla u_h^e)^s \cdot (\nabla z_h^e)^s \operatorname{div} V_h^e \, d\Omega \\
& - \sum_{e=1}^{nel} \int_{\Omega^e} (\dot{b} \cdot z_h^e + b \cdot z_h^e \operatorname{div} V_h^e)\, d\Omega - \sum_{e=1}^{nel} \int_{\Gamma_f^e} (\dot{f} \cdot z_h^e + f \cdot z_h^e \operatorname{div}_\Gamma V_h^e)\, d\Gamma.
\end{aligned} \tag{26}
$$

Finally, the constraint,

$$
\text{meas}\,(\Omega_h) = \sum_{e=1}^{nel} \int_{\Omega^e} d\Omega, \quad (\text{ meas}\,(\Omega_h))^{\cdot} = \sum_{e=1}^{nel} \int_{\Omega^e} \operatorname{div} V_h^e \, d\Omega, \tag{27}
$$

Now it is necessary to define V_h in Ω_h and a simple rule to do this is described as follows: at any nodal point x_r the velocity $V_h(x_r)$ is given by

$$
V_h(x_r) = \begin{cases} 0, & \text{if } x_r \notin I \\ V_i^k(x_r(\xi)), & \text{if } x_r \in I \end{cases} \tag{28}
$$

if the design variables are the coordinates X_i^k, $k = 1, 2$, of the B-Spline control points or

$$
V_h(x_r) = \begin{cases} 0, & \text{if } x_r \notin I \\ V_i^\eta(x_r(\xi)), & \text{if } x_r \in I \end{cases} \tag{29}
$$

if the we take preferential directions η over the i-control points.
Roughly speaking, the above expression means that the velocity field V_h is equal to zero except in a one-element layer along the boundary Γ_c. This is a straight-forward form of defining V_h in the interior but not the best in order to obtain accurate results. A layer with more than one element of thickness, will give less errors in the integration process Masmoudi (1987).
Finally, the discrete optimization problem takes the form:

Find $\chi \in R^m$ solution of:

$$
\begin{gathered}
\min_{\chi} \psi_h(\chi, u_h(\chi)), \\
\underline{\chi_i} \le \chi_i \le \overline{\chi_i}, \quad i = 1, \ldots, m, \\
\text{meas}\,(\Omega_h(\chi)) - \text{ meas}\,(\Omega_h(\chi_0)) = 0,
\end{gathered} \tag{30}
$$

where χ_0 is the initial value of the control points and $u_h(\chi)$ is the solution of the approximated Signorini problem at domain $\Omega_h(\chi)$.
If we replace the equality constraint of area for the inequality,

$$\text{meas}\,(\Omega_h(\chi)) - \text{meas}\,(\Omega_h(\chi_0)) \geq 0\,, \tag{31}$$

where no decrease of area is allowed, it was observed that the minimum is reached at a saturation point, i.e., the original constraint is satisfied. For this reason we decided to use inequation (31) instead of (30) in numerical applications.
Hence, the structure of the general nonlinear optimization problem we need to solve is the following:

$$\begin{aligned} &\text{Find } x^* \in R^n \text{ solution of:} \\ &\min_x f(x)\,, \\ &g(x) \leq 0 \end{aligned} \tag{32}$$

where $f: R^n \rightarrow R^1$ is an objective function and $g: R^n \rightarrow R^m$ defines the constraints.
There exist several methods for solving this problem (Luenberger (1973), Glowinski (1984)), many of them based on the recursive formula

$$x_{k+1} = x_k + t_k d_k\,, \tag{33}$$

where x_k, x_{k+1} are the design variables for k and $k+1$ th iterations respectively, d_k is the descent direction of the function f and t_k is the step to be done in direction d_k. In the present case we use the interior point algorithm proposed by Herskovits (1991). In this method a feasible direction d_k is calculated using the Karush-Khun-Tucker (KKT) first order optimality conditions. The algorithm generates a convergent sequence of interior points with decreasing values of the objective function. Inequality constraints are satisfied at each iteration while equality constraints are satisfied at the limit, together with the whole set of KKT optimality conditions. During the different steps of the algorithm, evaluation of f, g and their derivatives with respect to the design variables is required. We will reproduce here a general procedure of the algorithm but a complete description of it, including convergence results can be seen in Herskovitz (1991) while some results of an implementation in Hevsukoff (1991).
A scheme of the computational procedure used to solve our optimal shape design problem can be cast as follows:

0. Definition of geometric data, b.c., load system, control parameters,
1. Given an admissible initial point χ_0, evaluate:
   - $u_h(\chi_0)$, solution of the contact problem,
   - $\psi_h(\chi_0, u_h(\chi_0))$, meas $(\Omega_h(\chi_0))$,
   - $\dot{\psi}_h(\chi_0, u_h(\chi_0))$, (meas $(\Omega_h(\chi_0)))'$.
2. Activation of optimization algorithm, being necessary at point χ^k, $k = 1, 2, ...$,(Max. Num. Iterations) the evaluation of:
   - $u_h(\chi_k)$, solution of the contact problem,
   - $\psi_h(\chi_k, u_h(\chi_k))$, meas $(\Omega_h(\chi_k))$,
   - If necessary: $\dot{\psi}_h(\chi_k, u_h(\chi_k))$, (meas $(\Omega_h(\chi_k)))'$.
3. If convergence is achieved, STOP. Else, $k = k + 1$, GOTO 2.

6. NUMERICAL RESULTS

The aim of this section is to show two examples, the behavior of the contact stress distribution associated to each cost function. In all these examples, the body force b is equal to zero and linear triangular elements are used.

6.1 Example 1

Let us have an elastic beam clamped on the left and supported below by a rigid foundation $\mathcal{F}$ (see Figure 1) Original configuration Ω and rigid foundation $\mathcal{F}$ are given by:
$\Omega = \{(x, y) \in R^2 : 0 < x < 4,\ 0 < y < 1\}$,

$\mathcal{F} = \{(x, y) \in R^2 : (y + r)^2 + (x - 2)^2 < r^2\}$, $r = 40.025$.
Material constants are $E = 2150$, $\nu = 0.29$ and distributed force
$f = (-5.78, -5.78)$, $\Gamma_f = \{(x, y) \in R^2 : 2 < x < 4, \; y = 1\}$.
The initial shape produces a stress concentration over a narrow region at the center of the beam. The final shapes for each cost function produce the normal stress distributions shown in Figure 1.

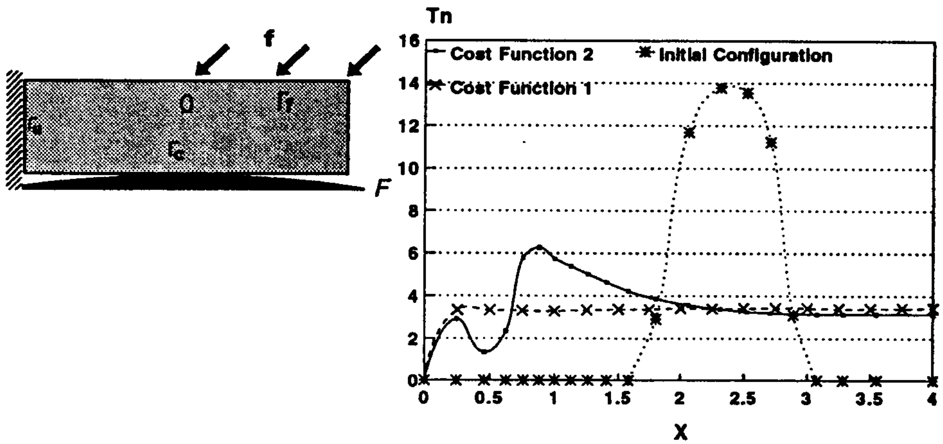

Figure 1. Example 1.

6.2. Example 2

The Figure 2 shows the geometry of this example. The boundary Γ_c and rigid foundation $\mathcal{F}$ can be described as $\Gamma_c = \{(x, y) \in R^2 : 0 < x < 4, \; y = 0\}$; $\mathcal{F} = \{(x, y) \in R^2 : 0 < x < 4, \; y < -0.0125x\}$.
Material constants are $E = 2150$, $\nu = 0.29$. The boundary forces are
$f = (0.0, -5.0)$, $\Gamma_f = \{(x, y) \in R^2 : 0 < x < 6, \; y = 2.5\}$.
The results obtained for the two cost functions are given in Figure 2.

7. CONLUDING REMARKS

The definition of geometry through geometric entities provides a great versatility on the characterization and the control of the body shape. Moreover, this approach works as a natural "link element" between the continuum formulation of sensitivity analysis and its approximation: given a configuration and a direction of change in geometry, velocity V is completely defined, being independent of the finite element discretization. Its corresponding discrete version V_h is just a consequence of a particular mesh. This approach is also powerful in terms of software implementation since each module of the process (solid modeling, mesh generation, analysis, adaptivity, sensitivity analysis, optimization algorithm) can be treated and, eventually, improved as independent objects.

From the presented examples, it can be seen that the first cost function leads to a uniform contact pressure distribution, as it is known in the literature. However, for the second cost function, this property is only reached for the second example. Our interpretation about this behavior is the following. Under the adopted particular case, this cost function is nothing but the work of contact reactions if no other forces are present. Thus, contact stress distribution obtained by the minimization of this functional is "weighted" by the displacements this distribution produces along boundary Γ_c. Because of the particular shape of the last example, all the points on the boundary Γ_c translate the same. Thus, the "weight" is constant, allowing to reach an almost constant stress distribution. A possible alternative to improve this behavior is taking a more general adjoint space $\mathcal{V}_M$, different from $\mathcal{V}$. Unfortunately, with this attitude the differentiability of cost function is lost and, as pointed in Haslinger and Klarbring (1993), non-smooth optimization techniques are needed.

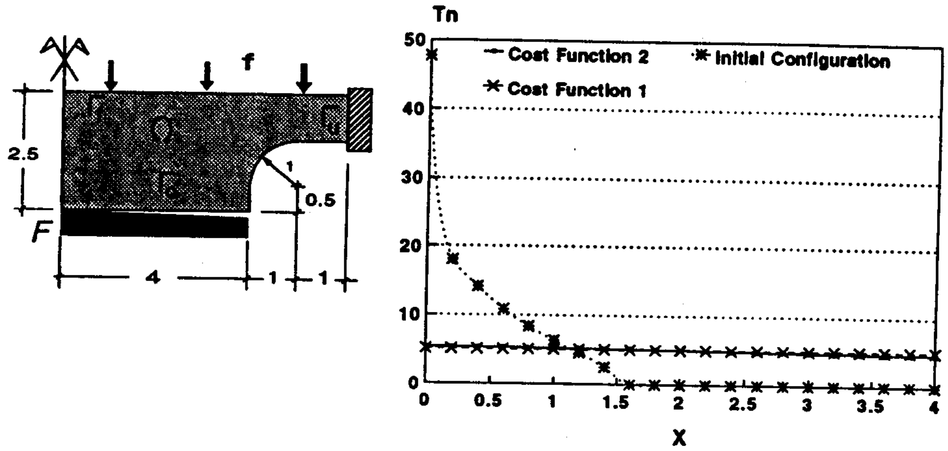

Figure 2. Example 2.

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