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GEOMETRIC NON-LINEAR ANALYSIS OF THIN-WALLED SPATIAL FRAMES

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ABSTRACT

An incremental and piecewise nonlinear finite element approach is developed for the large displacement , large strain regime with particular reference to elastic-plastic behavior in metal structures A large displacement, small strain formulation (as applicable to problems of structural stability) is obtained from this theory by assuming that changes in lenght of line elements and relative rotations are negligible when compared to unity. A consistent Updated Lagrangian formulation is derived from the energy balance equation in reference to propper configuration. Differences between the existing formulations and similar ones in the literarure are found to be in specific geometric nonlinear terms in the final incremental equation as well as in the definition of the load increment vector. A complete formulation for the equilibrium of a thin walled member of arbitrary open cross-section is used and a complete displacement fied (axial and transversal) is developed including higher order terms. A more restricted approach under the hypothesis hypotesis of linearized field of displacements is adopted in order to show an application and a new stiffness matrix for geometrically non-linear incremental analysis of three dimensional beam-column with bisymmetrical, thin walled, I-type cross section is presented. Corrections in the element matrix are made to propperly consider the behavior under finite rotations. The formulation implemented in a computer program uses the Newton-Raphson scheme for nonlinear incremental analysis.

INTRODUCTION AND REVIEW OF LITERATURE

Much previous research has been conducted to obtain the governing differential equations for beams and beam-columns in three dimensions based on consideration of equilibrium, virtual work, or total potential energy. Studies in the literature consider only some effects as uniform torsion [2], whilst others [6] [13] studied both uniform and nonuniform torsion. Several other effects have been taken into account such as the effect of initial imperfections, constitutive nonlinearities and geometric nonlinearities, the latter being due to large displacements. The most commonly used nonlinear material is the elastic-plastic model, for which the linearity of the incremental stress-strain law forms the basis of the equations. Its most direct application is in the incremental type solution, where the solution is built up as a series of linear increments. Geometric nonlinearities were first included by means of incremental geometric stiffness (initial stress stiffness matrix). The earlier results were obtained on basis of equilibrium at nodes. The derivation of the initial stress stiffness matrix was finally placed on a firm basis by the use of the

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Lagrangian or Green's strain. More recent analysis has stabilished the importance of additional terms wich take the form of an initial displacement matrix in the incremental solution. The inclussion of this matrix is very important even within the small strain approximation.

PROCEDURE

The development of the equilibrium equations for a three dimensional beam column of thin-walled open cross section requires that attention be given to:

(1) Basic assumptions
(2) Kinematics of a section and equilibrium equations
(3) Stress-Strain relationship.
(4) Generalized forces and displacements
(5) Selection of shape functions

And for analyses of spatial structures composed by non-collinear members, an additional requirement of kinematic continuity that must be satisfied [2], [5]:

(6) Finite Rotations

BASIC ASUMPTIONS

(a) The beam-column has a general open cross section and no distortion of the cross section occurs appart of warping (rigid transversal body assumption).
(b) Effects of transversal stresses are negligible.
(c) For transversal displacements we will consider moderate rotations hypotesis.
(d) For axial displacements we will consider:
  (i) Transversal displacements are much larger than the longitudinal ones.
  (ii) The thin-walled beam element is composed by individual plate elements.
  (iii) The shearing strain in the middle surface and in the plane normal to the individual plate elements is neglected.

KINEMATICS OF A SECTION AND EQUILIBRIUM EQUATIONS

Hipotesis of Large Rotation in Flexure

In large displacements and small strains, it is important to know the order of magnitude of the rotation of the cross section caused by the coupling of bending and torsion. In order to deal correctly with approximations and error estimations, we begin to consider the complete set of the field of displacements caused by large rotations, considering all the terms without approximations. The complete development of the field of displacements obtained from Vlasov's theory for a general thin-walled open-cross section beam (Figure 1), can be found in reference [7].

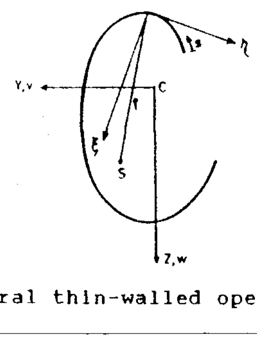


Figure 1. A general thin-walled open-cross section beam

The displacements in the interior of the element considering large rotations are:

$$u_x(x,y,z) = u_x(x) - y\,[\,u'_z(x)\sin\varphi(x) + u'_y(x)\cos\varphi(x)\,] - z\,[\,u'_z(x)\cos\varphi(x) - u'_y(x)\sin\varphi(x)\,] - \bar{\omega}(y,z)\,\varphi'(x)$$
$$u_y(x,y,z) = u_y(x) - (z - z_s)\sin\varphi(x) - (y - y_s)(1 - \cos\varphi(x))$$
$$u_z(x,y,z) = u_z(x) + (y - y_s)\sin\varphi(x) - (z - z_s)(1 - \cos\varphi(x)) \qquad (1)$$

Where y-z is the principal axis of the cross section, $\bar{\omega}(y,z)$ is the sectorial area and φ' is a measure of the warping of the section. With this set of displacements, we can now deal with the hypothesis of moderate rotations.

Hypothesis of Moderate Rotations in Beam Theory

We shall assume that the rotations about the normal to the midplane are small and of the order of magnitude of the strains, while the rotations of the normal are moderate. These assumptions are consistent with the real features of these strutures which exhibit a large in-plane rigidity and some transverse flexibility. In mathematical terms, the assumption of small strain means

$$\varphi(x) << 1 \quad ; \sin\varphi(x) = \varphi(x) \quad ; \quad \cos\varphi(x) = 1 \qquad (2)$$

we will disregard the terms

$$\varphi(x)\,\varphi'(x)\,u'_y(x) \quad ; \quad \varphi(x)\,\varphi'(x)\,u'_z(x) \qquad (3)$$

With this last assumptions, the field of displacements reduces to:

$$u_x(x,y,z) = u_x(x) - y(\,u'_y(x) + u'_z(x)\,\varphi(x)\,) - z(\,u'_z(x) - u'_y(x)\,\varphi(x)\,) - \bar{\omega}(y,z)\,\varphi'(x)$$
$$u_y(x,y,z) = u_y(x) - (z - z_s)\,\varphi(x)$$
$$u_z(x,y,z) = u_z(x) + (y - y_s)\,\varphi(x) \qquad (4)$$

Hipothesis of linearized field of displacements

It is shown that the choice of this hypotesis represents an simplified and particular case of a variety of nonlinear refined thin-walled beam theories. In the light of the present theory, these approximate variants may be derived in a unified manner and related to each other in the basis of order of magnitude considerations, which accordingly makes it possible to clarify the range of validity of this hypothesis. It is also shown that under coherent assumptions and constraints, the present use of linearized field of displacements reduces the hard work of deal with a complete set of field displacements and a stress-strain relationships. Several authors [6], [8], [9] have adopted the linearized field of displacements directly and not as a simplified hypothesis through the complete field of displacements.

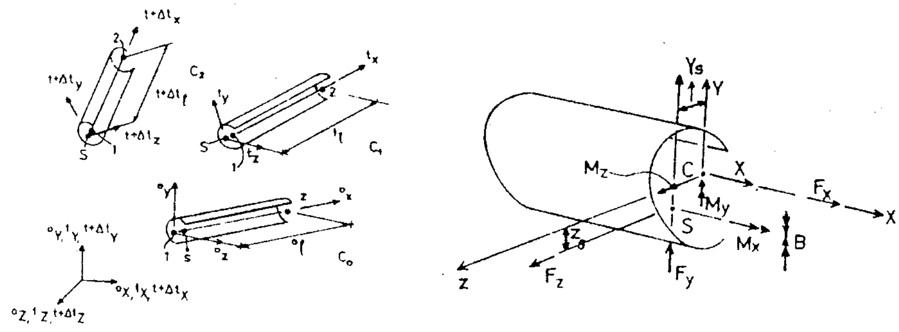
Taking the relations (4) as a basis, we have:

$$u_x(x,y,z) = u_x(x) - y\,u'_y(x) - z\,u'_z(x) - \bar{\omega}(y,z)\,\varphi'(x)$$
$$u_y(x,y,z) = u_y(x) - (z - z_s)\,\varphi(x)$$
$$u_z(x,y,z) = u_z(x) + (y - y_s)\,\varphi(x) \qquad (5)$$

This hypothesis is completed with the small strains hypothesis.

Equilibrium Equations for Thin-Walled Beam-Columns

In the following development one may consider the motion of a beam element in a stationary Cartesian co-ordinate system, as shown in Figure 2. There are three configurations to consider: C_0 the initial or undeformed, C_1 is a current deformed and known deformed state, C_2 is the neighbouring unknown deformed state. In the present development we will use the notation of Bathe [3]: left subscript denotes the configuration in which the quantity is measured; left superscript denotes the configuration in which the quantity occurs. Quantities with no superscript denotes increment between configuration 1 and 2.



(a) (b)

Figure 2. Configurations of a thin-walled beam and stress resultants

Total and Incremental Virtual Work

The principle of virtual displacement may be used to write the equilibrium requirements for any stress field. Using Cartesian tensor notation, neglecting body forces, and expressing the internal work as the product of the associated components of the Kirchhoff stress tensor, σ_{ij}, and Green's strain tensor ε_{ij}, the principle of virtual displacements may be written in the deformed configuration 2 as

$$\int_{{}^{t}V} {}^{t+\Delta t}_{\ \ t}S_{ij}\ \delta\ {}^{t+\Delta t}_{\ \ t}\varepsilon_{ij}\ {}^{t}dV = {}^{t+\Delta t}_{\ \ t}\mathfrak{R} \tag{6}$$

where $\mathfrak{R}$ is the expression of the external total virtual work, S is the 2nd Piola-Kirchhoff stress tensor and ε is the Green-Lagrange strain tensor. An incremental decomposition of these tensors is:

$$ {}^{t+\Delta t}_{\ \ t}S_{ij} = {}^{t}\tau_{ij} + {}_{t}S_{ij}\ ; \qquad {}^{t+\Delta t}_{\ \ t}\varepsilon_{ij} = {}^{t}\varepsilon_{ij} + {}_{t}\varepsilon_{ij} \tag{7}$$

where ${}^{t}\tau_{ij}$ and ${}^{t}\varepsilon_{ij}$ are the Cauchy and Green-Lagrange tensors in C_1.

Due to equation (6) we can see that

$$\delta\ {}^{t+\Delta t}_{\ \ t}\varepsilon_{ij} = \delta\ {}_{t}\varepsilon_{ij}\ ; \qquad {}_{t}\varepsilon_{ij} = {}_{t}e_{ij} + {}_{t}\eta_{ij} \tag{8}$$

where

$$ {}_{t}e_{ij} = \frac{1}{2}\left[\left({}_{t}u_{i,j} + {}_{t}u_{j,i}\right) + \left({}^{t}u_{k,i}\ {}_{t}u_{k,j} + {}^{t}u_{k,j}\ {}_{t}u_{k,i}\right)\right] \tag{9}$$

and

$$ {}_{t}\eta_{ij} = \frac{1}{2}\left({}_{t}u_{k,i}\ {}_{t}u_{k,j}\right) \tag{10}$$

are the linear and nonlinear components of the strain tensor.If we use the constitutive tensor to relate the strain tensor to the 2nd Piola-Kirchhoff, then we have

$$ {}_{t}S_{ij} = {}_{t}C_{ijrs}\, {}_{t}\varepsilon_{rs} \tag{11} $$

and equation [6] can now be expressed as

$$ \int_{{}^{t}V} {}_{t}C_{ijrs}\, {}_{t}\varepsilon_{rs}\, \delta_{t}\varepsilon_{ij}\, {}^{t}dV + \int_{{}^{t}V} {}^{t}\tau_{ij}\, \delta_{t}\eta_{ij}\, {}^{t}dV = {}^{t+\Delta t}\Re - \int_{{}^{t}V} {}^{t}\tau_{ij}\delta_{t}e_{ij}\, {}^{t}dV \tag{12} $$

that is linear in displacements u_i. If the time increment between configurations 1 and 2 is very small , we can rewrite the last equation in the form

$$ \int_{{}^{t}V} {}_{t}C_{ijrs}\, {}_{t}e_{rs}\, \delta_{t}e_{ij}\, {}^{t}dV + \int_{{}^{t}V} {}^{t}\tau_{ij}\, \delta_{t}\eta_{ij}\, {}^{t}dV = {}^{t+\Delta t}\Re - {}^{t}_{t}F \tag{13} $$

in which high order terms have been dropped.

It is usually more efficient to evaluate all magnitudes and matrices in local co-ordinate system and then perform a transormation to global system, before the element assembly proccess, then, we will identify this magnitudes with an upper bar. The last equation is now

$$ \int_{{}^{t}V} {}_{t}\bar{C}_{ijrs}\, \delta({}_{t}\bar{e}_{rs})^{2}\, {}^{t}dV + \int_{{}^{t}V} {}^{t}\bar{\tau}_{ij}\, \delta_{t}\bar{\eta}_{ij}\, {}^{t}dV = {}^{t+\Delta t}\bar{\Re} - {}^{t}_{t}\bar{F} \tag{14} $$

STRESS-STRAIN RELATIONSHIPS

Due to characteristics of the selected bi-symmetrical thin-walled beam (Figure 2b), we have only three non-zero stress components, then

$$ \langle {}^{t}\bar{\tau} \rangle = \langle {}^{t}\sigma_{xx} \quad {}^{t}\tau_{yx} \quad {}^{t}\tau_{zx} \rangle $$

$$ \langle {}_{t}\bar{e} \rangle = \langle {}_{t}e_{xx} \quad 2\, {}_{t}e_{yx} \quad 2\, {}_{t}e_{zx} \rangle $$

$$ \langle {}_{t}\bar{\eta} \rangle = \langle {}_{t}\eta_{xx} \quad 2\, {}_{t}\eta_{yx} \quad 2\, {}_{t}\eta_{zx} \rangle \tag{15} $$

and if we neglect the square of the Poisson effect,

$$ \left[{}_{t}\bar{C} \right] = \mathrm{diag}\ [\ E\ G\ G\ ] \tag{16} $$

Incremental Equilibrium Equation of a 3D Beam-Column

In substituting the equations (15) and (16) into (14), we obtain

$$ \int_{{}^{t}V} [\ \tfrac{E}{2}\, \delta({}_{t}u_{x,x})^{2} + \tfrac{G}{2}\, \delta({}_{t}u_{x,y} + {}_{t}u_{y,x})^{2} + \tfrac{G}{2}\, \delta({}_{t}u_{x,z} + {}_{t}u_{z,x})\ ]\ {}^{t}dV + $$

$$ \int_{{}^{t}V} [\ \frac{{}^{t}\sigma_{xx}}{2}\, \delta({}_{t}u^{2}_{x,x} + {}_{t}u^{2}_{y,x} + {}_{t}u^{2}_{z,x}) + $$

$$ {}^{t}\tau_{xy}\, \delta({}_{t}u_{x,x}\, {}_{t}u_{x,y} + {}_{t}u_{y,x}\, {}_{t}u_{y,y} + {}_{t}u_{z,x}\, {}_{t}u_{z,y}\ ) + $$

$$ {}^{t}\tau_{xz}\, \delta({}_{t}u_{x,x}\, {}_{t}u_{x,z} + {}_{t}u_{y,x}\, {}_{t}u_{y,z} + {}_{t}u_{z,x}\, {}_{t}u_{z,z}\ )\ ]\ {}^{t}dV = $$

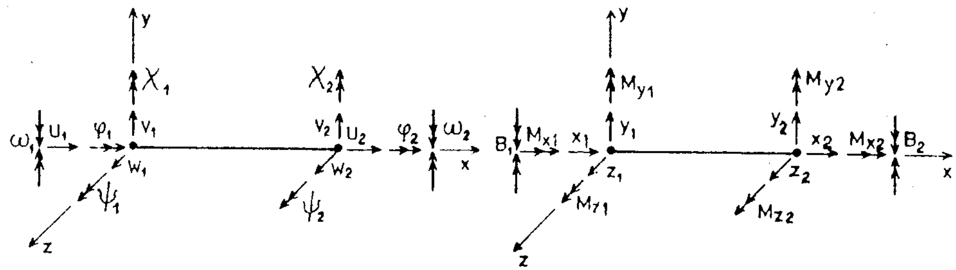
$$ {}^{t+\Delta t}_{t}\bar{\Re} - {}^{t}_{t}\bar{F} \tag{17} $$

We can rewrite this last equation after some operations, in a matrix

${}^{t}_{t}\{ \tilde{P} \}$ are the load vectors at the end and begining of the incremental proccess.

GENERALIZED NODAL FORCES AND DISPLACEMENTS

In order to obtain the geometrical stiffness matrix of a thin-walled bi-symmetrical I beam, the selected local beam axis are chosen so that x- coincides with centroidal axis and y-z are the cross-sectional ones. Next, figures 3a and 3b shows the Cartesian degrees of freedom and forces.



a) Displacements b) Forces

Figure 3 Cartesian degrees of freedom and forces

where

$$\langle \tilde{\rho} \rangle = \langle\, u_1\ v_1\ w_1\ \varphi_1\ \chi_1\ \psi_1\ u_2\ v_2\ w_2\ \varphi_2\ \chi_2\ \psi_2\ \omega_1\ \omega_2 \,\rangle \tag{19}$$

is the Cartesian displacement vector and

$$\langle \tilde{P} \rangle = \langle\, F_{X1}\ F_{Y1}\ F_{Z1}\ M_{X1}\ M_{Y1}\ M_{Z1}\ F_{X2}\ F_{Y2}\ F_{Z2}\ M_{X2}\ M_{Y2}\ M_{Z2}\ B_1\ B_2 \rangle \tag{20}$$

is the Cartesian load vector.

DISPLACEMENT FUNCTIONS

Usual elements for spatial frame analisis are formulated with Hermitian interpolations. We use linear function for axial displacement and cubic and quadratic functions for flexure, torsional and flexural rotation.

$$\underset{\sim}{u} = \langle H_1 \rangle \,\underset{\sim}{\bar{u}} \ ; \quad \underset{\sim}{v} = \langle H_3 \rangle \,\underset{\sim}{\bar{v}} \ ; \quad \underset{\sim}{w} = \langle H_3 \rangle \,\underset{\sim}{\bar{w}} \ ; \quad \underset{\sim}{\varphi} = \langle H_3 \rangle \,\underset{\sim}{\bar{\varphi}} \tag{21}$$

where

$$\langle H_1 \rangle = \langle (1-t)\ \ t \rangle$$

$$\langle H_3 \rangle = \langle (1-3t^2+2t^3)\ \ (t-2t^2+t^3)\ \ (3t^2-2t^3)\ \ (t^3-t^2) \rangle \tag{22}$$

with $t = x/l$. The upper bar on the displacement vectors is used to indicate a nodal displacement at the end nodes 1 and 2 of the element,

$$\underset{\sim}{\bar{u}} = \langle\, u_1\ u_2 \rangle; \qquad \underset{\sim}{\bar{v}} = \langle\, v_1\ l\varphi_1\ v_2\ l\varphi_2 \rangle,$$

$$\underset{\sim}{\bar{w}} = \langle w_1\ -l\chi_1\ w_2\ -l\chi_2 \rangle, \qquad \underset{\sim}{\bar{\varphi}} = \langle\, \varphi_1\ l\varphi'_1\ \varphi_2\ l\varphi'_2 \,\rangle \tag{23}$$

FINITE ROTATIONS

The rotational degrees of freedom adopted as $<\varphi\ \chi\ \phi> = <\theta_X,\ -u'_Z,\ u'_Y>$ does not ensure the kinematic continuity at nodes when finite rotations occurs [2],[5]. In order to obtain a consistent formulation, that describe the generalized nodal load vector, it is shown the necessity for the inclussion of the load correction matrix. The argumentation is based in the fact that the stiffness geometric matrices obtained through the expression of strain energy associated to the Green-Lagrange strain tensor components, does not consider the non-linear work produced by the applied loads, and it must be included as additional correcting matrices, called "load correction matrix" It means that for the correct selection of generalized displacements (translation and rotation), finite rotation vectors that are commutative up to second order terms, must be used to ensure the kinematic continuity in the nodes of the structure [2].

Generalized rotations $<\varphi> = <\varphi\ \ \chi\ \ \psi>$ must be related to Euler, Milenkovic or Rodrigues angles $<\Phi> = <\phi_X\ \phi_Y\ \phi_Z>$ as pointed out by Kane [4] and Surana [17] as

$$<\varphi\ -u'_Z\ u'_Y> = <\varphi\ \chi\ \psi> = <\phi_X\quad (\phi_Y - \tfrac{1}{2}\phi_X\phi_Z)\quad (\phi_Z + \tfrac{1}{2}\phi_X\phi_Y)> \tag{24}$$

The generalized virtual incremental rotation must be related to the finite incremental rotations as

$$\begin{Bmatrix}\delta\varphi\\ \delta\chi\\ \delta\psi\end{Bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -\phi_Z/2 & 1 & -\phi_X/2\\ \phi_Y/2 & \phi_X/2 & 1\end{bmatrix}\begin{Bmatrix}\delta\phi_X\\ \delta\phi_Y\\ \delta\phi_Z\end{Bmatrix} = [\,\Lambda\,]\ \delta\underset{\sim}{\phi} \tag{25}$$

Relationships between generalized moments $\{M\}$ and $\{M_\phi\}$ must be obtained from the last equation as

$$\{M_\phi\} = [\,\Lambda\,]^T\{M\} \tag{26}$$

The differential form of this last equation is

$$\{\delta M_\phi\} = [\,\Lambda\,]^T\{\delta M\} + [\,\delta\Lambda\,]^T\{M\} \tag{27}$$

We can rewrite the differential moments as a product of a stiffness matrix and a vector of incremental rotations as

$$\{\delta M_\phi\} = [K_\Phi]\{\delta\Phi\}\ ; \qquad \{\delta M\} = [K]\{\delta\varphi\} \tag{28}$$

Then, (24) is now:

$$[K_\Phi]\{\delta\Phi\} = [\,\Lambda\,]^T[K][\,\Lambda\,]\{\delta\Phi\} + \delta[\,\Lambda\,]^T\{M\} \tag{29}$$

where

$$[\delta\Lambda]^T\{M\} = \frac{1}{2}\begin{bmatrix}0 & -\delta\phi_Z & \delta\phi_Y\\ 0 & 0 & \delta\phi_X\\ 0 & -\delta\phi_X & 0\end{bmatrix}\begin{Bmatrix}M_X\\ M_Y\\ M_Z\end{Bmatrix} = \frac{1}{2}\begin{bmatrix}0 & M_Z & -M_Y\\ M_Z & 0 & 0\\ -M_Y & 0 & 0\end{bmatrix}\begin{Bmatrix}\delta\phi_X\\ \delta\phi_Y\\ \delta\phi_Z\end{Bmatrix} \tag{30}$$

We can say then

$$[\,\delta\Lambda\,]^T\{M\} = M_X\frac{1}{2}[\,0\,]\begin{Bmatrix}\delta\phi_X\\ \delta\phi_Y\\ \delta\phi_z\end{Bmatrix} + M_Y\begin{bmatrix}0 & 0 & -\frac{1}{2}\\ 0 & 0 & 0\\ -\frac{1}{2} & 0 & 0\end{bmatrix}\begin{Bmatrix}\delta\phi_X\\ \delta\phi_Y\\ \delta\phi_z\end{Bmatrix} +$$

$$M_Z \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \delta\phi_X \\ \delta\phi_Y \\ \delta\phi_Z \end{Bmatrix} \qquad (31)$$

The second member of this equation may be written in compact form as:

$$[\ \delta\Lambda\ ]^T \{\ M\ \} = [\ C^i_{\chi\psi}\ ]\ \{\ \delta\Phi\ \} \qquad (32)$$

where i is the node element and $\chi\ \psi$ the rotational degrees of freedom .

The expression of this matrix is

$$[\ C_{\chi\psi}\ ] = \lceil\ [0_{3\times3}]\ [C^1_{\chi\psi}]\ [0_{3\times3}]\ [C^2_{\chi\psi}]\ [0_{2\times2}]\ \rfloor \qquad (33)$$

This matrix added to the stiffness matrix, turns the latter suitable for finite rotations.

NUMERICAL RESULTS

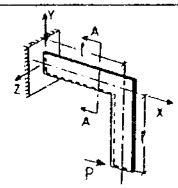
EXAMPLE 1 . Spatial L Frame

In this flexural-torsional example we have used 15 elements per side in order to ensure the convergence to linearized critical load and to obtain the performance of the element compared with those available in the literature.

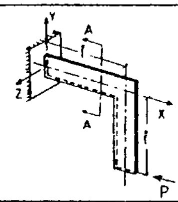
A-A

L=240 mm
b= 30 mm
t=0.6 mm
ν= 0.31
E=71240 N/mm
$P_x = \pm 1$N

A



B



Reference	$[K_G]$	A: Critical Load P_{cr} [N]	B: Critical Load P_{cr} [N]
[7]	$[K_G^{sq}]$	0,4303	0,5607
[7]	$[K_G^{ss}]$	0,6940	1,0972
[2]	$[K_G^{q}]_{c.c}$	0,4217	0,5505
[2]	$[K_G^{s}]_{c.c}$	0,6808	1,0873
[2]	$[K_G^{q}]_{s.c}$	0,8236	1,2234
[2]	$[K_G^{s}]_{s.c}$	1,0775	2,0965
[5]	$[K_{GB}]$	0,8236	1,2234
[5]	$[K_{GB}]_M$	0,8236	1,2234
[11]	$[K_{GG}]$	0,7101	1,0079
[6]	$[K_{GY}^{sq}]$	0,4217	0,5505
[6]	$[K_{GY}^{ss}]$	0,6808	1,0873
[18]	$[K_G]_{BL}$	0,4217	0,5505
[15]	$[K_{GA}^{sq}]$	0,4403	0,5697
[15]	$[K_{GA}^{ss}]$	0,7000	1,1000
[12]	ELIAS	0,6818*	1,0847*
[13]	FLDZ	0,6804**	-***
[14]	SIMO	-	1,09

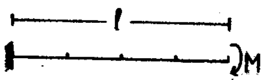
Table 1. Results for load cases A and B

* analytical results using nonlinear rotations
** results using 10 elements per side
*** results without indication of discretization employed
\- case not solved for the authors

The letters used inside the brackets are:"ss" and "sq" superscripts represents geometrical stiffness matrices with and without nodal correction matrix added. For the author in ref. [2] "s" is same as "ss", and "q" is same as "sq". The subscripts c.c and s.c outside the brackets represents the consideration or not of shear forces, and "M" represents the variation of flexure moments along the element. "BL" represents an element obtained with linear Lagrangian functions.

EXAMPLE 2. CANTILEVER BEAM WITH END MOMENT

This example was taken from refs. [16] and [17] and solved analytically through elliptic integrals [19]. Geometric and material properties are in Figure 4. Convergence tolerance used is 10^{-4} in all cases. A four three node element is used. The total load is applied in 100 equal steps.



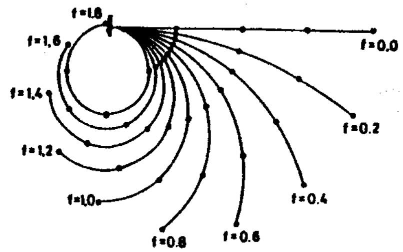


Figure 4. Cantilever beam model and a load-deflection behavior

Load factor	Δ [17] / Δ Anal.	Δ [16] / Δ Anal.	Δ [7] / Δ Anal.	Iter.[15]	Iter.[16]	Iter.[7]
0.2	0.99	1.00	1.00	7	4	3
0.4	1.00	1.01	1.00	7	5	3
0.6	1.00	1.01	1.00	7	4	3
0.8	1.00	1.02	1.00	7	5	3
1.0	1.01	1.02	1.00	7	4	3
1.2	1.01	1.01	1.01	8	5	3
1.4	0.99	1.00	1.00	8	5	3
1.6	0.96	0.97	0.98	8	6	3
1.8	0.91	0.92	0.93	7	5	3

Table 2 Comparison of results of cantilever beam under moment

CONCLUSIONS

The three dimensional thin-walled beam formulation presented here is based on a consistent derivation of a complete displacements field, performs very well in problems of linearized stability and for nonlinear incremental analysis. The inclussion of warping effects, a nodal correction matrix and an improved algorithm for the proccess of forces recovery, offers extremely good convergence characteristics during equilibrium iterations. The displacement of the elements is comparable with isoparametric beam elements.

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