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DYNAMICAL PROBLEMS FOR ELASTIC–PLASTIC BODIES WITH LINEAR HARDENING

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RESUMEN

Es posible probar que el problema dinámico de un material elastoplástico con endurecimiento lineal (en particular elastoplasticidad perfecta) es límite del modelo que responde a la ley viscoplástica de Perzyna. Se demuestra además en este trabajo resultados de regularidad adicionales para la velocidad que surge de la hipótesis de endurecimiento positivo.

ABSTRACT

It is possible to prove that the dynamical problem of an elastic plastic materials with linear hardening (in particular the perfect case) is limit of the clasic viscoplastic Perzyna model. We also demonstrate in this paper the additional regularity result for the velocity that gives the assumption of positive hardening.

1 - DYNAMIC EVOLUTION OF ELASTIC PLASTIC MATERIALS WITH LINEAR HARDENING

Notations and definitions

The body occupies a bounded region Ω in $\mathbf{R}^3$, with a smooth (C^1) boundary.

- v is the velocity field, $v(x,t) \in \mathbf{R}^3$, σ is the stress tensor $\sigma(x,t) \in \mathbf{R}^9_s$ and f are the body forces $f(x,t) \in \mathbf{R}^3$ with $x \in \Omega$, $t \in [0,T]$, $T > 0$.
- Let $\omega \in D'(\Omega)^3$, we define $\varepsilon(\omega)$ the strain tensor associated with ω

$$\varepsilon_{ij}(\omega) = \frac{1}{2}\left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i}\right) .$$

- Define Σ, the generalized stress tensor:

$$\Sigma = (\sigma, q) ,$$

and E, the generalized deformation:

$$E = (\varepsilon, 0)\ 0 \in \mathbf{R}^m ,$$

where $q(x,t) \in \mathbf{R}^m$, $(x,t) \in \Omega \times [0,T]$ is the hardening parameter.

- Denote by K the generalized closed convex set of plasticity, which belongs to $\mathbf{R}^9_s \times \mathbf{R}^m$ and delimits the set of physically admissible general stress states in the elastic plastic case.
- We will study the case where the free energy is a quadratic and strictly convex function (see for instance [11] and [6]). Therefore the constitutive law can be written as

$$E(v) \in (A\dot{\sigma}, H\dot{q}) + \partial I_K(\Sigma)$$

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where A is the elastic compliance tensor, and $H \in \mathbf{R}^{n\times n}$ is symmetric, constant and positive definite.
• Denote by $\mathbf{L}^p(\Omega) = (L^p(\Omega))^9_s \times (L^p(\Omega))^m$, $1 \le p \le \infty$.
• We recall the definition of the space of velocity fields with a bounded deformation on an open set Ω of $\mathbf{R}^3$ (see for instance [9], [10], [12])

$$BD(\Omega) = \{u \in L^1(\Omega)^3 \text{ with } \varepsilon_{ij}(u) \in M^1(\Omega) \quad 1 \le i,\, j \le 3\}$$

where $M^1(\Omega)$ is the space of bounded measures on Ω.
$BD(\Omega)$ is a Banach space with the norm

$$\|u\|_{BD(\Omega)} = \|u\|_{L^1(\Omega)^3} + \|\varepsilon(u)\|_{L^1(\Omega)^9_s}\,.$$

• Let adopt the notation $\chi' = P_K(\chi)$, $\chi \in \mathbf{R}^9_s \times \mathbf{R}^m$, where P_K is the projection on K for the usual scalar product of $\mathbf{R}^9_s \times \mathbf{R}^m$.

Existence and uniqueness result

In perfect plasticity (see [1],[5], [6], [8],[9]), in order to obtain an existence result we shall consider first a viscoplastic case. For plasticity with linear hardening, we will consider first the elastic viscoplastic *Perzyna's law*:

$$E(v) = (A\dot{\sigma}, H\dot{q}) + \partial\varphi_\mu(\sigma) \quad \text{if } x \in \Omega \text{ and } t \in [0,T]\,, \tag{1}$$

where

$$\varphi_\mu(\Sigma) = \frac{1}{2\mu}\|\Sigma - P_K\Sigma\|^2_{\mathbf{R}^9_s\times\mathbf{R}^m}\,,$$

together with the motion equation

$$\dot{v} = \operatorname{div}\sigma + f \quad (\text{we assume } \rho = 1 \text{ for simplicity})\,. \tag{2}$$

Initial conditions

$$\begin{cases} \Sigma(0) = \sigma_0 & x \in \Omega \\ v(0) = v_0 & x \in \Omega \end{cases} \tag{3}$$

and boundary conditions

$$\begin{cases} \sigma.n = F^d & \text{if } x \in \partial_F\Omega \quad t \in [0,T] \\ v = v^d & \text{if } x \in \partial_v\Omega \quad t \in [0,T] \end{cases} \tag{4}$$

are prescribed, $\partial\Omega = \partial_v\Omega \cup \partial_F\Omega$

Using the semi–group theory(see [2], [3], [4]) we can obtain the existence and uniqueness of the visco plastic solution:

$$\begin{cases} (\Sigma_\mu, v_\mu) \in W^{1,\infty}(0,T;\mathbf{L}^2(\Omega)\times L^2(\Omega)^3)\,, \\ \operatorname{div}\sigma_\mu(t) \in L^2(\Omega)^3 \quad t\in[0,T], \qquad v_\mu(t) \in H^1(\Omega)^3\,. \end{cases}$$

Making priori estimates (see [8]) and passing to the limit as μ tends to 0, we obtain the existence and uniqueness of (Σ, v):

$$\begin{cases} \Sigma \in W^{1,\infty}(0,T,\mathbf{L}^2(\Omega)),\ \operatorname{div}\sigma \in L^\infty(0,T,L^2(\Omega)^3) \\ v \in W^{1,\infty}(0,T;L^2(\Omega)^3) \cap L^\infty(0,T;BD(\Omega)^3)\,, \end{cases}$$

that satisfies the weak formulation of the elastic plastic law:

$$\int_\Omega (A\dot{\sigma}(t), H\dot{q}(t))(\Sigma(t) - \tau(t))dx + \int_\Omega (v(t) - v^*(t))\operatorname{div}(\sigma(t) - \tau_1(t))dx \le$$
$$\le \int_\Omega E(v^*(t))(\Sigma(t) - \tau(t))dx\,, \tag{5}$$

where $\tau(x,t) = (\tau_1(x,t), \tau_2(x,t)) \in \mathbf{R}^9_s \times \mathbf{R}^m$, $\tau(x,t) \in K$ a.e. , $\tau_1 \in L^\infty(0,T;L^2(\Omega)^9_s)$, $\operatorname{div} \tau_1(t) \in L^2(\Omega)^3$ $t \in [0,T]$, $\tau_2 \in L^\infty(0,T;L^2(\Omega)^m)$ $\tau_1.n = F^d$ in $\partial_F\Omega \times [0,T]$,
together with the motion equation,

$$\operatorname{div} \sigma + f = \dot{v} \quad (\text{we assume } \rho = 1 \text{ for simplicity}), \tag{6}$$

the initial conditions,

$$\begin{cases} \Sigma(0) = \Sigma_0 \\ v(0) = v_0 \end{cases} \tag{7}$$

and boundary conditions on σ,

$$\sigma.n = F^d \quad \text{if } x \in \partial_F\Omega \quad t \in [0,T]. \tag{8}$$

Remark

It is necessary to impose, (see [1], [8],[9]) regularity assumptions for the boundary and initial conditions, in order to obtain the results we have enunciated.

2 - DYNAMIC EVOLUTION OF ELASTIC PLASTIC MATERIALS WITH POSITIVE LINEAR HARDENING

In this section we shall assume that

$$\begin{cases} K = \{(\tau,\eta) \in \mathbf{R}^9_s \times \mathbf{R}^m / F(\tau,\eta) \le 0\} \\ \text{where} \\ \eta = \binom{\eta_1}{\eta_2}, \eta_1 \in \mathbf{R}^9_s,\ \eta_2 \in \mathbf{R}^k,\ \mathbf{R}^m = \mathbf{R}^9_s \times \mathbf{R}^k \\ F(\tau,(\eta_1,\eta_2)) = ((\tau-\eta_1)^D(\tau-\eta_1)^D)^{\frac{1}{2}} - Y(\eta_2), \\ \text{where } (\tau-\eta_1)^D \text{ is the deviator tensor of } (\tau-\eta_1), \\ Y \text{ is a concave function, } Y \in C^1(\mathbf{R}^k) \end{cases} \tag{9}$$

and there is a positive constant d such that

$$d \le \left|\frac{\partial Y}{\partial \eta_2}\right| \le \frac{1}{q} \quad \text{in } \mathbf{R}^k. \tag{10}$$

Remark that K is a natural generalization of the Von Mises' Convex.

Definition of positive hardening (see for instance [11])

There is a positive constant $a > 0$, such that if $(\tau,\eta) \in \partial K$ then

$$\frac{\frac{\partial F}{\partial \eta}(\tau,\eta) H^{-1} \frac{\partial F}{\partial \eta}(\tau,\eta)}{\frac{\partial F}{\partial \tau}(\tau,\eta) A^{-1} \frac{\partial F}{\partial \tau}(\tau,\eta)} \ge a.$$

Remark

Notice that H^{-1} and A^{-1} are constant and positive definite, therefore if the positive hardening assumption holds,

$$\frac{\left|\frac{\partial F}{\partial \eta}(\tau,\eta)\right|^2}{\left|\frac{\partial F}{\partial \tau}(\tau,\eta)\right|^2} \ge a' \qquad \text{with } a' > 0. \tag{11}$$

Some auxiliary results

Lemma 1

If χ belongs to $\mathbf{R}^9_s \times \mathbf{R}^m$, there is $\lambda_\mu(\chi) \geq 0$ such that

$$\partial\varphi_\mu(\chi) = \frac{1}{\mu}(\chi - P_K\chi) = \lambda_\mu(\chi)\frac{\partial F}{\partial\chi}(\chi') \,. \tag{12}$$

PROOF: The result holds because of the smoothness of F, and the convexity of K. Notice that $\lambda_\mu(\chi) = 0$ if $\chi \in K$ and $\lambda_\mu(\chi) > 0$ if $\chi \notin K$.

Lemma 2

Consider Σ_μ, v_μ solution of (1)-(4) and Σ, v solution of (5)-(8) ,therefore

$$\Sigma'_\mu \stackrel{\mu\to 0}{\longrightarrow} \Sigma \text{ in } L^\infty(0,T;\mathbf{L}^2(\Omega)) \tag{13}$$

PROOF: It can be proved, that

$$\Sigma_\mu \to \Sigma \text{ in } L^\infty(0,T;\mathbf{L}^2(\Omega))$$

thus the result holds, because the projection on K is a non-expansive function in $L^\infty(0,T;\mathbf{L}^2(\Omega))$.

Lemma 3

$$\frac{\partial F}{\partial\chi}(\chi)\frac{\partial F}{\partial\chi}(\chi') \geq \left|\frac{\partial F}{\partial\chi}(\chi')\right|^2 \quad \chi \in \mathbf{R}^9_s \times \mathbf{R}^m \,. \tag{14}$$

PROOF:

If $\chi \in K$, $\chi' = \chi$. If $\chi \notin K$, from the convexity of F we obtain

$$\left(\frac{\partial F}{\partial\chi}(\chi) - \frac{\partial F}{\partial\chi}(\chi')\right)\left(\chi - \chi'\right) \geq 0 \,,$$

and $\chi - \chi' = \lambda\frac{\partial F}{\partial\chi}(\chi')$ with $\lambda > 0$,
therefore

$$\left(\frac{\partial F}{\partial\chi}(\chi) - \frac{\partial F}{\partial\chi}(\chi')\right)\frac{\partial F}{\partial\chi}(\chi') \geq 0 \,.$$

Regularity result

Consider (Σ_μ, v_μ) the solution of (1)-(4). From (12) it yields

$$\begin{cases} A\dot\sigma_\mu = \varepsilon(v_\mu) - \lambda_\mu(\sigma_\mu, q_\mu)\dfrac{\partial F}{\partial\tau}(\sigma'_\mu, q'_\mu) \\ H\dot q_\mu = -\lambda_\mu(\sigma_\mu, q_\mu)\dfrac{\partial F}{\partial\eta}(\sigma'_\mu, q'_\mu) \end{cases} \tag{15}$$

where $(\sigma'_\mu, q'_\mu) = P_K(\sigma_\mu, q_\mu)$.

Define

$$\begin{cases} S_1 = \{(x,t) \in \Omega \times [0,T] / \Sigma_\mu(x,t) \in K\} \\ S_2 = \{(x,t) \in \Omega \times [0,T] / \Sigma_\mu(x,t) \notin K\} \end{cases} \tag{16}$$

so that $\Omega \times (0,T) = S_1 \cup S_2$.

If $(x,t) \in S_1$, from (15) we obtain

$$\varepsilon(v_\mu(x,t)) = A\dot{\sigma}_\mu(x,t) . \tag{17}$$

If $(x,t) \in S_2$, let define

$$H_\mu = \frac{\partial F}{\partial \tau}(\sigma_\mu, q_\mu) A\dot{\sigma}_\mu + \frac{\partial F}{\partial \eta}(\sigma_\mu, q_\mu) H \dot{q}_\mu , \tag{18}$$

where H_μ is bounded in $L^\infty(0,T;L^2(\Omega))$.

From (15) together with (18) we deduce

$$H_\mu = \frac{\partial F}{\partial \tau}(\sigma_\mu, q_\mu)(\varepsilon(v_\mu) - \lambda_\mu(\sigma_\mu, q_\mu)\frac{\partial F}{\partial \tau}(\sigma'_\mu, q'_\mu)) - \\ - \lambda_\mu(\sigma_\mu, q_\mu)\frac{\partial F}{\partial \eta}(\sigma_\mu, q_\mu)\frac{\partial F}{\partial \eta}(\sigma'_\mu, q'_\mu) , \tag{19}$$

therefore

$$\lambda_\mu(\Sigma_\mu) = \frac{\frac{\partial F}{\partial \tau}(\Sigma_\mu)\varepsilon(v_\mu) - H_\mu}{\frac{\partial F}{\partial \chi}(\Sigma_\mu)\frac{\partial F}{\partial \chi}(\Sigma'_\mu)} , \tag{20}$$

where $\dfrac{\partial F}{\partial \chi} = \left(\dfrac{\partial F}{\partial \tau}, \dfrac{\partial F}{\partial \eta}\right)$.

Define

$$c(\Sigma_\mu) = 1 - \frac{\frac{\partial F}{\partial \tau}(\Sigma_\mu)\frac{\partial F}{\partial \tau}(\Sigma'_\mu)}{\frac{\partial F}{\partial \chi}(\Sigma_\mu)\frac{\partial F}{\partial \chi}(\Sigma'_\mu)} , \tag{21}$$

from (15) the following expression yields:

$$A\dot{\sigma}_\mu = c(\Sigma_\mu)\varepsilon(v_\mu) + H_\mu \frac{\frac{\partial F}{\partial \tau}(\Sigma'_\mu)}{\frac{\partial F}{\partial \chi}(\Sigma_\mu)\frac{\partial F}{\partial \chi}(\Sigma'_\mu)} . \tag{22}$$

Using (14), the Cauchy–Schwartz's inequality, and properties of the deviator tensor (see [7]), that implies $\left|\dfrac{\partial F}{\partial \tau}(\Sigma'_\mu)\right| = \left|\dfrac{\partial F}{\partial \tau}(\Sigma_\mu)\right| = 1$ we get

$$c(\Sigma_\mu) \geq 1 - \frac{\left|\frac{\partial F}{\partial \tau}(\Sigma'_\mu)\right|^2}{\left|\frac{\partial F}{\partial \chi}(\Sigma'_\mu)\right|^2} ,$$

and from (10), there is a constant $0 < \delta < 1$, such that

$$c(\Sigma_\mu) \geq 1 - \delta , \tag{23}$$

$$\varepsilon(v_\mu) = c(\Sigma_\mu)^{-1} A\dot{\sigma}_\mu - c(\Sigma_\mu)^{-1} R_\mu \,, \tag{24}$$

where

$$R_\mu = H_\mu \frac{\dfrac{\partial F}{\partial \tau}(\Sigma'_\mu)}{\dfrac{\partial F}{\partial \chi}(\Sigma_\mu)\dfrac{\partial F}{\partial \chi}(\Sigma'_\mu)} \,.$$

Thus from (14) the following inequality yields:

$$|R_\mu| \le |H_\mu| \frac{\left|\dfrac{\partial F}{\partial \tau}(\Sigma'_\mu)\right|}{\left|\dfrac{\partial F}{\partial \chi}(\Sigma_\mu)\dfrac{\partial F}{\partial \chi}(\Sigma'_\mu)\right|} \le |H_\mu| \,. \tag{25}$$

If $(x,t) \in S_1$, from (17), there exists a positive constant k such that

$$|\varepsilon(v_\mu(x,t))|_{\mathbf{R}^9_s} \le k |\dot{\sigma}_\mu(x,t)|_{\mathbf{R}^9_s} \,, \tag{26}$$

If $(x,t) \in S_2$, from (24) and (25) we deduce

$$|\varepsilon(v_\mu(x,t))|_{\mathbf{R}^9_s} \le \frac{1}{1-\delta}\left(k|\dot{\sigma}_\mu|_{\mathbf{R}^9_s} + |H_\mu(x,t)|_{\mathbf{R}^9_s}\right) . \tag{27}$$

Therefore we obtain

$$\|\varepsilon(v_\mu(t))\|_{L^2(\Omega)^9_s} \le \frac{1}{1-\delta}\left(k\|\dot{\sigma}_\mu(t)\|_{L^2(\Omega)^9_s} + \|H_\mu(t)\|_{L^2(\Omega)^9_s}\right) . \tag{18}$$

From (26) and (18):

$$\varepsilon(v_\mu) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)^9_s) \,. \tag{29}$$

Using the Korn's inequality (see for instance [5]), we finally get

$$v_\mu \text{ is bounded in } L^\infty(0,T;H^1(\Omega)^3) \,. \tag{30}$$

Therefore, it is possible to extract a subsequence from (v_μ) such that

$$\begin{cases} v_\mu \to v & \text{in } L^\infty(0,T;H^1(\Omega)^3) \text{ weak } * \\ \varepsilon(v_\mu) \to \varepsilon(v) & \text{in } L^\infty(0,T;\mathbf{L}^2(\Omega)) \text{ weak } * \end{cases} \tag{31}$$

where (Σ, v) is the solution of (5)-(8).

From (31) it follows as an importanr result that v satifies the boundary condition:

$$v = v^d \quad \text{on } \partial_v\Omega \times (0,T) \,,$$

so that (Σ, v) satisfies not only the weak constitutive law (5) for elastoplasticity with hardening, it also satisfies the strong one:

$$E(v) \in (A\dot{\sigma}, H\dot{q}) + \partial I_K(\Sigma) \,.$$

In the quasi–static case the same result has been obtained by some authors (see for instance [6],[11]).

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