SPATIAL BUCKLING ANALYSIS: A COMPARISON OF FORMULATIONS

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ABSTRACT

A geometrical stiffness matrix for a spatial beam element is derived in detail using linear Lagrangian interpolation of translations and rotations along the element. It is applied in linearized buckling analyses. For comparison with other geometric matrices derived with cubic Hermitian interpolations for transverse displacements, the same elastic stiffness matrix is used in the examples. It is concluded that both formulations lead to erroneous results, and require similar corrections.

INTRODUCTION

Usual elements for spatial frame analysis are formulated with Hermitian and Lagrangian interpolations. In the case of Hermitian interpolation the usual scheme for slender beams is to use cubic functions for flexure. The flexural rotations are obtained by derivation of transverse displacements, that is, interpolated with quadratic functions. The axial displacement is interpolated with linear functions as well as the torsional rotation. For the Lagrangian interpolation polynomials of the same degree are used independently for rotations and translations. Here we deal with geometrical stiffness matrices, also called initial stress matrices, fo spatial frame elements. Explicit expressions for such matrices are found in literature only in case of elements with Hermitian interpolation. For that reason here we detail the method of obtaining the geometrical matrix for an element with linear Lagrangian interpolation, that will serve as a means of general presentation of the way of obtaining the geometrical matrix allowing computational analysis of problems in which nonlinearities occur associated with changes in the geometry of the structure. These matrices are particularly useful for the evaluation of critical loads. This paper considers only the problem of linearized stability of elastic beams and frames in space, subject to conservative loads elastic and small displacements. This leads to a generalized linear eigenvalue problem with simmetrical matrices.

GEOMETRICAL STIFFNESS MATRIX DERIVATION

Bathe's [1] notation is used in conjunction with a total Lagrangian formulation (T.L.). Considering the equilibrium of a deformable body at time $t+\Delta t$, the principle of virtual work assumes the following form

$$\int_{{}^{o}V} {}^{t+\Delta t}_{\ o}S_{ij}\ \delta\ {}^{t+\Delta t}_{\ o}\varepsilon_{ij}\ {}^{o}dv = {}^{t+\Delta t}R \tag{1}$$

where R is the increment of the total external work and

$${}^{t+\Delta t}_{\ o}S_{ij} = {}^{t}_{o}S_{ij} + {}_{o}S_{ij} \tag{2}$$

$$\delta\ {}^{t+\Delta t}_{\ o}\varepsilon_{ij} = \delta\ {}^{t}_{o}\varepsilon_{ij} + \delta\ {}_{o}\varepsilon_{ij}$$

where the tensors ${}_{o}S_{ij}$ and ${}_{o}\varepsilon_{ij}$ are the stress and strain increments, and ${}^{t}_{o}\varepsilon_{ij}$ are the stress and strains at time t, referred to the state t=0.

Because of $\delta\ {}^{t}_{o}\varepsilon_{ij} = 0$, we have $\delta\ {}^{t+\Delta t}_{\ o}\varepsilon_{ij} = \delta\ {}_{o}\varepsilon_{ij}$ in equation (3).

The strain increment component can be separated into linear and non-linear parts as a function of displacement increments ${}_{o}u_{k}$:

$${}_{o}\varepsilon_{ij} = {}_{o}e_{ij} + {}_{o}\eta_{ij} \tag{4}$$

where

$${}_{o}\eta_{ij} = \frac{1}{2}({}_{o}u_{k,i}\ {}_{o}u_{k,j}) \tag{5}$$

is the nonlinear part of the component of the strain increment.

For the virtual internal work we have the following expression

$$\delta W = \int_{{}^{o}v} ({}^{t}_{o}S_{ij} + {}_{o}S_{ij})\ \delta\,{}_{o}\varepsilon_{ij}\ {}^{o}dv \tag{6}$$

that can be approximated as follows, disregarding products of stress and strain increments.

$$\delta W \doteq \int_{{}^{o}v} {}^{t}_{o}S_{ij}\ \delta\,{}_{o}\varepsilon_{ij}\ {}^{o}dv = \int_{{}^{o}v} {}^{t}_{o}S_{ij}(\delta\,{}_{o}e_{ij} + \delta\,{}_{o}\eta_{ij})\ {}^{o}dv \tag{7}$$

In order to obtain the geometrical stiffness we consider only the work associated with the nonlinear part of the strain increments.

$$\delta W_{NL} = \int_{{}^{o}v} {}^{t}_{o}S_{ij}\ \delta\,{}_{o}\eta_{ij}\ {}^{o}dv \tag{8}$$

Substituting equation (5) into equation (8) and omitting for simplicity the indexes on the left of the variables we obtain

$$\delta W_{NL} = \int_{v} S_{ij}\ \frac{1}{2}(\delta\,u_{k,i}\ \ u_{k,j} + u_{k,i}\ \ \delta u_{k,j})dv \tag{9}$$

Figure 1 shows a linear Lagrangian element, which has two end nodes

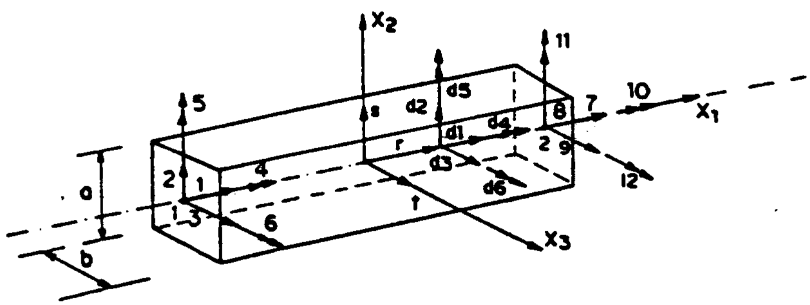


Figure 1. Linear Lagrangian Element

with six degrees-of-freedom per node. The principal moment of inertia axes of the beam element define the local co-ordinates system r,s,t.

The interpolation functions are

$$h_1 = \frac{1}{2}(1 - r), \qquad h_2 = \frac{1}{2}(1 + r) \tag{10}$$

where r is the non-dimensional co-ordinate in the axial direction.

The generalized displacement vector of a cross section of the Lagrangian linear element of Figure 1. is

$$\underline{d}(r) = \frac{1-r}{2}\ \underline{d}(-1) + \frac{1+r}{2}\ \underline{d}(1) \tag{11}$$

Hence we define an interpolation matrix $\underline{N}$ such that

$$\underline{d}(r) = \underline{N}(r)\ g \tag{12}$$

where g is the nodal displacement vector.

The displacement of a point in a cross section in the X_k direction is

$$u_k = \underline{N}_k\ g \qquad (k = 1,2,3) \tag{13}$$

where

$$\underline{N}_k = \underline{N}_k^*\ (s,t)\ \underline{N}(r) \tag{14}$$

and then

$$u_k = \underline{N}_k^*\ (s,t)\ \ \underline{N}(r)\ g \tag{15}$$

Notice we introduced a new combined matrix-index notation for convenience. This last equation is valid, too, for increments $\delta\underline{u}$ and δg.

Finally, substituting equation (12) into (5), we obtain

$$u_k = \underline{N}_k^*(s,t)\ \underline{d}(r) \tag{16}$$

The derivative of equation (15) is

$$u_{k,i} = \underline{N}_{k,i}^*(s,t)\ \underline{N}(r)\ g + \underline{N}_k^*(s,t)\ \underline{N}_{k,i}(r)\ g \tag{17}$$

After grouping the terms and simplifying the notation, we obtain

$$u_{k,i} = (\underline{N}_{k,i}^*\ \underline{N} + \underline{N}_k^*\ \underline{N}_{,i});\ \underline{u}_{k,j} = (\underline{N}_{k,j}^*\ \underline{N} + \underline{N}_k^*\ \underline{N}_{,j}) \tag{17}$$

Using equation (17) into equation (9), we obtain

$$\delta W_{NL} = \int_v S_{ij}\ \frac{1}{2}\ \ [\delta g^T(\underline{N}^T\ \underline{N}_{k,i}^{*T} + \underline{N}_{,i}^T\ \underline{N}_k^{*T}\ )(\underline{N}_{k,j}^*\ \underline{N} + \underline{N}_k^*\ \underline{N}_{,j})g$$

$$+\ \ g^T(\underline{N}^T\ \underline{N}_{k,i}^{*T} + \underline{N}_{,i}^T\ \underline{N}_k^{*T})(\underline{N}_{k,j}^*\ \underline{N} + \underline{N}_k^*\ \underline{N}_{,j})\,\delta\, g]\ dv \tag{18}$$

We evaluate next the interpolation matrices.

The first component of $\underline{d}$ in equation (12) is

$$d_1 = [\ \frac{1-r}{2}\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ \frac{1+r}{2}\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]\ g \tag{19}$$

The non-dimensional coordinates are

$$r = \frac{2X_1}{L}\ \ ,\ s = \frac{2X_2}{a}\ \ ,\ \ t = \frac{2X_3}{b} \tag{20}$$

Substituting the first equation (20) into (19), we obtain

$$d_1 = [(1/2 - \frac{X_1}{L})\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ (1/2 + \frac{X_1}{L})\ 0\ \ 0\ \ 0\ \ 0\ \ 0]\ \underset{\sim}{q} \qquad (21)$$

Generalizing for the other five components, we obtain $\underset{\sim}{N}$.

In order to express the section displacements in the global co-ordinate system, we use the interpolation matrix $\underset{\sim}{N}^*$. As we can see in Figure 2, the displacements of a point in a cross section depend on the trans-

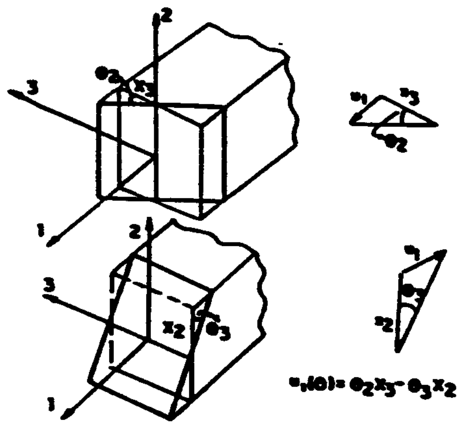


Figura 2. Displacements of a point in a cross section

lational degrees of freedom d_1, d_2, d_3 and the rotational degrees of fredom, d_4, d_5, d_6 and they will be given by equation (16), which can be written as

$$\begin{aligned} u_1 &= [1\ \ \theta\ \ 0\ \ 0\ \ X_3\ -X_2\ ]\ \underset{\sim}{d} \\ u_2 &= [0\ \ 1\ \ 0\ -X_3\ \ 0\ \ \ 0\ ]\ \underset{\sim}{d} \\ u_3 &= [0\ \ 0\ \ 1\ \ 1\ \ X_2\ \ 0\ ]\ \underset{\sim}{d} \end{aligned} \qquad (22)$$

Hence the complete matrix $\underset{\sim}{N}^*$ is

$$\underset{\sim}{N}^* = \begin{bmatrix} 1 & 0 & 0 & 0 & X_3 & -X_2 \\ 0 & 1 & 0 & -X_3 & 0 & 0 \\ 0 & 0 & 1 & X_2 & 0 & 0 \end{bmatrix} \qquad (23)$$

In order to evaluate equation (18), we need the expressions of the derivatives of the matrices $\underset{\sim}{N}$ and $\underset{\sim}{N}^*$

$$\underset{\sim}{N}_{,1} = \frac{1}{2} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (24)$$

$$\underset{\sim}{N}_{,2} = 0$$

$$\underset{\sim}{N}_{,3} = 0$$

In a similar way, the derivatives of $\underset{\sim}{N}^*$ are

$$\underset{\sim}{N}^*_{1,1} = \underset{\sim}{N}^*_{2,1} = \underset{\sim}{N}^*_{2,2} = \underset{\sim}{N}^*_{3,1} = \underset{\sim}{N}^*_{3,3} = 0$$

$$\underset{\sim}{N}^*_{1,2} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1] \,, \quad \underset{\sim}{N}^*_{1,3} = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0]$$

$$\underset{\sim}{N}^*_{2,3} = [0 \quad 0 \quad 0 \quad -1 \quad 0 \quad 0] \,, \quad \underset{\sim}{N}^*_{3,2} = [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0] \qquad (25)$$

We define matrix $\underset{\sim}{H}$ as

$$\underset{\sim}{H} = \underset{\sim}{N}^* \; \underset{\sim}{N} \qquad (26)$$

then, the derivative of this matrix is

$$\underset{\sim}{H}_{,j} = \underset{\sim}{N}^*_{,j} \; \underset{\sim}{N} + \underset{\sim}{N}^* \; \underset{\sim}{N}_{,j} \qquad (27)$$

Using this last expression in equation (18), we obtain

$$\delta W_{NL} = \int_v S_{ij} \frac{1}{2} \left(\delta g^T \; \underset{\sim}{H}^T_{,i} \; \underset{\sim}{H}_{,j} \; g + g^T \; \underset{\sim}{H}^T_{,i} \; \underset{\sim}{H}_{,j} \, \delta g \right) dv \qquad (28)$$

Because of the symmetry of the tensor S_{ij}, we can write

$$\delta W_{NL} = \frac{1}{2} \int_v S_{ij} [\delta g^T (\underset{\sim}{H}^T_{,i} \; \underset{\sim}{H}_{,j} + \underset{\sim}{H}^T_{,i} \; \underset{\sim}{H}_{,j}) \; g] \; dv \qquad (29)$$

and

$$\delta W_{NL} = \delta g^T \left(\int_v \underset{\sim}{H}^T_{,i} \; S_{ij} \; \underset{\sim}{H}_{,j} \; dv \right) g \qquad (30)$$

that can be written in a different form

$$\delta W_{NL} = \delta g^T \; \underset{\sim}{K}_G \; g \qquad (31)$$

For any arbitrary g, we have

$$K_G = \int_v H^T_{,i}\, S_{ij}\, H_{,j}\, dv \qquad \begin{array}{l} i=1,2,3 \\ j=1,2,3 \end{array} \tag{32}$$

where K_G is the geometrical stiffness matrix. Notice that tensor S_{ij} is symmetrical with components $S_{22} = S_{23} = S_{33} = 0$ for the beam element. Hence the expanded form of equation (32) is

$$K_G = \int_v (H^T_{,1}\, S_{11}\, H_{,1} + H^T_{,2}\, S_{21}\, H_{,1} + H^T_{,1}\, S_{12}\, H_{,2} +$$

$$+ H^T_{,3}\, S_{31}\, H_{,1} + H^T_{,1}\, S_{13}\, H_{,3})\, dv \tag{33}$$

Substituting equations (24) and (25) into this last expression, we obtain

$$K_G = \int_v (S_{11} \frac{1}{L^2} A + S_{21} \frac{1}{L} \bar{B} + S_{12} \frac{1}{L} B^T + S_{31} \frac{1}{L} \bar{C} +$$

$$S_{13} \frac{1}{L} \bar{C}^T)\, dv \tag{34}$$

We can define

$$B = \bar{B} + \bar{B}^T \quad , \qquad C = \bar{C} + \bar{C}^T \tag{25}$$

With the above relations, equation (34) is rewritten as follows

$$K^L_G = \int_v (S_{11} \frac{1}{L^2} A + S_{21} \frac{1}{L} B + S_{31} \frac{1}{L} C)\, dv \tag{36}$$

The expressions of A, B, $\bar{B}$, C, $\bar{C}$, for the linear Lagrangian functions are found in reference [2] here obviated for reasons of space.

To integrate equation (36), we have to take into account that

$$S_{11} = \frac{N}{A} + \frac{M_2 X_3}{I_2} - \frac{M_3 X_2}{I_3} \tag{37}$$

where N is the normal force, I_2 and I_3 are the principal moments of inertia, M_2 and M_3 are the bending moments in direction of co-ordinates 2 and 3, defined as

$$N = \int_A S_{11}\, dA, \quad M_2 = \int_A S_{11}\, X_3\, dA, \quad M_3 = -\int_A S_{11}\, X_2\, dA \tag{38}$$

In addition, we define the shear forces as

$$Q_2 = \int_A S_{12}\, dA, \quad Q_3 = \int_A S_{13}\, dA \tag{39}$$

and the torque as

$$T = \int_A (-S_{12}\, X_3 + S_{13}\, X_2)\, dA \tag{40}$$

For reasons of convenience in the computer implementation we use the Argyris' [3] notation that express the resulting forces in a cross section in terms of natural forces described as follows

$$N = P_{n_1}, \quad Q_2 = -\frac{2}{L} P_{n_3}, \quad Q_3 = -\frac{2}{L} P_{n_5}, \quad T = P_{n_6}$$

$$M_2 = P_{n_4} - \frac{2X_1}{L} P_{n_5}, \quad M_3 = -P_{n_2} + \frac{2X_1}{L} P_{n_3} \tag{41}$$

The final expression of the geometrical stiffness matrix K_G^L, valid for doubly-simmetrical cross sections, is expressed in terms of natural forces as shown in table I.

$$[K_G^L] = \begin{bmatrix}
a & & & & f & -e & -a & & & & -h & g \\
 & a & & -f & & & & a & & h & & \\
 & & a & e & & & & & -a & -g & & \\
 & & & b & & & & f & -e & -b & & \\
 & & & & c & & -f & & & & -c & \\
 & & & & & d & e & & & & & -d \\
 & & & & & & a & & & & h & -g \\
 & & & & & & & a & & -h & & \\
 & \text{symmetric} & & & & & & & a & g & & \\
 & & & & & & & & & b & & \\
 & & & & & & & & & & c & \\
 & & & & & & & & & & & d
\end{bmatrix}$$

Table I. Geometrical stiffness matrix

where $a = \frac{1}{L} P_{n_1}$, $b = \frac{I_1}{AL} P_{n_1}$, $c = \frac{I_2}{AL} P_{n_1}$, $d = \frac{I_3}{AL} P_{n_1}$ and

$$e = \frac{1}{L}(P_{n_2} + P_{n_3}), \quad f = \frac{1}{L}(P_{n_4} + P_{n_5}), \quad g = \frac{1}{L}(P_{n_2} - P_{n_3}), \text{ and}$$

$$h = \frac{1}{L}(P_{n_4} - P_{n_5}).$$

The general method could be applied to non-symmetrical sections, with more cumbersome integration.

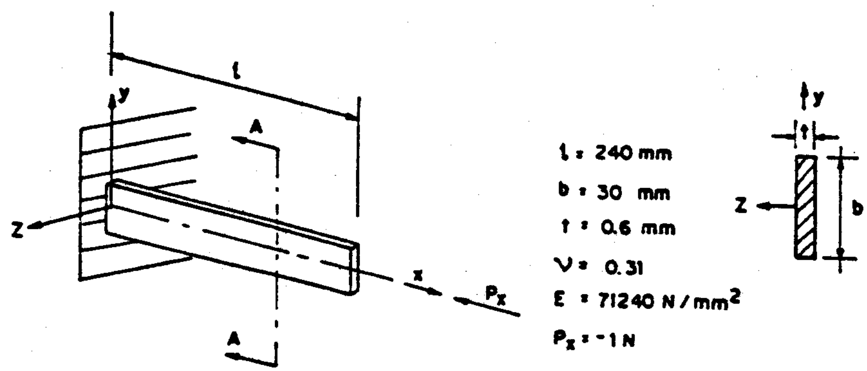
NUMERICAL RESULTS

We adopted as an analysis tool an available and tested program [2], in which were implemented several geometric matrices, formulated with Hermitian interpolation. The matrices are here in designated as $[K_G]_{c.c}$, $[K_G]_{s.c}$, $[K_G^S]_{s.c}$, and $[K_G^L]$, where the subscript c.c means consideration of the shear forces, s.c. no shear forces, the superscript S means matrix with semi-tangencial correction. The last one is developed in this work with linear Lagrangian interpolation.

We used the same element elastic stiffness matrix with Hermitian interpolation, in all calculations.

Example 1. Axially Loaded Cantilever beam.

This classical buckling example serves to show the convergence to the theoretical critical load value, when the beam is divided into several elements. It is noted that the obtained results through the matrices with Hermitian interpolation, converge more quickly than with Lagrangian interpolation. The results are summarized in Table II.



$$P_{cr.} = \frac{\pi}{4}\,\frac{E\,I_y}{\ell^2} = 1.6470\ \text{N}$$

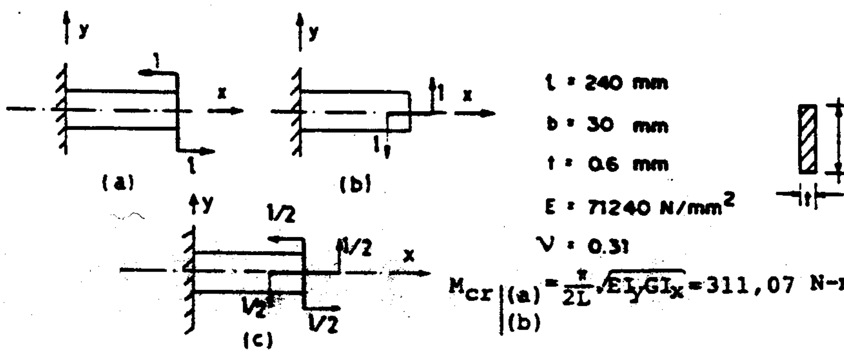
Figure 3. Axially Loaded Cantilever Beam.

Nº of Elements	P_{cr} (n)				
	$[K_G]_{c.c}$	$[K_G^S]_{c.c}$	$[K_G]_{s.c}$	$[K_G^S]_{s.c}$	$[K_G^L]$
1	1.6601	1.6601	1.6601	1.6601	2.0019
3	1.6479	1.6479	1.6479	1.6479	1.6845
5	1.6479	1.6479	1.6479	1.6479	1.6601
6	1.6479	1.6479	1.6479	1.6479	1.6570
10	1.6479	1.6479	1.6479	1.6479	1.6501
15	1.6479	1.6479	1.6479	1.6479	1.6479

Table II. Values of convergence test

Example 2. Cantilever beam loaded with quasitangential and semi-tangential bending moments.

It should be mentioned that both quasi- and semi-tangential bending moments can be generated by the mechanisms represented in Figure 4. (a), (b) and (c), consisting in rigid levers and constant directional forces as



ℓ = 240 mm

b = 30 mm

t = 0.6 mm

E = 71240 N/mm²

ν = 0.31

$$M_{cr}\Big|_{(a),(b)} = \frac{\pi}{2L}\sqrt{EI_yGI_x} = 311,07 \text{ N-mm}$$

$$M_{cr}\Big|_{(c)} = \frac{\pi}{L}\sqrt{EI_yGI_x} = 622.21 \text{ N-mm}$$

Figure 4. Cantilever with different load cases

pointed out by Ziegler [4] and Argyris [3].

In case (a), where a 90° angle exists between the bar and beam axis the, results provided by the matrices $[K_G]_{c.c}$ and $[K_G^L]$ converged to a value that is the half of the theoretical value, while the matrix $[K_G^S]_{s.c}$ gave a result twice greater than the theoretical. In case (b), in which the angle between the bar and the beam axis is null, the matrices $[K_G]_{c.c}$ and $K_G^L]$ results converged to the theoretical value, $[K_G^S]_{s.c}$ converged to the same erroneous value of case and those (a).

In case (c), with semitangential bending moment application, only the matrices $[K_G^S]_{c.c}$ and $[K_G^S]_{s.c}$ converged to the theoretical value. All the results for other matrices converged to the quasitangential value.

The last line of Table III, corresponds to the obtained results after modifying the data for the program through a matrix of nodal correction for

case	Nº of Elem.	M_{cr} (N-mm)				
		$[K_G^S]^*_{c.c}$	$[K_G^S]_{c.c}$	$[K_G]_{s.c}$	$[K_G^S]_{s.c}$	$[K_G^L]$
(a)	15	127.418	311.097	311.097	623.078	127.418
(b)	15	311.097	311.097	311.097	623.078	311.097
(c)	15	311.097	623.078	311.097	623.078	311.097
(c)*	15	623.078	-	623.078		623.078

Table III. Results for Example 2

the cases of matrices with quasitangential formulation, and linear Lagrangian interpolation. This correction was made to consider the application of the moment in a semitangential manner. We see that after such correction, all results agree with the theoretical value.

FINAL CONSIDERATIONS

Example 1 shows the limitations of the element with linear Lagrangian interpolation in the convergence test. Although in linearized stability problems higher order formulations provide more accurate solutions, difficulties may arise in nonlinear analysis. Therefore, it is recommended to use lower order polynomials and more elements, rather than sophisticated elements.

Example 2 has shown that only the $[K_G^S]_{c.c}$ matrix, that includes the so-called semitangential correction, leads to results consistent with the form of the load application when flexural-torsional buckling occurs.

We observe that the results obtained through matrix $[K_G^L]$ converge to the same obtained results with matrix $[K_G]_{c.c}$. This indicates that to obtain coherent results it is necessary to make a correction in $[K_G^L]$ similar to the one found in $[K_G]_{c.c}$. This occurs in spite of the fact that $[K_G^L]$ and $[K_G]_{c.c}$ are deduced using different kinematic variables to represent the rotations due to flexure.

It should be noted that the results obtained through matrix $[K_G^S]_{s.c}$ show that not only the nodal correction is relevant but also that for flexural-torsional buckling the consideration of the shear forces is very important in a geometrical stiffness matrix consistent formulation, even in the case of frames composed of slender beams.

REFERENCES

[1] Bathe, K.J., "Finite Element Procedure in Engineering Analysis". Prentice-Hall, Inc., Englewood Cliffs, N.J., 1982.

[2] Ronda Vásquez, J.A., "Estudo Comparativo de Matrizes Geométricas para Análise da Estabilidade de Pórticos Espaciais". MSc. Thesis, Pontifícia Universidade Católica do Rio de Janeiro, R.J., December, 1987.

[3] Argyris, J.H. Hilbert, O., Malejannakis, G.A., Scharpf, D.W., "On the Geometrical Stiffness of a Beam in Space - a Consistent V.W. Approach", Computer Methods in Applied Mechanics and Engineering, vol. 20, 1979, pp. 105-131.

[4] Ziegler, H., "Principle of Structural Stability, 2nd ed., Birkhauser" Verlage, Basel und Stuttgart, 1977.