SOLUTION OF MINDLIN PLATES WITH UNILATERAL CONTACT SUPPORTS BY A DIRECT FORMULATION

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RESUMEN

El presente trabajo trata del comportamiento no lineal de placas con apoyos unilaterales dentro de la teoría elástica clásica. El análisis se lleva a cabo usando una formulación directa a través de la minimización del funcional de la energía potencial total con restricciones. La introducción de la técnica de elementos finitos conduce a un problema de programación cuadrático a ser resuelto con el algoritmo de Lemke.

ABSTRACT

The present work deals with the nonlinear behavior of plates with unilateral supports within the classic elastic theory. The analysis is carried out by using a direct formulation through the minimization of the total potential energy functional with restrictions. The introduction of the finite element technique leads to a quadratic programming problem to be solved by Lemke's algorithm.

INTRODUCTION

The contact of one body with another is the manner through which a structure transmits forces to its supports.

Some structures present the so-called ambiguous or unilateral supports; that is, they have the possibility to lose contact with their supports under certain load conditions (Signorini's Problem [1, 2]).

Contact problems in solid mechanics are nonlinear because the contact area is not known prior to the application of loads and require special mechanical and mathematical considerations.

These problems in elastostatics (small displacements) can be studied within the theory of variational inequalities whose solution can be obtained by finite element approximations and mathematical programming techniques.

This work deals with the nonlinear behavior of plates with unilateral supports. The analysis is carried out by using a direct formulation, avoiding in the discretization, the introduction of artificial finite elements required for incremental techniques.

The solution of the problem can be determined in two ways: using, either a variational formulation (theorem of virtual work); or by the constrained minimization potential energy functional. Once the finite element method is applied [3], the first one leads to a system of variational inequalities and the second one to a quadratic programming problem.

Up to this point, both procedures can be transformed into a linear complementarity problem to be solved by Lemke's algorithm [4], through which the exact solution of the discrete problem is obtained in a finite number of steps.

This work is based in others, for example the studies of Feijóo and Barbosa [5], and is developed under the point of view of the minimization of the energy functional (primal problem). By the introduction of the Lagrange multipliers, which release the restrictions, the primal problem is carried to its dual. Its solution is associated with the linear complementarity problem.

Numerical results are obtained by modification of a Mindlin plate finite element program [6] with reduced integration, where unilateral restrictions and Lemke algorithm for solving linear complementarity problems, are introduced.

The advantages can be appreciated in the inputs and in the outputs whose transparency allows us to recognize immediately the type of support. Some examples were carried out giving satisfactory results.

FORMULATION OF THE UNILATERAL PROBLEM

Consider an elastic deformable body which occupies an open region Ω in the three-dimensional Euclidean space $\mathbb{R}^3$, Figure 1.

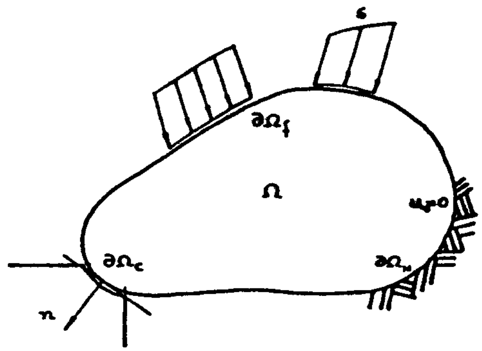

FIGURE 1

The boundary $\partial\Omega$ of the body Ω is regular and consists of three disjoint parts: $\partial\Omega_u$, $\partial\Omega_f$ and $\partial\Omega_c$. The portion $\partial\Omega_u$ identifies the part of the boundary with prescribed displacements. On the part $\partial\Omega_f$ forces are prescribed. And the part $\partial\Omega_c$ is the actual contact surface that is unknown because of the unilateral nature of the restriction. Thus, the points on $\partial\Omega_c$ can either remain in contact with the support or detach from it. The body is under the action of the body forces b, boundary forces s on the boundary $\partial\Omega_f$, and prescribed displacements u_0 on $\partial\Omega_u$.

The problem of finding the deformation and contact force for equilibrium configurations of the body Ω, under certain boundary and loading conditions, is called Signorini's problem. In this work the theoretical mechanics Signorini's problem is discussed in regard to the special case in which no friction exists on the contact surface.

For the case of small deformations, the displacement field is small enough so that higher-order terms of the displacement gradient can be neglected in the equilibrium and strain-displacement equations.

With all the considerations stated above, Signorini's equilibrium problem without friction assumes the form of a boundary value problem whose strong formulation takes the form:

$$\operatorname{div} T(u) + b = 0 \qquad \text{in } \Omega \qquad (1.a)$$

$$u = u_s = 0 \qquad \text{on } \partial\Omega_u \qquad (1.b)$$

$$T(u)n = s \qquad \text{on } \partial\Omega_f \qquad (1.c)$$

$$r_n \le 0;\ u_n \le 0;\ r_n.u_n = 0 \qquad \text{on } \partial\Omega_c \qquad (1.d)$$

where T is the stress tensor and n represents the outward normal to the surface of the body, so that $r_n = r.n$, $u_n = u.n$, are the normal components of the reaction r and the displacement u on the boundary $\partial\Omega_c$. Conditions (1.d) are called complementarity conditions and express the alternative that when $u_n = 0$, contact occurs ($T(u)n = r_n.n$) and when $r_n = 0$, there is no contact. Besides T and u are related by the elasticity tensor $\mathbb{C}$ in the constitutive equation:

$$T = \mathbb{C}E(u)$$

with E(u) the strain tensor.

The weak formulation of the problem is established through the principle of virtual work by multiplying (1.a) by an arbitrary smooth test function v such that v = 0 on $\partial\Omega_u$. Integrating by parts, we have:

$$\int_\Omega \mathbb{C}E(u).E(v)\ d\Omega = \int_\Omega b.v\ d\Omega + \int_{\partial\Omega_f} s.v\ d\partial\Omega +$$

$$+ \int_{\partial\Omega_c} r.v\ d\partial\Omega \qquad \forall\ v \in H \qquad (2)$$

where H is the set of all functions of $(H^1(\Omega))^3$ which vanish on $\partial\Omega_u$. The last integral depends on the contact zone which is unknown, therefore it cannot be evaluated. If the set of admissible displacements is restricted to:

$$\mathbb{K} = \Big\{ v \in H\text{: } v \text{ sufficiently regular in } \Omega,$$

$$v = 0 \text{ on } \partial\Omega_u,\ v_n \le 0 \text{ on } \partial\Omega_c \Big\} \qquad (3)$$

and $r_n \le 0$, then $r.v = r_n.v_n \ge 0$ on $\partial\Omega_c$ and the virtual work (2) can be substituted by a variational inequality:

$$\int_{\Omega} \mathbb{C}E(u).E(v)\ d\Omega \geq \int_{\Omega} b.v\ d\Omega + \int_{\partial\Omega_f} s.v\ d\partial\Omega \qquad \forall\ v \in \mathbb{K} \qquad (4)$$

The set $\mathbb{K}$ of all admissible displacements is a closed convex set and a linear cone. Both are useful conditions for the existence and uniqueness of the solution of (4).

If $\mathbb{C}$ is symmetric there exists a function of the form:

$$\phi(E(u)) = \frac{1}{2}\ \mathbb{C}E(u).E(u) \qquad (5)$$

from which can be derived the associated stress state of E(u):

$$T = T(E) = \frac{\partial\phi}{\partial E} = \mathbb{C}E(u)$$

(5) is called the potential energy function which is a quadratic form positive definite and convex.

Introducing the functional:

$$F(v) = \int_{\Omega} \phi(v)\ d\Omega - \int_{\Omega} b.v\ d\Omega - \int_{\partial\Omega_f} s.v\ d\partial\Omega \qquad (6)$$

it can be proved that the solution $u \in \mathbb{K}$ of (4) is also the solution of the constrained minimization problem:

$$F(u) = \min\ (F(v);\ v \in \mathbb{K}\ ) \qquad (7)$$

THEORY OF PLATES

The theory of plates with transverse shear deformations, assumes that particles on the plate originaly on a line that is normal to the undeformed middle surface remain on a straight line during deformation, but this line is not necessarily normal to the deformed middle surface, Figure 2.

The displacement components of a point coordinates x, y, z, are:

$$u = z\ \beta_x(x,y) \qquad (8.a)$$
$$v = -z\ \beta_y(x,y) \qquad (8.b)$$
$$w = w(x,y) \qquad (8.c)$$

where w is the transverse displacement, β_x and β_y are the rotations of the normal to the undeformed middle surface in the x-z and y-z planes, respectively, Figure 3.

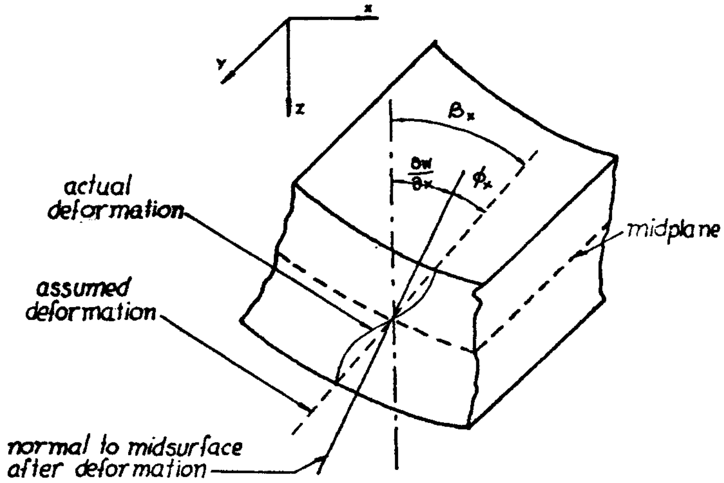

FIGURE 2

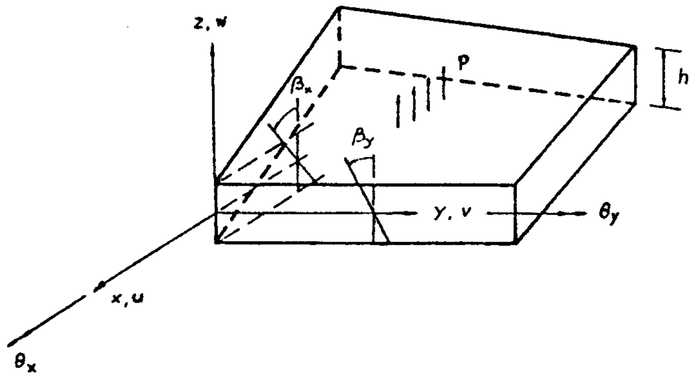

FIGURE 3

The bending strains ε_{xx}, ε_{yy}, γ_{xy}, vary linearly through the plate thickness and are given by the curvature of the plate:

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{bmatrix} = \begin{bmatrix}\partial u/\partial x\\ \partial v/\partial y\\ \partial u/\partial y + \partial v/\partial x\end{bmatrix} = z\begin{bmatrix}\partial\beta_x/\partial x\\ -\partial\beta_y/\partial y\\ \partial\beta_x/\partial y - \partial\beta_y/\partial x\end{bmatrix} \qquad (9)$$

whereas the transverse shear strains are assumed to be constant through the thickness:

$$\begin{bmatrix}\gamma_{yz}\\ \gamma_{zx}\end{bmatrix} = \begin{bmatrix}\partial w/\partial y - \beta_y\\ \partial w/\partial x + \beta_x\end{bmatrix} \qquad (10)$$

Considering the plate element of Figure 3, the expression for the total potential $F(w,\beta_x,\beta_y)$ is:

$$F(w,\beta_x,\beta_y) = \frac{1}{2}\int_A\int_{-h/2}^{h/2}\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy}\end{bmatrix}\begin{bmatrix}\tau_{xx}\\ \tau_{yy}\\ \tau_{xy}\end{bmatrix} dz\, dA\, +$$

$$+\frac{k}{2}\int_A\int_{-h/2}^{h/2}\begin{bmatrix}\gamma_{yz} & \gamma_{zx}\end{bmatrix}\begin{bmatrix}\tau_{yz}\\ \tau_{zx}\end{bmatrix} dz\, dA - \int_A w\, p\, dA \qquad (11)$$

where k is a constant to account for the actual nonuniformity of the shearing stresses.

The state of stress in the plate corresponds to plane stress conditions ($\tau_{zz} = 0$) and considering an isotropic material we can write:

$$\begin{bmatrix}\tau_{xx}\\ \tau_{yy}\\ \tau_{xy}\end{bmatrix} = z\,\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\partial\beta_x/\partial x\\ -\partial\beta_y/\partial y\\ \partial\beta_x/\partial y - \partial\beta_y/\partial x\end{bmatrix} \qquad (12)$$

$$\begin{bmatrix}\tau_{yz}\\ \tau_{zx}\end{bmatrix} = \frac{E}{2(1+\nu)}\begin{bmatrix}\partial w/\partial y - \beta_y\\ \partial w/\partial x + \beta_x\end{bmatrix} \qquad (13)$$

Substituting (9), (10), (12), (13), into (11), we obtain:

$$F(w,\beta_x,\beta_y) = \frac{1}{2}\int_A K^T C_b K \, dA + \frac{1}{2}\int_A \gamma^T C_s \gamma \, dA - \int_A w.p \, dA \qquad (14)$$

where:

$$K = \begin{bmatrix} \partial\beta_x/\partial x \\ -\partial\beta_y/\partial y \\ \partial\beta_x/\partial y - \partial\beta_y/\partial x \end{bmatrix} \qquad (15.a)$$

$$\gamma = \begin{bmatrix} \partial w/\partial y - \beta_y \\ \partial w/\partial x + \beta_x \end{bmatrix} \qquad (15.b)$$

$$C_b = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad (15.c)$$

$$C_s = \frac{Ehk}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (15.d)$$

In the finite element analysis of an assemblage of elements, we only need to enforce interelement continuity on w, β_x and β_y, and we use:

$$w_N = h_i w_i \qquad (16.a)$$

$$\beta_x^N = h_i \theta_i \qquad (16.b)$$

$$\beta_y^N = h_i \theta_i \qquad (16.c)$$

where the h_i are the interpolation functions.

Besides, in order to reduce the size of the problem we make use of a substructuring technique by segregating constrained degrees of freedom.

By replacing (16) into (14) we arrive to the following segregated matrix expression of the total potential:

$$F(w_N,\theta_x^N,\theta_y^N) = \frac{1}{2}\begin{bmatrix} w_i,\theta_x^i,\theta_y^i \end{bmatrix} \begin{bmatrix} A_{ij} & D_{ij} & E_{ij} \\ D^T_{ij} & B_{ij} & F_{ij} \\ E^T_{ij} & F^T_{ij} & C_{ij} \end{bmatrix} \begin{bmatrix} w_j \\ \theta_x^j \\ \theta_y^j \end{bmatrix} -$$

$$- \begin{bmatrix} w_i,\theta_x^i,\theta_y^i \end{bmatrix} \begin{bmatrix} Q_{Aj} \\ Q_{Bj} \\ Q_{Cj} \end{bmatrix} \qquad (17)$$

By calling:

$$D = \frac{Eh^3}{12(1-\nu^2)} \qquad K = \frac{Ehk}{2(1+\nu)}$$

we distinguish:

$$A_{ij} = \int_A K \left[\frac{\partial h_i}{\partial y}\frac{\partial h_j}{\partial y} + \frac{\partial h_i}{\partial x}\frac{\partial h_j}{\partial x}\right] dA$$

$$B_{ij} = \int_A D \left[\frac{\partial h_i}{\partial y}\frac{\partial h_j}{\partial y} - \frac{1-\nu}{2}\frac{\partial h_i}{\partial x}\frac{\partial h_j}{\partial x}\right] dA$$

$$C_{ij} = \int_A \left\{ D \left[\frac{\partial h_i}{\partial x}\frac{\partial h_j}{\partial x} + \frac{1-\nu}{2}\frac{\partial h_i}{\partial y}\frac{\partial h_j}{\partial y}\right] + K \left[h_i h_j\right] \right\} dA$$

$$D_{ij} = \int_A K \left[- \frac{\partial h_i}{\partial y} h_j \right] dA$$

$$D^T_{ij} = \int_A K \left[- h_i \frac{\partial h_j}{\partial y} \right] dA$$

$$E_{ij} = \int_A K \left[\frac{\partial h_i}{\partial x} h_j\right] dA$$

$$E^T_{ij} = \int_A K \left[h_i \frac{\partial h_j}{\partial x} \right] dA$$

$$F_{ij} = \int_A D \left[- \nu \frac{\partial h_i}{\partial y} \frac{\partial h_j}{\partial x} - \frac{1-\nu}{2} \frac{\partial h_i}{\partial x} \frac{\partial h_j}{\partial y} \right] dA$$

$$F^T_{ij} = \int_A D \left[- \nu \frac{\partial h_i}{\partial x} \frac{\partial h_j}{\partial y} - \frac{1-\nu}{2} \frac{\partial h_i}{\partial y} \frac{\partial h_j}{\partial x} \right] dA$$

$$Q^j_A = \int_A p^{\cdot} h_j \, dA$$

MATHEMATICAL QUADRATIC PROBLEM

$F(w_N, \theta^N_x, \theta^N_y)$ is now a function of a finite number of degrees of freedom w_i, θ^N_x, θ^N_y, whose minimization would provide the coefficients of the solution appoximation.

It is convenient to construct a reduced primal problem by condensation of all degrees of freedom non related to the contact surface.

The condensation process is possible provided that the de formable body is properly restrained. The procedure consists in the steps explained below.

The mathematical quadratic problem is to minimize the objective function $F(w_N, \theta^N_x, \theta^N_y)$ subject to the inequality constraints: $Gw \le c$, and non-negativity restrictions: $w \ge 0$.

$$\min_{w,\theta_x,\theta_y} \left\{ \begin{array}{l} F(w,\theta_x,\theta_y) \\ Gw \le 0 \\ \;\; w \ge 0 \end{array} \right\} \qquad (18)$$

As the minimization over θ_x and θ_y is unconstrained:

$$\frac{\partial F(w,\theta_x,\theta_y)}{\partial \theta_x} = 0 \qquad (19.a)$$

$$\frac{\partial F(w,\theta_x,\theta_y)}{\partial \theta_y} = 0 \qquad (19.a)$$

we can eliminate both unknowns from the functional. They will be calculated later, once the value of w be obtain, from the following expresions:

$$\theta_x = H^{-1}(T - Ww) \qquad (20.a)$$

$$\theta_y = Xw - Y \qquad (20.b)$$

where:

$$H = B - FC^{-1}F^T$$

$$T = Q_B - FC^{-1}Q_C$$

$$W = D^T - FC^{-1}E^T$$

$$X = C^{-1}F^TH^{-1}W - C^{-1}E^T$$

$$Y = C^{-1}F^TH^{-1}T - C^{-1}Q_C$$

The expression (17) can be rewritten in absolute notation as:

$$F(w,\theta_x,\theta_y) = \left\{\frac{1}{2}\left[w^TAw + w^TD\theta_x + w^TE\theta_y\right] - w^TQ_A + \right.$$

$$+ \frac{1}{2}\left[\theta_x^TAw + \theta_x^TD\theta_x + \theta_x^TE\theta_y\right] - \theta_x^TQ_B +$$

$$\left. + \frac{1}{2}\left[\theta_y^TAw + \theta_y^TD\theta_x + \theta_y^TE\theta_y\right] - \theta_y^TQ_C \right\} \qquad (21)$$

By replacing (20.a) and (20.b) into (21), this reduces to the primal problem:

$$F(w) = \frac{1}{2} w^TKw - w^TV + const \qquad (22)$$

with:

$$K = A - DH^{-1}W - W^T H^{-T} D^T + EX + X^T E^T + W^T H^{-T} BH^{-1} W -$$
$$- W^T H^{-T} FX - X^T F^T H^{-1} W + X^T CX$$

$$V = - \Big[DH^{-1}T - EY - Q_A - W^T H^{-T} BH^{-1} T + X^T F^T H^{-1} T +$$
$$+ W^T H^{-T} FY - W^T H^{-T} Q_B - X^T CY + X^T Q_C \Big]$$

and the inequality constraints.

REDUCTION TO A LINEAR COMPLEMENTARITY PROBLEM

The quadratic problem is now:

$$\min_{\substack{Gw \le c \\ w \ge 0}} \{F(w)\} \qquad (23)$$

For the solution of (23) we use Lemke's algorithm which solve the linear complementarity problem associated to (23). Then, the above problem will be reduced to such a problem which is the purpose of this section.

Under the assumptions on the convexity of the objective function and the symmetry of the stiffness matrix K which is also positive definite, the called primal problem (23) is equivalent to the saddle-point problem:

$$\min_{w} \max_{\lambda \ge 0} \left\{ \frac{1}{2} w^T K w - w^T V + (Gw - c)^T \lambda \right\} \qquad (24)$$

where $\lambda \in R^m$ is the Lagrange multiplier introduced in order to release the constraints $Gw \le c$.

As the minimization over w is now unconstrained, it is obtained:

$$w = K^{-1}(V - G^T \lambda) \qquad (25)$$

Substituting (25) into (24) leads to the dual problem:

$$\min_{\lambda \geq 0} \left\{ \frac{1}{2} \lambda^T P \lambda - \lambda^T s \right\} \qquad (26)$$

which corresponds to the minimum of the complementary energy, and where $P = GK^{-1}G^T$ is an $m \times m$ symmetric matrix and $s = GK^{-1}V - c$ is an m vector.

The resulting quadratic problem has the simpler constraint set $\lambda \geq 0$.

If λ is the solution of (26), then:

$$(P\lambda - s)(\lambda^* - \lambda) \geq 0 \qquad \forall \lambda^* \geq 0$$

Taking $\lambda^* = 2\lambda$ and $\lambda^* = 0$ successively, and setting $P\lambda - s = \rho$, the linear complementarity problem associated with (26) is stated as follows:

$$\begin{cases} \rho - P\lambda = -s \\ \rho \geq 0 \\ \lambda \geq 0 \\ \rho . \lambda = 0 \end{cases} \qquad (27)$$

Lemke's algorithm is a complementary pivoting method which solves linear complementarity problems such as that defined in (27). An interesting property of the Lemke's algorithm is that gives the exact solution of the discrete problem in a finite number of steps. Also, it is a pivoting method which seems to be one of the most efficient developed for linear and quadratic problems and then extended to linear complementarity problems. Finally, Lemke's algorithm has an easy computer implementation.

COMPUTER IMPLEMENTATION

A finite element program is implemented to solve plates with bilateral and unilateral supports. These unilateral supports can restrain displacements, either negative,positive or both. Plates are modelled with quadrangular elements of 8 nodes and can be loaded with distributed and concentrated loads.

Once the global stiffness matrix is found, its rows and columns associated with unilateral restrictions are segregated in such a way that the upper part of the stiffness matrix is associated with the restrictions and the lower part is associated with the non restricted variables (such as the rotations).

Working with matrix algebra operations, we arrive at the matrix P and vector s of the dual problem (26) which are needed for solving the linear complementarity problem by Lemke's algorithm. Its solution allows us to obtain the remainder displacements and the rotations.

NUMERICAL EXAMPLES

In order to show the feasibility of the developed variational formulation and algorithm we present in this section two numerical examples.

The first example consists in a square clamped plate subject to a distributed load. Figure 4 shows a quarter of the plate (for its symmetry) modelled by four rectangular isoparametric elements of 8 nodes. The plate is clamped at nodes 1, 2, 3, 4, 5, 6, 9, 14 and 17, and has an unilateral support at node 21, placed at a distance of 0.48×10^{-4} mm from the axis.

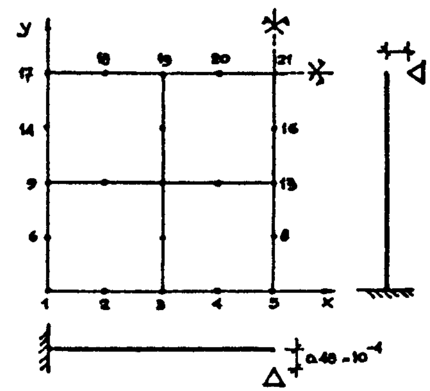

FIGURE 4

To prove the program a solution is obtained by a superposition method using a classic finite element program which solves plates.

As it is shown in Figure 5, the first step considers the clamped plate with any unilateral support. We look for the load for which the plate reaches the displacement of 0.48×10^{-4} mm.

A second stage consists in the same plate with a support in the middle point 21 subject to a distributed load such that, added to the previous one, results in the total load of the unilateral supported plate of the example.

Applying the superposition method the solution is the sum of both, while with the unilateral program the solution is obtained in one step.

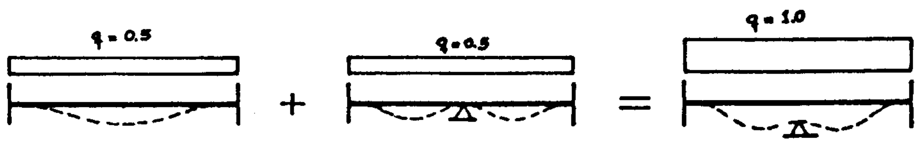

FIGURE 5

The corresponding support reactions are listed in Table 1, where it is possible to appreciate the agreements of the results.

Node	R^1	R^2	R^1+R^2	R^*
1	-0.23	-0.76	-0.99	-0.999
2	-2.15	-1.96	-4.11	-4.118
3	-2.76	-2.58	-5.34	-5.34
4	-7.16	-5.46	-12.62	-12.624
5	-3.80	-2.34	-6.14	-6.14
6	-2.15	-1.96	-4.11	-4.118
8	0.0	0.0	0.0	0.0
9	-2.76	-2.58	-5.34	-5.34
13	0.0	0.0	0.0	0.0
14	-7.16	-5.46	-12.62	-12.624
16	0.0	0.0	0.0	0.0
17	-3.80	-2.33	-6.14	-6.14
18	0.0	0.0	0.0	0.0
19	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0	0.0
21	0.0	-6.56	-6.56	-6.557

* Results of the unilateral program

TABLE 1

The second example is that of a circular plate of external radius R = 390.0 mm and mean thickness e = 110.0 mm, resting on elastic supports in accordance with radius R = 298.75 mm and R = 340.0 mm respectively. The plate is under the action of bolt forces and an inner pressure $p = 11.5\ N/mm^2$.

The discrete model adopted corresponds to a quarter of plate (by the symmetry) as it is shown in Figure 6. The mesh consists in 78 nodes and 21 quadrangular elements of 8 nodes.

We solve for 6 load states. Starting from a bolt load of 60 Kgm, following with 90 Kgm, 120 Kgm , 150 Kgm, 180 Kgm and 180 Kgm plus the action of the pressure.

Figure 7 shows the deformation of the plate (in each state) along a radius and Figure 8 shows the deflection of the central point (node 21) in each step respectively.

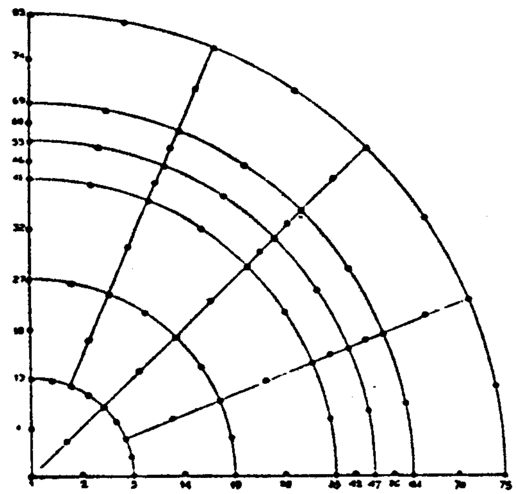

FIGURE 6

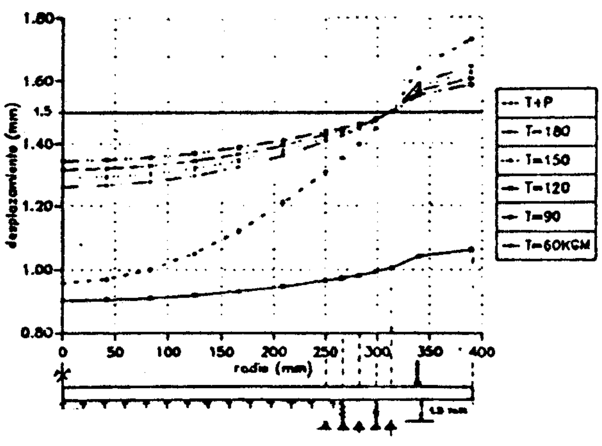

FIGURE 7

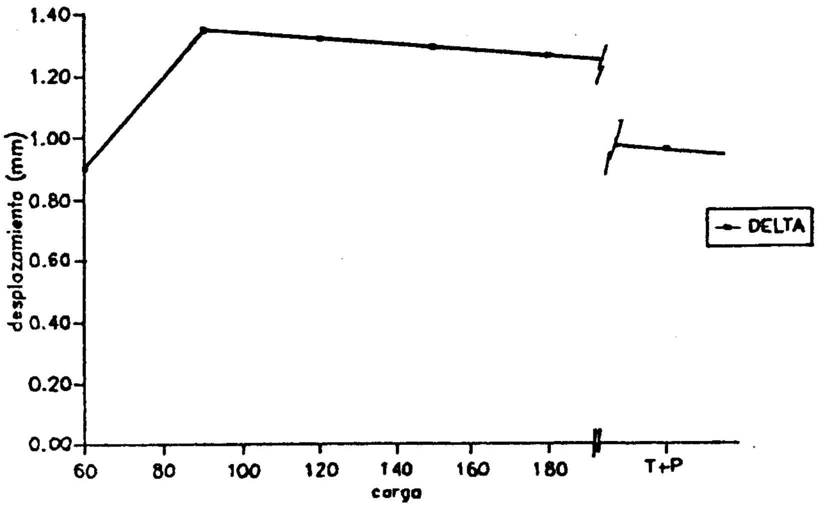

FIGURE 8

CONCLUSIONS

Unilateral constraints problems are more frequent than bilateral, but their formulation involves inequalities and this means mathematical difficulties. Developments in linear programming, convex analysis and variational inequalities, provide new techniques for the solution of unilateral problems in structural engineering.

The authors want to remark that this work must be considered as an application of unilateral contact problems within the area of structural analysis. It is taken into account the equilibrium problem for plates in terms of displacements whereas the nature of the material concerns an elastic one.

We try to show a rather simple procedure to modify a computer program to solve contact problems. For this, we basically use a plate finite element program and, by making specific modifications it becomes avaiable for our needs.

Thus, the problem is stated by the constrained minimization energy functional (which is quadratic), and approximate solutions can be obtained by the introduction of the finite element technique.

Up to this point, it does not seem to be a simple formulation, neither in its mathematical procedure, nor in terms of computational storage. The next step consists in a condensation process introduced in order to arrive at the so-called dual problem with a simpler set of restrictions. Then, we are able to reduce our problem to a linear complementarity problem and, therefore, to save storage capacity.

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