STRESS FUNCTIONS FOR EQUILIBRIUM, HYBRID AND MIXED MODELS

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RESUMO

Em vários modelos do método dos elementos finitos é necessário partir de um campo de tensões em equilíbrio. Em certos problemas sabe-se de antemão quais as componentes do tensor das tensões importantes e quais as sem importância. Anulando-se as sem importância, os cálculos são simplificados.

Mostram-se neste trabalho doze funções de tensão em coordenadas cilíndricas e dez em coordenadas esféricas, com várias componentes do tensor das tensões nulas.

ABSTRACT

In several models of the finite element method it is necessary to start with a stress field in equilibrium. In some problems it is known, beforehand, important and non important components of the stress tensor. Considering zero the non important components calculations are simplified.

In this paper twelve stress functions in cylindrical coordinates and ten in spherical coordinates are shown, with several components of the stress tensor considered zero.

INTRODUCTION

In several problems the finite element method using a polynomial as an interpolation function for displacements is not the best possible solution. More refined stress fields can be necessary to use in zones of high-stress gradients. Then other formulations of the method, based on hybrid or complementary energy principles are more advantageous. Using those principles it is necessary to start with a stress field in equilibrium with body forces, and satisfying boundary conditions.

To solve an elasticity problem it is necessary to satisfy equilibrium, compatibility and boundary conditions. Usually equilibrium is considered the most important condition, in the sense that if it is infringed, the solution is not accepted even as an approximation. However the solutions given in any text-book in strength of materials don't satisfy compatibility.

In several problems it is possible to know what are the most important components of the stress field and components of secondary interest. If those small components can be considered zero, it is possible to choose a stress function with the important components. Other stress functions can be added, if necessary, to satisfy boundary conditions.

An equation can be satisfied in two ways: if all terms appearing in that equation are zero or if at least two terms are different from zero.

Equilibrium can always be satisfied with one stress component equal to zero.

With two stress components equal to zero, however equilibrium cannot always be satisfied.

The existence of a stress function, however, is guaranteed only if the number of stresses different from zero is greater than the number of equilibrium equations not identically satisfied.

STRESS FUNCTIONS

Equilibrium is satisfied if

$$\bar{\bar{\sigma}} = \vec{\nabla} \times \bar{\bar{P}} \qquad (1)$$

where $\bar{\bar{\sigma}}$ is the stress tensor, $\bar{\bar{P}}$ is any second order tensor, and

$$\nabla = \underset{\sim}{e}_i \frac{\partial}{\partial s_i} \qquad (2)$$

To guarantee the symmetry of the tensor field $\bar{\bar{\sigma}}$,

$$\bar{\bar{P}} = \bar{\bar{Q}} \times \vec{\nabla} \qquad (3)$$

and

$$\bar{\bar{\sigma}} = \vec{\nabla} \times \bar{\bar{Q}} \times \vec{\nabla} \qquad (4)$$

Note that

$$\bar{\bar{Q}} \times \vec{\nabla} = -(\vec{\nabla} \times \bar{\bar{Q}})^T \qquad (5)$$

When some stresses are considered zero, several equations are of the type.

$$\frac{1}{A(x)} \frac{\partial}{\partial x}(B\sigma_i) + \frac{1}{C(y)} \frac{\partial}{\partial y}(D\sigma_j) = 0 \qquad (6)$$

This equations is satisfied if

$$B\sigma_i = \frac{1}{C} \frac{\partial f}{\partial y}, \quad D\sigma_j = -\frac{1}{A} \frac{\partial f}{\partial x} \qquad (7)$$

where f is an arbitrary stress function.

The procedure to obtain stress functions is the following.

All possible combinations of zero stress components able of satisfying equilibrium are studied. The stress functions are supposed to depend on three variables. Of these cases, those that don't permit a stress function are discarded. The remaining cases are then studied with eq. (4) or (7). The arbitrary second order tensor $\bar{\bar{Q}}$ is considered symmetric with one or two components different from zero.

STRESS FUNCTIONS IN CYLINDRICAL COORDINATES

The equilibrium equations in cylindrical coordinates are:

$$\frac{1}{r} \frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} - \frac{\sigma_{\theta\theta}}{r} + R = 0 \qquad (8)$$

$$\frac{1}{r^2} \frac{\partial}{\partial r}(r^2\sigma_{r\theta}) + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + T = 0 \qquad (9)$$

$$\frac{1}{r} \frac{\partial}{\partial r}(r\sigma_{rz}) + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + Z = 0 \qquad (10)$$

In these equations R,T and Z are the components of body force per volume.

Equilibrium can be satisfied in nine cases with two stress components zero four cases with three stresses and one with four stresses. If

$$\sigma_{rr} = \sigma_{rz} = \sigma_{\theta z} = \sigma_{zz} = 0$$

there are two stresses to satisfy two equations. Clearly it is not possible to write relations among $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ and an arbitrary stress function.

All cases with two stress components zero can be used to get a stress function; really, then four stresses must satisfy three equilibrium equations.

If body forces are not zero, a particular solution of eqs. (8) to (10) can be obtained considering that all shear stresses are zero. Then it is easy to verify that the normal stresses are given by:

$$\sigma_{zz} = - \int Z \, dz \qquad (11)$$

$$\sigma_{\theta\theta} = - \int T \, r \, d\theta \qquad (12)$$

$$\sigma_{rr} = - \frac{1}{r} \int_0^r \left[\int_0^\theta T \, d\theta \right] r \, dr - \frac{1}{r} \int_0^r Rr \, dr \qquad (13)$$

<u>Case 1</u> $$\sigma_{rr} = \sigma_{r\theta} = 0 \qquad (14)$$

The stress function can be obtained using eq. (4) with $Q_{11} = f_1$ and all others $Q_{ij} = 0$. The final result is.

$$\sigma_{\theta\theta} = - \frac{\partial^2 f_1}{\partial z^2} \qquad (15)$$

$$\sigma_{rz} = \frac{-1}{r} \frac{\partial f_1}{\partial z} \qquad (16)$$

$$\sigma_{\theta z} = \frac{1}{r} \frac{\partial^2 f_1}{\partial z \partial \theta} \qquad (17)$$

$$\sigma_{zz} = \frac{1}{r} - \frac{\partial f_1}{\partial r} - \frac{1}{r^2} \frac{\partial^2 f_1}{\partial \theta^2} \qquad (18)$$

<u>Case 2</u> $$\sigma_{rr} = \sigma_{\theta\theta} = 0 \qquad (19)$$

The stress function can be obtained from eq. (4)

considering $Q_{12} = Q_{21} = f_2$ and açç others $Q_{ij} = 0$

$$\sigma_{r\theta} = \frac{\partial^2 f_2}{\partial z^2} \qquad (20)$$

$$\sigma_{rz} = \frac{-1}{r} \frac{\partial^2 f_2}{\partial \theta \partial z} \qquad (21)$$

$$\sigma_{\theta z} = \frac{-2}{r} \frac{\partial f_2}{\partial z} - \frac{\partial^2 f_2}{\partial z \partial r} \qquad (22)$$

$$\sigma_{zz} = \frac{2}{r^2} \frac{\partial^2}{\partial \theta \partial r} (r f_2) \qquad (23)$$

Case 3 $\sigma_{zz} = 0$ (24)

Proceeding as before with $Q_{13} = Q_{31} = f_3$, it is obtained:

$$\sigma_{rr} = \frac{2}{r} \frac{\partial f_3}{\partial z} \tag{25}$$

$$\sigma_{r\theta} = \frac{-1}{r} \frac{\partial^2 f_3}{\partial z \partial \theta} \tag{26}$$

$$\sigma_{\theta\theta} = 2 \frac{\partial^2 f_3}{\partial z \partial r} \tag{27}$$

$$\sigma_{rz} = \frac{1}{r^2} \frac{\partial^2 f_3}{\partial \theta^2} \tag{28}$$

$$\sigma_{\theta z} = \frac{-\partial^2}{\partial z \partial r} \left(\frac{f_3}{r}\right) \tag{29}$$

Case 4 $\sigma_{\theta\theta} = \sigma_{zz} = 0$ (30)

Obtained with $Q_{23} = Q_{32} = f_4$.

$$\sigma_{rr} = \frac{2}{r} \frac{\partial^2 f_4}{\partial \theta \partial z} \tag{31}$$

$$\sigma_{r\theta} = - r \frac{\partial^2}{\partial r \partial z} \left(\frac{f_4}{r}\right) \tag{32}$$

$$\sigma_{rz} = \frac{-1}{r^2} \frac{\partial^2}{\partial r \partial \theta} (r f_4) \tag{33}$$

$$\sigma_{\theta z} = \frac{\partial}{\partial r} \left(\frac{f_4}{r} + \frac{\partial f_4}{\partial r}\right) \tag{34}$$

Case 5 $\sigma_{r\theta} = \sigma_{\theta\theta} = \sigma_{\theta z} = 0$ (35)

The results below were got with $Q_{22} = f_5$

$$\sigma_{rr} = \frac{\partial^2 f_5}{\partial z^2} \tag{36}$$

$$\sigma_{rz} = \frac{1}{r} \frac{\partial^2}{\partial r \partial z} (r f_5) \tag{37}$$

$$\sigma_{zz} = \frac{-1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f_5}{\partial r}\right) \tag{38}$$

Case 6 $\sigma_{rz} = \sigma_{\theta z} = \sigma_{zz} = 0$ (39)

The plane stress system can be solved with $Q_{33} = f_6$, all others $Q_{ij} = 0$.

$$\sigma_{rr} = \frac{-1}{r^2}\frac{\partial^2 f_6}{\partial \theta^2} - \frac{1}{r}\frac{\partial f_6}{\partial r} \quad (40)$$

$$\sigma_{r\theta} = \frac{\partial^2}{\partial\theta\partial r}\left(\frac{f_6}{r}\right) \quad (41)$$

$$\sigma_{\theta\theta} = -\frac{\partial^2 f_6}{\partial r^2} \quad (42)$$

<u>Case 7</u> $\sigma_{rr} = \sigma_{zz} = 0$ (43)

Add cases (3) and (4) with

$$f_3 = -\frac{\partial f_4}{\partial \theta} \quad (44)$$

And introduce $f_7 = f_4$

$$\sigma_{r\theta} = \frac{1}{r}\frac{\partial}{\partial z}\left[\frac{\partial^2 f_7}{\partial\theta^2} - r\frac{\partial f_7}{\partial r} + f_7\right] \quad (45)$$

$$\sigma_{\theta\theta} = -2\frac{\partial^3 f_7}{\partial r\,\partial\theta\,\partial z} \quad (46)$$

$$\sigma_{rz} = \frac{-1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{\partial^2 f_7}{\partial\theta^2} + r\frac{\partial f_7}{\partial r} + f_7\right] \quad (47)$$

$$\sigma_{\theta z} = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial^2 f_7}{\partial\theta^2} + \frac{f_7}{r} + \frac{\partial f_7}{\partial r}\right] \quad (48)$$

<u>Case 8</u> $\sigma_{rr} = \sigma_{rz} = 0$ (49)

Add cases (1) and (2) with

$$f_1 = \frac{\partial f_2}{\partial \theta} \quad (50)$$

and introduce $f_8 = f_2$

$$\sigma_{r\theta} = \frac{\partial^2 f_8}{\partial z^2} \quad (51)$$

$$\sigma_{\theta\theta} = \frac{\partial^3 f_8}{\partial\theta\partial z^2} \quad (52)$$

$$\sigma_{\theta z} = \frac{\partial}{\partial z}\left[-\frac{1}{r}\left(\frac{\partial^2 f_8}{\partial\theta^2} + 2f_8\right) + \frac{\partial f_8}{\partial r}\right] \quad (53)$$

$$\sigma_{zz} = \frac{\partial}{\partial r}\left[\frac{1}{r^2}\left(\frac{\partial^2 f_8}{\partial\theta^2} + 2f_8\right) + \frac{1}{r}\frac{\partial f_8}{\partial r}\right] \quad (54)$$

<u>Case 9</u> $$\sigma_{rr} = \sigma_{\theta z} = 0 \tag{55}$$

The results below were found with a systematic use of eq. (6).

$$\sigma_{r\theta} = \frac{1}{r^2}\,\frac{\partial^3 f_9}{\partial\theta\partial z^2} \tag{56}$$

$$\sigma_{\theta\theta} = -\frac{1}{r}\,\frac{\partial^3 f_9}{\partial r\partial^2 z} \tag{57}$$

$$\sigma_{rz} = -\frac{1}{r^2}\,\frac{\partial}{\partial r}\left[\frac{\partial f_9}{\partial r} + \frac{1}{r}\,\frac{\partial^2 f_9}{\partial\theta^2}\right] \tag{58}$$

$$\sigma_{zz} = \frac{1}{r}\,\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial f_9}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f_9}{\partial\theta^2}\right] \tag{59}$$

<u>Case 10</u> $$\sigma_{\theta r} = \sigma_{zr} = 0 \tag{60}$$

$$\sigma_{rr} = \frac{\partial^2 f_{10}}{\partial z^2} \tag{61}$$

$$\sigma_{\theta\theta} = \frac{-\partial^3(r\,f_{10})}{\partial r\partial z^2} \tag{62}$$

$$\sigma_{z\theta} = \frac{1}{r}\,\frac{\partial^3(r\,f_{10})}{\partial r\partial\theta\partial z} \tag{63}$$

$$\sigma_{zz} = -\frac{1}{r^2}\,\frac{\partial^3\,(r\,f_{10})}{\partial\theta^2\,\partial r} \tag{64}$$

<u>Case 11</u> $$\sigma_{\theta r} = \sigma_{zz} = 0 \tag{65}$$

$$\sigma_{rr} = \frac{\partial}{\partial z}\left[\frac{\partial f_{11}}{\partial r} - \frac{f_{11}}{r} - \frac{1}{r}\,\frac{\partial^2 f_{11}}{\partial\theta^2}\right] \tag{66}$$

$$\sigma_{\theta\theta} = r\,\frac{\partial^3 f_{11}}{\partial r^2\,\partial z} \tag{67}$$

$$\sigma_{zr} = \frac{1}{r}\,\frac{\partial^3 f_{11}}{\partial r\,\partial\theta^2} \tag{68}$$

$$\sigma_{z\theta} = -\frac{\partial^3 f_{11}}{\partial r^2\partial\theta} \tag{69}$$

<u>Case 12</u> $$\sigma_{\theta\theta} = \sigma_{zr} = 0 \tag{70}$$

$$\sigma_{rr} = \frac{1}{r}\,\frac{\partial^3 f_{12}}{\partial\theta\partial z^2} \tag{71}$$

$$\sigma_{\theta r} = -\frac{\partial^3 f_{12}}{\partial r\partial z^2} \tag{72}$$

$$\sigma_{z\theta} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \frac{\partial^2 f_{12}}{\partial r \partial z}) \tag{73}$$

$$\sigma_{zz} = \frac{-1}{r^2} \frac{\partial^2}{\partial r \partial z}(r^2 \frac{\partial f_{12}}{\partial r}) \tag{74}$$

STRESS FUNCTIONS IN SPHERICAL COORDINATES

Consider the spherical coordinates ϕ, θ, R; ϕ is the angle between the position vector and the z axis; R is the distance from the point to the origin and θ the angle between the x axis and the projection of the position vector in the x y plane. Body force per unit volume have componentes F_θ, F_ϕ, F_R. Equilibrium equations are:

$$\frac{1}{R} \frac{\partial}{\partial R}(R^2 \sigma_{RR}) + \frac{1}{sen\phi} \frac{\partial \sigma_{\theta R}}{\partial \theta} - (\sigma_{\theta\theta} + \sigma_{\phi\phi}) + \frac{1}{sen\phi} \frac{\partial}{\partial \phi}(sen\phi \sigma_{R\phi}) +$$
$$+ R F_R = 0 \tag{75}$$

$$\frac{1}{R^2} \frac{\partial}{\partial R}(R^3 \sigma_{R\theta}) + \frac{1}{sen\phi} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{sen^2\phi} \frac{\partial}{\partial \phi}(sen^2\phi\ \sigma_{\theta\phi}) + R F_\theta = 0 \tag{76}$$

$$\frac{1}{R^2} \frac{\partial}{\partial R}(R^3 \sigma_{R\phi}) + \frac{1}{sen\phi} \frac{\partial \sigma_{\phi\theta}}{\partial \theta} + \frac{1}{sen\phi}(sen\phi\ \sigma_{\phi\phi}) - cot\phi\ \sigma_{\theta\theta} + R F_\phi = 0 \tag{77}$$

To get a particular solution easily, consider $\sigma_{\theta\theta} = 0$, $\sigma_{\theta R} = 0$, $\sigma_{R\phi} = 0$. Then equilibrium equations simplify; the remaining stresses can be found without any difficulty:

$$\sigma_{\theta\phi} = - \frac{R}{sen^2\phi} \int sen^2\phi F_\theta \ d\phi \tag{78}$$

$$\sigma_{\phi\phi} = - \frac{1}{sen\phi} \int (\frac{\partial \sigma_{\theta\phi}}{\partial \phi} + Rsen\phi\ F_\phi) d\phi \tag{79}$$

$$\sigma_{RR} = \frac{1}{R^2} \int R(\sigma_{\phi\phi} - R F_R) dR \tag{80}$$

When some stresses are considered zero the system simplifies and in eleven of thirteen cases there is an equation with two unknowns, like eq. (6)

The eleven cases are the following

<u>Case 1</u>

$$\sigma_{\theta\theta} = \sigma_{R\theta} = \sigma_{\theta\phi} = 0 \tag{81}$$

$$\sigma_{RR} = \frac{-1}{R^2 sen\phi} \left[f_1 + R \frac{\partial f_1}{\partial R} + \frac{\partial^2 f_1}{\partial \phi^2} \right] \tag{82}$$

$$\sigma_{R\phi} = \frac{1}{Rsen\phi} \frac{\partial^2 f_1}{\partial R \partial \phi} \tag{83}$$

$$\sigma_{\phi\phi} = \frac{-1}{R^2 sen\phi} \frac{\partial}{\partial R}\left(R^2 \frac{\partial f_1}{\partial R}\right) \tag{84}$$

<u>Case 2</u>:

$$\sigma_{\phi\phi} = \sigma_{\theta\theta} = 0 \tag{85}$$

$$\sigma_{RR} = \frac{2}{R^2 sen^3\phi} \frac{\partial}{\partial\theta}\left[cotg\phi \; f_2 - \frac{\partial f_2}{\partial\theta} \right] \tag{86}$$

$$\sigma_{R\theta} = \frac{1}{R sen^2 \phi} \frac{\partial^2 f_2}{\partial\phi\partial R} \tag{87}$$

$$\sigma_{R\phi} = \frac{1}{R sen^3\phi} \frac{\partial^2 f_2}{\partial\theta\partial R} \tag{88}$$

$$\sigma_{\phi\theta} = \frac{-1}{R^2 sen^2\phi} \frac{\partial}{\partial R}\left(R^2 \frac{\partial f_2}{\partial R}\right) \tag{89}$$

<u>Case 3</u>:

$$\sigma_{\phi\phi} = \sigma_{R\phi} = 0 \tag{90}$$

$$\sigma_{RR} = \frac{1}{R\cos\phi} \frac{\partial^2 f_3}{\partial R\partial\theta} + \frac{(1+\cos^2\phi)}{R^2 sen^2\phi\cos\phi} \frac{\partial f_3}{\partial\theta} + \frac{1}{R^2 sen^2\phi\cos\phi} \frac{\partial^3 f_3}{\partial\theta^3} + \frac{1}{R^2 sen\phi} \frac{\partial^2 f_3}{\partial\phi\partial\theta} \tag{91}$$

$$\sigma_{R\theta} = \frac{-1}{R sen\phi\cos\phi} \frac{\partial^3 f_3}{\partial R\partial\theta^2} - \frac{2\; cotg\phi}{R} \frac{\partial f_3}{\partial R} - \frac{1}{R} \frac{\partial^2 f_3}{\partial R\partial\phi} \tag{92}$$

$$\sigma_{\phi\theta} = \frac{1}{R^2} \frac{\partial}{\partial R}\left(R^2 \frac{\partial f_3}{\partial R}\right) \tag{93}$$

$$\sigma_{\theta\theta} = \frac{1}{R\cos\phi} \frac{\partial}{\partial R}\left(R^2 \frac{\partial^2 f_3}{\partial R\partial\theta}\right) \tag{94}$$

<u>Case 4</u>:

$$\sigma_{\phi\phi} = \sigma_{R\theta} = 0 \tag{95}$$

$$\sigma_{RR} = \frac{1}{R sen\phi} \frac{\partial^2 f_4}{\partial\phi\partial R} - \frac{1}{R^2 sen\phi} \frac{\partial}{\partial\phi}\left[cotg\phi \frac{\partial f_4}{\partial\phi} + \frac{1}{sen^2\phi} \frac{\partial^2 f_4}{\partial\theta^2} \right] + \frac{1}{R^2 sen\phi} \frac{\partial f_4}{\partial\phi} \tag{96}$$

$$\sigma_{R\phi} = \frac{cotg\phi}{R sen\phi} \frac{\partial^2 f_4}{\partial R\partial\phi} + \frac{1}{R sen^3\phi} \frac{\partial^3 f_4}{\partial R\partial\theta^2} \tag{97}$$

$$\sigma_{\theta\theta} = \frac{1}{R^2 sen\phi} \frac{\partial}{\partial R}\left(R^2 \frac{\partial^2 f_4}{\partial\phi\partial R}\right) \tag{98}$$

$$\sigma_{\theta\phi} = \frac{-1}{R^2 sen\phi} \frac{\partial}{\partial R}\left(R^2 \frac{\partial^2 f_4}{\partial R\partial\theta}\right) \tag{99}$$

<u>Case 5:</u> $$\sigma_{\phi\phi} = \sigma_{\theta\phi} = 0 \tag{100}$$

$$\sigma_{RR} = \frac{1}{R}\frac{\partial f_5}{\partial R} + \frac{2f_5}{R^2} + \frac{1}{R^2 sen^2\phi}\frac{\partial^2 f_5}{\partial\theta^2} - \frac{cotg\phi}{R^2}\frac{\partial f_5}{\partial\phi} \tag{101}$$

$$\sigma_{\theta\theta} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial f_5}{\partial R}\right) \tag{102}$$

$$\sigma_{R\phi} = \frac{cotg\partial}{R}\frac{\partial f_5}{\partial R} \tag{103}$$

$$\sigma_{R\theta} = \frac{-1}{Rsen\phi}\frac{\partial^2 f_5}{\partial R\partial\theta} \tag{104}$$

<u>Case 6:</u> $$\sigma_{\theta\theta} = \sigma_{R\phi} = 0 \tag{105}$$

$$\sigma_{RR} = \frac{-1}{R^2 sen\phi}\frac{\partial^3 f_6}{\partial\phi^2\partial\theta} - \frac{2cotg\phi}{R^2 sen\phi}\frac{\partial^2 f_6}{\partial\phi\partial\theta} + \frac{1}{Rsen\phi}\frac{\partial^2 f_6}{\partial R\partial\phi} + \frac{1}{R^2 sen\phi}\frac{\partial f_6}{\partial\theta} \tag{106}$$

$$\sigma_{R\theta} = \frac{1}{Rsen^2\phi}\frac{\partial}{\partial\phi}\left(sen^2\phi\frac{\partial^2 f_6}{\partial R\partial\phi}\right) \tag{107}$$

$$\sigma_{\theta\phi} = \frac{-1}{R^2}\frac{\partial^2}{\partial R\partial\phi}\left(R^2\frac{\partial f_6}{\partial R}\right) \tag{108}$$

$$\sigma_{\phi\phi} = \frac{1}{R^2 sen\phi}\frac{\partial}{\partial R}\left(R^2\frac{\partial^2 f_6}{\partial\theta\partial R}\right) \tag{109}$$

<u>Case 7:</u> $$\sigma_{R\phi} = \sigma_{R\theta} = 0 \tag{110}$$

$$\sigma_{RR} = \frac{1}{R^2 sen\phi}\left[\frac{\partial f_7}{\partial\theta} + \int\frac{1}{sen^2\phi}\frac{\partial^2 f_7}{\partial\theta^2}\,d\phi + \int cotg\phi\frac{\partial f_7}{\partial\phi}\,d\phi\right] \tag{111}$$

$$\sigma_{\phi\phi} = \frac{1}{Rsen\phi}\left[\int cotg\phi\frac{\partial^2 f_7}{\partial R\partial\phi}\,d\phi + \int\frac{1}{sen^2\phi}\frac{\partial^3 f_7}{\partial R\partial\theta^2}\,d\phi\right] \tag{112}$$

$$\sigma_{\theta\theta} = \frac{1}{Rsen\phi}\frac{\partial^2 f_7}{\partial R\partial\phi} \tag{113}$$

$$\sigma_{\theta\phi} = \frac{-1}{Rsen^2\phi}\frac{\partial^2 f_7}{\partial R\partial\theta} \tag{114}$$

<u>Case 8:</u> $$\sigma_{R\phi} = 0 \quad \sigma_{\theta\phi} = 0 \tag{115}$$

$$\sigma_{RR} = \frac{1}{R^2}\int\left[\frac{-1}{R\cos\phi}\frac{\partial^2 f_8}{\partial R\partial\phi} - \frac{1}{Rsen\phi}\frac{\partial f_8}{\partial R} - \frac{1}{R^2 sen^2\phi\cos\phi}\frac{\partial^3 f_8}{\partial\phi\partial\theta^2}\,dR\right] \tag{116}$$

$$\sigma_{R\theta} = \frac{1}{R^3 \mathrm{sen}\phi\cos\phi} \quad \frac{\partial^2 f_2}{\partial\phi\partial\theta^2} \tag{117}$$

$$\sigma_{\theta\theta} = \frac{-1}{R^2\cos\phi} \quad \frac{\partial^2 f_2}{\partial R\partial\phi} \tag{118}$$

$$\sigma_{\phi\phi} = \frac{-1}{R^2 \mathrm{sen}\phi} \quad \frac{\partial f_2}{\partial R} \tag{119}$$

Case 9

$$\sigma_{RR} = \sigma_{\phi\phi} = 0 \tag{120}$$

$$\sigma_{R\theta} = \frac{1}{R^3}\left[\frac{\partial U}{\partial\phi} + \frac{W}{\mathrm{sen}\phi\cos\phi}\right] \tag{121}$$

$$\sigma_{R\phi} = \frac{-1}{R^3 \mathrm{sen}\phi} \quad \frac{\partial U}{\partial\theta} \tag{122}$$

$$\sigma_{\theta\theta} = \frac{1}{R^3 \mathrm{sen}^2\phi\cos\phi} \quad \frac{\partial W}{\partial\theta} \tag{123}$$

$$\sigma_{\phi\theta} = \frac{1}{R^2}\frac{\partial U}{\partial R} + \frac{1}{R^3 \mathrm{sen}^2\phi} W \tag{124}$$

In eqs. (121) to (124) W is chosen and U is given by

$$U = \frac{-\cos\phi}{2}\left[\int\frac{W}{\cos^3\phi}\,d\phi + \iint\frac{1}{R\,\mathrm{sen}^2\phi\cos^3\phi}\frac{\partial^2 W}{\partial\theta^2}\,dR\,d\phi + \right.$$
$$\left. + \iint\frac{1}{R\cos^2\phi\,\mathrm{sen}\phi}\frac{\partial W}{\partial\phi}\,dR\,d\phi\right] \tag{125}$$

Case 10:

$$\sigma_{RR} = 0 \quad \sigma_{\theta\theta} = 0 \tag{126}$$

$$\sigma_{\theta\phi} = \frac{-1}{R^2 \mathrm{sen}^2\phi} \quad \frac{\partial}{\partial R}\left(R^2\frac{\partial f_2}{\partial R}\right) \tag{127}$$

$$\sigma_{\theta R} = \frac{1}{R\,\mathrm{sen}^2\phi} \quad \frac{\partial^2 f_2}{\partial\phi\partial R} \tag{128}$$

$$\sigma_{\phi\phi} = \frac{-1}{R^2 \mathrm{sen}\phi} \quad \frac{\partial}{\partial R}\left(R^2\frac{\partial f_{1\theta}}{\partial R}\right) \tag{129}$$

$$\sigma_{\phi R} = \frac{1}{R\,\mathrm{sen}\phi} \quad \frac{\partial}{\partial R}\left[\frac{\partial f_{1\theta}}{\partial\phi} + \frac{1}{\mathrm{sen}^2\phi}\frac{\partial f_2}{\partial\theta}\right] \tag{130}$$

The stress function $f_{1\theta}$ is arbitrary and f_2 is obtained with eq. (131):

$$f_2 = -\frac{\mathrm{sen}\,\phi}{2}\iint\left[\mathrm{sen}\,\phi\,\frac{\partial}{\partial R}\left(R\,f_{1\theta}\right) + \frac{\partial^2 f_{1\theta}}{\partial\phi^2}\right]d\phi\,d\theta \tag{131}$$

Case 11:

$$\sigma_{RR} = 0 \qquad \sigma_{R\phi} = 0 \tag{132}$$

<u>Case 12:</u> $$\sigma_{RR} = 0 \qquad \sigma_{R\theta} = 0 \tag{133}$$

<u>Case 13:</u> $$\sigma_{RR} = 0 \qquad \sigma_{\theta\phi} = 0 \tag{134}$$

Those three cases can be considered together. One equilibrium equation is satisfied with a function U and another with a function V. The third equation relates U and V, and is of the form

$$R\,\frac{\partial V}{\partial R} + \frac{\partial^2 V}{\partial \phi^2} = L(U) \tag{135}$$

where L is a differential operator. With the change of variables

$$z = \operatorname{colog} R \tag{136}$$

eq. (93) is written

$$\frac{-\partial V}{\partial z} + \frac{\partial^2 V}{\partial \phi^2} = L'(U) \tag{137}$$

This equation, studied by several authors (1), (2), is the heat conduction equation. A relation of the form

$$V = L''(U) \tag{138}$$

is possible to get, but it is so complicated that certainly will be impracticable.

APPLICATION

Determine a stress field satisfying equilibrium in a cantilever beam with circular axis with a radius of curvature R subjected to a torsion moment at the free end.

Shear stress $\sigma_{\theta r}$ and $\sigma_{\theta z}$ are the most important. Normal stresses are of secondary importance and then eqs. (43) to (48) she selected.

The desired solution must give the well known results for a straight beam when R tends to infinity; therefore the torsion stress function must be present in the expression of the stresses.

The torsion stress function ψ satisfies

$$\nabla^2\psi = -2 \quad \text{in } D \tag{139}$$

$$\psi = 0 \quad \text{in } \partial D \tag{140}$$

The torsion moment of inertia is given by

$$J_T = 2\int_A \psi\, dA \tag{141}$$

If the stress field adopted is proportional to sen θ or cos θ,

$$\frac{d^2f}{d\theta^2} + f = 0 \qquad (142)$$

Adopting

$$f_7 = \frac{T_\ell}{J_T} \cos\theta \qquad \phi\, dr \qquad (143)$$

the stress field is

$$\sigma_{r\theta} = - \frac{T_\ell}{J_T} \cos\theta \, \frac{\partial\phi}{\partial z} \qquad (144)$$

$$\sigma_{\theta\theta} = + \frac{2\, T_\ell}{J_T} \, \mathrm{sen}\theta \frac{\partial\phi}{\partial z} \qquad (145)$$

$$\sigma_{rz} = + \frac{1}{r} \, \frac{T_\ell}{J_T} \, \mathrm{sen}\theta \qquad (146)$$

$$\sigma_{\theta z} = \frac{T}{J_T} \cos\theta \, \frac{\partial\phi}{\partial r} \qquad (147)$$

If can be verified that boundary conditions are satisfied.

REFERENCES

[1] Sneddon, I., "Elements of partial differential equations", McGraw-Hill Book Company, New York, 1951.

[2] Tijonov, A. and Samarsky, A.,"Equaciones de la fisica matematica", Editorial Mir, Moscow 1972.