# FINITE ELASTOPLASTIC DEFORMATIONS : BASIC CONCEPTS AND APPLICATIONS

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RESUMEN

Este trabajo analisa la representación del comportamiento mecánico (isotérmico) de sólidos elastoplásticos con grandes deformaciones elásticas y plásticas y anisotropía en el rango plástico. Estado y orientación de elementos descargados son descritos por medio de tensores de rango irreducible. Es incluida la discusión de algunas dificultades usuales en análisis con grandes deformaciones así como aplicaciones de un programa de elementos finitos.

ABSTRACT

The paper is concerned with the representation of isothermal mechanical behavior of an elastoplastic solid that possesses a finite elastic range and exhibits anisotropy in its plastic behavior. The state and orientation of a stress free element is described by means of irreducible rank tensors. A discussion of some difficulties usual in finite strain analysis and applications using finite elements are included.

## 1. INTRODUCTION

In this work we describe some applications of the finite elements method to elastoplastic problems in the presence of finite strains. We consider inviscid plasticity and incremental (rate) solutions; this approach has been the most popular for the solution of practical problems because, by reducing the nonlinear problem to a series of linear associated problems, it can take advantage of available algorithms and software. Important alternatives not considered here are the viscoplastic formulations |1|,|2| and solutions by direct minimization of the functional relations |3|,|4|. The relatively new technique of boundary elements has been already applied to plasticity problems |5| and may have advantages in some situations. On the other hand, most of the results described here for constitutive modelling in the presence of finite strains should be valid for any solution procedure.

We begin this paper by recalling, in Section 2, some well-known results from small deformations plasticity theory. Then, in Section 3 we indicate an extension to the range of finite elastic and plastic deformations. As it is well-known, elastic deformations of ductile metals are bound to be small because of the relation between yield stress and elastic modulus. Nevertheless, the consideration of finite elastic deformations is important from a theoretical point of view and has already some applications for new synthetic materials |6|. In Section 4 this formulation is illustrated through examples for frame and continuous structures. Some applications to geomaterials and metalworking processes are discussed in Section 5.

Our work at the Curso de Pós-Graduação em Engenharia Civil, UFRGS, has been a joint effort with E.T.Onat from Yale University and A.G.Groehs. Some numerical results were taken from MSc. dissertions of our students, as indicated in the references. The research program was initiated under the coordination of A.J.Ferrante.

## 2. SMALL DEFORMATIONS FORMULATION

In order to establish a common background and notation we begin by reviewing the small deformations theory. The basic relations are:

i) Definition of plastic deformation

$$\varepsilon = \varepsilon^e + \varepsilon^p \qquad (1)$$

where $\varepsilon^p$ is the plastic strain and $\varepsilon^e$ the elastic (recoverable) one. In incremental plasticity this relation is normally written in the rate form $|(\dot{\ }) = \partial/\partial t|$

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p \qquad (2)$$

ii) Definition of yield condition

$$Y(\sigma, S) = 0 \qquad (3)$$

Eq(3) represents, in the 9-dimentional space $\Sigma$ (whose coordinates are the components of the (Cauchy) stress tensor $\sigma$), a closed surface that encloses the elastic domain. In general, this surface changes during the elastoplastic deformation process; this change is loosely referred to as hardening. Symbol $S$ in (3) represents a set of parameters that characterize the state of the material and models hardening.

iii) Loading-unloading criteria. From (3) we characterize unloading processes with

$$\dot{Y} = \frac{\partial Y}{\partial \sigma} \dot{\sigma} < 0 \tag{4}$$

and loading processes with

$$\dot{Y} = \frac{\partial Y}{\partial \sigma} \dot{\sigma} + \frac{\partial Y}{\partial S} \dot{S} = 0 \quad ; \quad \frac{\partial Y}{\partial \sigma} \dot{\sigma} \geq 0 \tag{5}$$

In the second (5) sign (=) corresponds to neutral loading that does not modify the material's state and (>) to active loading. The differentiation indicated in (4),(5) assumes that the yield surface is smooth; alternative formulations can be considered for yield surfaces with corners |7|.

iv) Incremental elastic relation:

$$\dot{\varepsilon}^{e} = E^{-1} \dot{\sigma} \tag{6}$$

where E is the elastic isotropic constitutive matrix.

v) Incremental relation for the plastic deformation

$$\dot{\varepsilon}^{p} = \Lambda^{\cdot} \frac{\partial g}{\partial \sigma} \left(\frac{\partial Y}{\partial \sigma} \dot{\sigma}\right) \quad ; \quad \Lambda^{\cdot} \geq 0 \tag{7}$$

where $g(\sigma,S)$ is a scalar function termed plastic potential; the term $(\partial Y/\partial\sigma.\dot{\sigma})$ assures continuity for neutral loading.

From (2),(5),(6),(7) we obtain the elastic constitutive relation

$$\dot{\sigma} = E_{ep} \dot{\varepsilon} \tag{8}$$

where

$$E_{ep} = E - \frac{E\bar{a}a^{T}E}{A+a^{T}Ea} \tag{9}$$

being

$$a = \frac{\partial Y}{\partial \sigma} \quad ; \quad \bar{a} = \frac{\partial g}{\partial \sigma} \quad ; \quad A = -\frac{1}{\Lambda}\frac{\partial Y}{\partial \sigma}.S \tag{10}$$

Relation (9), first proposed by Hill |8| was rediscovered by Yamada |9|.

## 3. FINITE STRAIN FORMULATION

In this section we shall limit ourselves to review only some important points, stressing the differences between small and large strain theories. A more detailed explanation can be found in the references given.

### 3.1. Kinematics

For finite strain we must obviously abandon the usual linearized definition of specific deformation $\epsilon$. Many alternatives have been proposed |10|,|11|,|12|, but it appears that, as in finite elasticity, it is convenient to begin with the deformation gradient F, such that the relation

$$x = F(t)\ X \quad ; \quad t \in [0,\infty) \tag{11}$$

indicates that F(t) carries the material point X from its position X in t=0 to the position x in t. For the deformations considered det F(t) > 0. During the deformation process a cubic element is transformed into a parallelepiped whose shape is characterized by $F^T F$. For rigid body rotations we have naturally F=I and det F=1.

In the elastoplastic case we define plastic deformations through the decomposition |13|

$$F = F_e\ F_p \tag{12}$$

This decomposition is made unique by choosing |14|

$$F_e = F_e^T \tag{13}$$

From (11) we may write

$$\dot{F}(t) = \frac{d}{dt}\ F(t) = F(t)\ F^{-1}(t) = D+\Omega \tag{14}$$

where $D = D^T$ is the rate of strain and $\Omega = -\Omega^T$ is the rate of rotation. We can define the corresponding elastic and plastic entities in the form

$$\begin{aligned} \dot{F}_p\ F_p^{-1} &= D_p + \Omega_p \\ F_e\ F_e^{-1} &= D_e + \Omega_e \end{aligned} \tag{15}$$

and observe that it is

$$D+\Omega = D_e+\Omega_e+F_e(D_p+\Omega_p)F_e^{-1} \tag{16}$$

Thus, when elastic strains are small, so that we can take $F_e \simeq I$ (16) yields

$$\begin{aligned} D &= D_e + D_p \\ \Omega &= \Omega_e + \Omega_p \end{aligned} \tag{17}$$

The first (17) is (incrementally) equivalent to (2).

In the general case we can find a form more convenient than (16). Defining elastic strain through the left Cauchy-Green strain tensor

$$B_e = F_e F_e^T = F_e^2 \tag{18}$$

and the new measure of plastic strain $\hat{D}_p$ with

$$\hat{D}_p B_e + B_e \hat{D}_p = 2F_e D_p F_e \qquad \hat{D}_p = \hat{D}_p^T \tag{19}$$

we obtain

$$\dot{B}_e = (D-\hat{D}_p+\Omega)B_e + B_e(D-\hat{D}_p-\Omega) \tag{20}$$

$$\Omega-\Omega_p = W(D+D_p) \tag{21}$$

where W is a linear mapping that depends on $F_e$.

Eq. (2) is the finite strain equivalent of (2) and keeps the additive pattern. Eq. (21) tells us that once D, $D_p$ and $\Omega$ are given $\Omega_p$ is determined.

3.2. Elastic relations

We consider the elastic constitutive relations for an elastic material which is isotropic in the elastic range. In this case it exists a strain energy function $\psi$ per unit mass

$$\psi = \psi(I_1,I_2,I_3) \tag{22}$$

where

$$I_i = tr(B_e^i) \tag{23}$$

are the basic invariants of the left Cauchy-Green tensor $B_e$. The corresponding Cauchy stress $\sigma$ is obtained from (22) and the relation

$$\sigma.D = \rho\,\dot{\psi} \tag{24}$$

in the form

$$\sigma = 2\rho\left(\frac{\partial\psi}{\partial I_1} B_e + 2\frac{\partial\psi}{\partial I_2} B_e^2 + 3\frac{\partial\psi}{\partial I_3} B_e^3\right) \tag{25}$$

where $\rho$ is the current density of the material. Using the continuity relation

$$\dot{\rho} + \rho\, tr\, D = 0 \tag{26}$$

and noticing that from (23) it is

$$\dot{I}_i = 2iB_e^i(D-\hat{D}_p) = 2itrB_e^i(D-\hat{D}_p) \tag{27}$$

we obtain

$$\dot{\sigma} = -\sigma trD + (D-\hat{D}_p+\Omega)\sigma + \sigma(D-\hat{D}_p-\Omega) + \bar{E}\,(D-\hat{D}_p) \tag{28}$$

where

$$\tilde{E}_{ijkl} = 2\rho[2\psi_{,2}(B_{ik}B_{lj}+B_{lj}+B_{il}B_{kj}) + \\ + 3\psi_{,3}(B^2_{ik}B_{lj}+B_{ik}B^2_{lj}+B^2_{il}B_{kj}+B_{il}B^2_{kj}) + 2\psi_{,11}B_{ij}B_{kl} + \\ + 4\psi_{,12}(B_{ij}B^2_{kl}+B^2_{ij}B_{kl}) + 8\psi_{,22}(B^2_{ij}B^2_{kl}) + \\ + 12\psi_{,23}(B^2_{ij}B^3_{kl}+B^3_{ij}B^2_{kl}) + 18\psi_{,33}B^3_{ij}B^3_{kl}] \tag{29}$$

In metal plasticity it may be assumed that plastic deformations are isochoric, i.e. $\mathrm{tr}D_p=0$ and thus (28) can be written

$$\dot{\sigma} = \Omega\sigma - \sigma\Omega + \hat{E}\ (D-\hat{D}_p) \tag{30}$$

3.3. State and orientation of stress-free previously deformed elements

As seen in Section 3.1 we may unload a plastically deformed element applying a deformation $F_e^{-1}$. The previous plastic deformation has produced some changes in the material's internal structure that may be represented using the concept of state |14,15|. The state $S$ can be characterized by n parameters

$$S = (q_1, \ldots q_n) \tag{31}$$

It can be shown |16| that the $q_i$'s must be irreducible even rank tensors.

We are also interested in the growth law for $S$ and in particular in its dependence on rotation which is important in the case of materials that develop anisotropy during a deformation process with finite strains and rotations. A rigid body rotation of the material element represented by $Q\in O^+(3)$ causes the state point to move to

$$P_Q S = (P_Q q_1, \ldots, P_Q q_N) \tag{32}$$

where $P_Q$ represents an ordinary tensor transformation adequate to the rank of $q_i$. As for the growth of $S$ we can write, for a rotation $\Omega$,

$$\frac{dS(t)}{dt} = A(S,\sigma,D) - T_{WD}S + T_\Omega S \tag{33}$$

where $T_\Omega$ is a linear mapping on $R^n$ that depends linearly on $\Omega$. A more detailed explanation may be found in refs. |17|, |18|.

3.4. Some fine points to be careful about

i) When linearized stress $\sigma$ and linearized $\epsilon$ are used in the classical infinitessimal theory it is certainly correct to calculate for incremental loading the corresponding stress increment $\Delta\sigma$ and add these stress increments in order to obtain an approximate expression for the stresses. For finite deformations, however, this must be done with care, keeping in mind that stresses can be added only if they are referred to and measured per unit area of the same reference configuration.

ii) Yield conditions must be written in terms of Cauchy stress which is the only one with real physical meaning. It is unacceptable to write yield conditions in terms of (for instance) the second Piola-Kirchhoff stress, that depends on an (arbitrary) initial configuration definition.

iii) In classical plasticity theory plastic deformations are assumed isochoric. In the case of small elastic and large plastic strain, total deformation is mostly incompressible. That is, the material behaves as if having a shear modulus much higher than the shear modulus. This type of behavior may give rise to important errors when working with some finite element types and meshes |19|,|20|.

Let us consider the four-nodes isoparametric element shown in Fig. 1 for the case of an equivolumetric deformation. With this type of element it is normal to use 2x2 Gauss integration to calculate the stiffness. For this situation we have volume increase around two integration points and volume decrease in the other two. As the material is almost incompressible these (opposite) changes of volume give rise to important stresses and the stiffness obtained results larger than the real one.

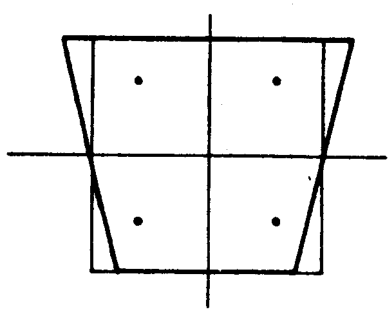


Fig. 1

A way out for this problem was proposed by Nagtegaal et al |19| using ideas already proposed by Herrmann |21| and Key |22|. It consists in defining a new degree of freedom for the volumetric strain. This procedure leads to a modification of the basic variational principle. From Fig. 1 it is also apparent that the problem may be avoided using (in the present situation) a single integration point at the center of the element. This observation leads to the method of reduced integration proposed by Zienkiewicz |23| in order to introduce a singularity in the volumetric component of the stiffness matrix. In the case of the linear element this method is not adequate because reduced integration leads frequently to a singularity also for the tangencial stiffness |24|. In ESFINGE |20|,|25| we have used the modified functional formulation for linear elements and reduced integration for quadratic elements. Fig. 2 shows an example comparing ESFINGE with compressible and incompressible elements and numerical and experimental results from ref. |26|, indicated by (H). The error becomes much larger for finite deformations.

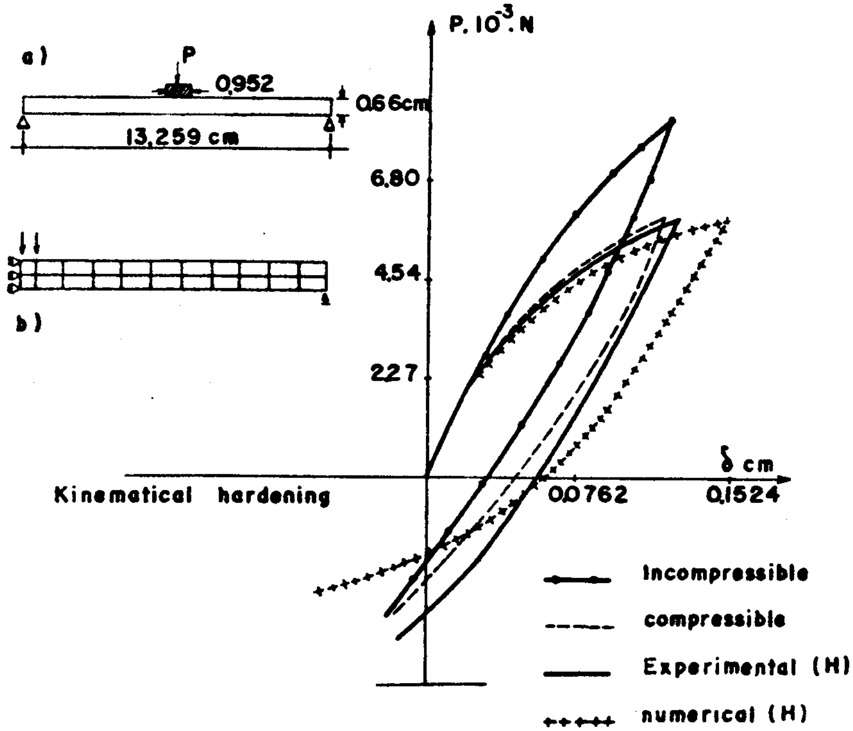
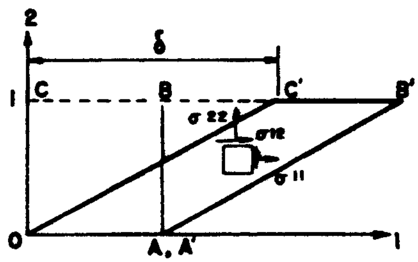



Fig. 2

iv) Computer studies by Nagtegaal and de Jong |27| have shown some surprising results in the analysis of large plastic deformations of kinematically hardening materials. These results show oscillating stresses under monotonically increasing strains. We will look at the problem following a simplified analysis by Onat |28|.

We consider a material element under simple shear, Fig. 3. The corresponding values of the rates of strain and rotation are

$$D = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad ; \quad \Omega = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad (34)$$



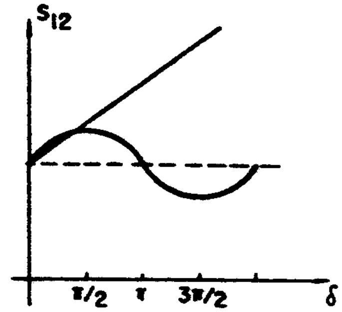




Fig. 3

We consider a rigid-perfectly plastic material with kinematic hardening

$$Y(\sigma,S) = \frac{1}{2}(s-\alpha)(s-\alpha) - k^2 \qquad (35)$$

where $s$ is the deviatoric stress

$$s = \sigma - \frac{1}{3}(\mathrm{tr}\sigma)I \qquad (36)$$

Using the associated flow rule

$$D = \lambda\ \partial Y/\partial s = \lambda\ (s-\alpha) \qquad (37)$$

we obtain

$$s = \alpha + \sqrt{2}\ k\ \frac{D}{\sqrt{D.D}} \qquad (38)$$

The growth law for $\alpha$ can be written in general (see 33)

$$\dot{\alpha} = \lambda A\ (s,\alpha) + \Omega\ \alpha - \alpha\ \Omega \qquad (39)$$

Prager's rule is a particular case of (39) with

$$\dot{\alpha} = \lambda a\ (s-\alpha) + \Omega\ \alpha - \alpha\ \Omega \qquad (40)$$

Integrating (40) for the shear problem (34) and substituting into (38) we obtain

$$s_{12} = \sigma_{12} = k + \frac{1}{2}\ a\ \mathrm{sen}\ \delta \qquad (41)$$

an oscillating stress as shown in Fig. 3, where one would expect an ever increasing stress. This behavior is caused by the rotation terms in dominate expression (40) and can be corrected with a better choice of A in (39). Onat proposes

$$\dot{\alpha} = aD + \left[\Omega + c(D\alpha - \alpha D)\right]\alpha - \alpha\left[\Omega + c(D\alpha - \alpha D)\right]; \quad a>0;\ c>0 \qquad (42)$$

The interplay of parameters a and c allows the modelling of a large variety of responses. The most adequate values have to be determined by experiments.

## 4. APPLICATIONS

### 4.1. Frame structures

One of the simplest applications corresponds to frame structures. The results are useful per se and from a didactical point of view.

Elastic relations: we must make some simplifying assumptions in order to obtain the usual nonlinear expressions corresponding to small deformations and large rotations. Thus, we disregard specific deformations in relation to rotations, writing

$$D + \Omega \simeq \Omega \tag{43}$$

Moreover, we consider that the most important component of the stress state in relation to the nonlinear behavior is the axial stress

$$\sigma \simeq N/A \quad ; \quad \text{A: element area} \tag{44}$$

With these simplifying assumptions, the element stiffness matrix reduces to a form

$$K = \int_{V_e} (B^T E B + \frac{N}{A} \bar{N}^T \bar{N}) dV \tag{45}$$

where B and $\bar{N}$ are the matrices used to calculate strain gradient D and velocity gradient D+Ω respectively from the node velocities.

Using (45) and the interpolation functions corresponding to simple beam theory we obtain

$$K = K_L + K_{NL} \tag{46}$$

$K_L$ and $K_{NL}$ are respectively the linear and nonlinear (or geometrical or initial stress) stiffness matrices. The condition

$$\det (K_L + K_{NL}) = 0 \tag{47}$$

is used to determine the load corresponding to elastic instability, |23|.

Plastic relations: to establish the plastic relations we need a yield function Y adequate for the problem. For plane frames, this relation may be written |29|,|30|

$$Y = (\frac{F_x}{N_p})^{\alpha} + (\frac{F_y}{Q_p})^{\beta} + \frac{|M_z|}{M_p} - 1 - q = 0 \tag{48}$$

where $N_p$, $Q_p$, $M_p$ are the limit plastic values for normal force, shear force and moment for the given section. This yield function represents the behavior of many simmetric metallic sections for adequate values of α and β. The state variable q may be used to model isotropic hardening. We may found in the literature yield functions for many sections, including reinforced concrete |31|,|32| and thin-walled metallic section |33|.

The elastoplastic matrix $E_{ep}$ is obtained using the same procedure as indicated in Section 3, |30|. Naturally, when we use the simplifyed plastic relation

$$\frac{|M|}{M_p} - 1 = 0 \qquad (49)$$

we obtain the stiffness matrix corresponding to a bar with bending moment discontinuity that corresponds to the classical plastic hinges analyses. The use of the generalized yield function (48) is mandatory whenever normal and shear forces are important. This is the case with arches and with some large deformations situations |35|,|29|.

4.2. Metal forming problems

One of the most promising fields for the application of finite strains elastoplastic analysis is probably the study of metal forming processes. In the past, approximate methods have been developed and used for several fabrication processes, based on the hipothesis of rigid-plastic materials without hardening |8|. These methods are useful to predict deformation loads, approximate global deformations and qualitative models of plastic flow. But we cannot control the effects of friction and hardening neither determine the internal stresses generated during the deformation process.

Only after the development of the finite elements technique it begun to be possible a more precise determination of these effects and the role played by the different parameters.

Fig. 4 indicates the results of the numerical analysis using ESFINGE |35|, with experimental results |36| for an upsetting process. We see that the approximation is fairly good even for finite deformations.

A difficult question in the analysis of metal forming is the adequate representation of unilateral contact and friction. A promissing approach has been proposed in ref. |37| that we shortly review in the following section.

Unilateral contact and friction analysis: the problem of unilateral contact may be expressed

$$\Delta u_n - \Delta g_n = 0 \qquad \text{on } \Gamma_c \qquad (50)$$

where $\Delta u_n$ is the increment of the normal displacement constrained by a given motion $\Delta g_n$ on the contact surface $\Gamma_c$. The constraint (50) may be approximated by a penalty condition |38|,|39|

$$k(\Delta u_n - \Delta g_n) = 0 \qquad (51)$$

where k is much larger than the element stiffness. Physically, this can be interpreted as adding to the degree of

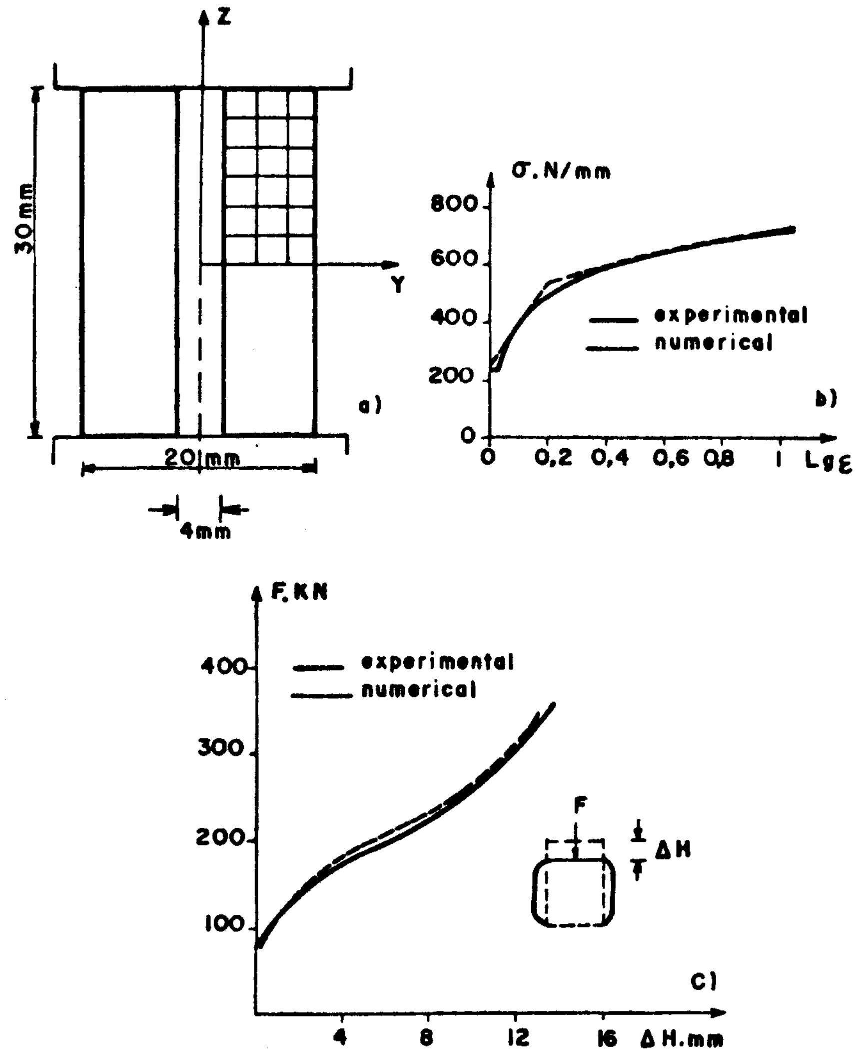



Fig. 4

freedom i a spring of large stiffness k and specifying a load which because of the relative rigidity produces the required displacement $\Delta g_n$. This procedure may eventually introduce large off-diagonal terms into the stiffness matrix, diminishing the solution accuracy, |40|. A different approach may be found in |41|.

For the friction problem, we may assume that there exists a Coulomb isotropic slip function |37|,|42|

$$f = \sqrt{t_T \cdot t_T} + \mu_F \, t_n \tag{52}$$

where $t_T$ and $t_n$ are the tangencial and normal components of the stress at the contact region. The situation of non-sliding and sliding may then be written as the conditions for loading in the general plastic case (Section 3)

$$f < 0 \quad ; \quad \Delta t_T = - k_T \, (\Delta u_T - \Delta g_T) \tag{53}$$

$$f = 0 \quad ; \quad \dot{f} = 0 \quad ; \quad \Delta u_T = - \Lambda \frac{\partial f}{\partial t_T} \tag{54}$$

Using (52),(53),(54) we can determine relations similar to a normal elastoplastic relation. The functional relation used in the finite elements formulation must naturally be completed with the work performed by the contact stress vector during the slip.

Plastic behavior of geomaterials: the analysis of elastoplastic behavior of geomaterials is a field of growing interest; for a recent review see for example |43|. One important problem concerns the failure behavior of soft rock masses in relation to the construction of tunnels and the excavation of mines |44|, which is strongly related to negative hardening (or softening) in the plastic range.

The simplest approach to softening behavior uses the same formulation of classical plasticity and a negative hardening parameter, |45|,|23|. In Fig. 5 we may see some results corresponding to a thick walled tube under external pressure in a plane strain condition |46|; the variation of radial stress $\sigma_x$ and circunsferential stress $\sigma_y$ is shown for three characteristic stages. Strain softening produces a concentration of stress in an outer ring in a state close to three-axial loading. Stresses are very low in the already degraded material near the central hole. In the load-displa-

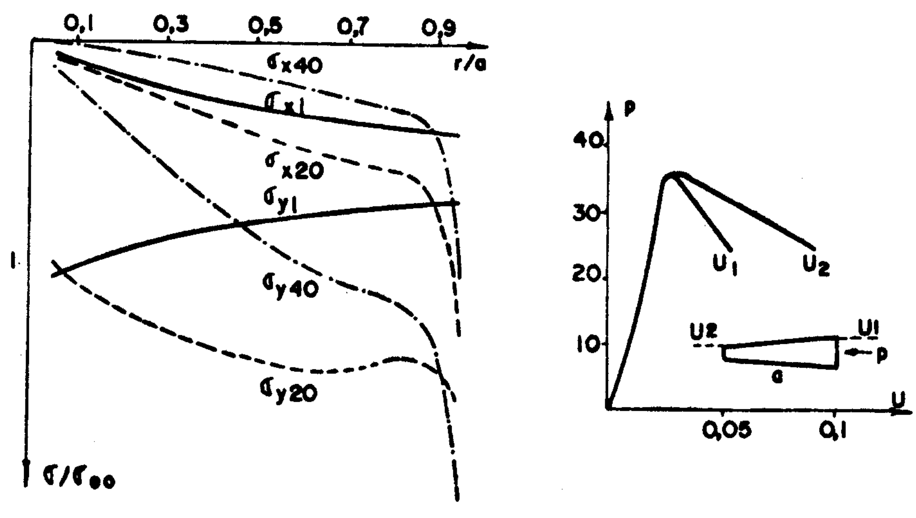



Fig. 5

cement behavior in Fig. 5 we may observe the larger displacements corresponding to the elements near the inner edge. This behavior and the general stress distribution seems to represent well that observed for deep tunnels in strain-softening materials.

The concentration of plastic zones in strain softening bodies is a well known phenomenon |23|,|45|, and can be physically interpreted. On the other hand, in finite elements analyses it produces numerical instability and sensitivity to mesh size |23|,|45|,|47| that seem unnatural. It has also been stated that dynamic problems with continuous strain softening are not mathematically well posed |48|. Several ways have been already proposed to circunvent these difficulties: the use of rate dependent (viscoplastic) relations, |48|, the consideration of concentrated shear bands |49| and the use of nonlocal theories |50|.

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