Mecánica Computacional Vol. XXIII, pp. 545-562
G.Buscaglia, E.Dari, O.Zamonsky (Eds.)
Bariloche, Argentina, November 2004

AN ENERGY PRESERVING SCHEME FOR CONSTRAINED FLEXIBLE NONLINEAR MULTIBODY SYSTEMS

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Key Words: Flexible multibody dynamics, Nonlinear beam, Energy conservation, Large finite rotations.

Abstract. *An energy conserving algorithm which was presented in previous works, is now being extended to flexible problems by developing the formulation of a nonlinear large rotations beam. Although the general formulation in previous works included flexibility, only rigid multibody dynamics problems were tested. Flexibility is dealt with easily in energy conserving algorithms only for finite element models with displacement degrees of freedom. However, beam models which have rotation degrees of freedom, are more cumbersome to be handled. The beam model which we introduce in this paper has been simplified and lead to quite compact expressions of its different terms. The time integration algorithm proves now to be able to deal with flexible multibody dynamics problems. This kind of algorithms have many advantages, both theoretical and practical, because of its unconditional stability which is warranted even in the nonlinear regime.*

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1 INTRODUCTION

It is a well known fact that the time integration of the second-order, index 2 DAE equations which arise in multibody dynamics may lead to numerical instability when we attempt to use a method of the Newmark type.[1–3] Several modifications to the Newmark scheme were proposed (HHT scheme,[2] α-Generalized method[4]), introducing high frequency algorithmic dissipation to remedy this situation. An alternative way to achieve stability is based on the energy preservation property of time integration schemes.[5] Several authors have introduced energy preserving and/or decaying schemes for a variety of problems: constrained and unconstrained rigid bodies dynamics,[6–10] nonlinear elastodynamics,[11–17] nonlinear dynamics of shells and beams.[18,19]

The main focus of this work is on the derivation of an algorithm for which unconditionally stability can be proved in the nonlinear regime. A scheme based on Time Continuous Galerkin approximations applied to the equations of motion is proposed in the frame of a variational formulation. The energy preservation argument is used to prove its unconditional stability.

We have tested the scheme for rigid multibody systems in previous works.[10] Kinematic constraints were enforced via the Lagrange multipliers technique. A general formulation was presented which takes account of most types of joints.

Now, we are extending the formulation to include flexibility of beams. Beams are modelled using the Finite Elements Method. A new flexible beam formulation is described, which is based on the idea of a multiplicative decomposition of incremental rotations along the beam length to interpolate the rotations field. The formulation leads to simple expressions, which are easier to implement than those of the formulation we proposed in previous works.[1,3] Also, this new beam formulation is suitable to be modified and implemented in the context of the energy conserving algorithm, as described in this paper.

2 GENERAL FORMULATION OF THE DYNAMICS OF A MULTIBODY SYSTEM

Given a conservative mechanical system described in terms of N generalized coordinates $\boldsymbol{q}$ and subjected to R algebraic constraints

$$\boldsymbol{\Phi}(\boldsymbol{q}) = \boldsymbol{0}, \tag{1}$$

its mechanical properties can be derived by an adequate description of its potential energy $\mathcal{V} = \mathcal{V}(\boldsymbol{q})$ and its kinetic energy which can be written without loss of generality in the quadratic form

$$\mathcal{K} = \frac{1}{2}\boldsymbol{v}^T \boldsymbol{M} \boldsymbol{v}. \tag{2}$$

The $(M \times M)$ inertia matrix $\boldsymbol{M}$ can be assumed constant, symmetric and positive definite provided that the velocities $\boldsymbol{v}$ are expressed in a *material frame*. The latter are treated as quasi-coordinates and thus take the form of linear combinations of time derivatives of generalized coordinates

$$\boldsymbol{v} = \boldsymbol{L}(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{3}$$

being $\boldsymbol{L}(\boldsymbol{q})$ a $(M \times N)$ matrix.[20]

The inequality $M \leq N$ covers the case where the angular velocities description is made in terms of redundant rotation parameters such as the Euler parameters. In this case the redundancy between parameters has to be removed by adding appropriate constraints to the global set (1).

The equations of motion result from the application of the Hamilton principle which can be written taking into account the system holonomic constraints (1) and the constraints which relate the material velocities and the time derivatives of generalized coordinates (3):

$$\delta \int_{t_1}^{t_2} \left\{ \frac{1}{2} \boldsymbol{v}^T \boldsymbol{M} \boldsymbol{v} - \boldsymbol{\mu}^T \left(\boldsymbol{v} - \boldsymbol{L}(\boldsymbol{q}) \dot{\boldsymbol{q}} \right) - \mathcal{V}(\boldsymbol{q}) - \boldsymbol{\lambda}^T \boldsymbol{\Phi}(\boldsymbol{q}) \right\} dt = 0 \tag{4}$$

If we successively perform the variations of $\boldsymbol{\mu}$, $\boldsymbol{\lambda}$, $\boldsymbol{v}$ and $\boldsymbol{q}$:

- the variation of the multipliers $\boldsymbol{\mu}$ restores the velocity equations (3)
- the variation of the multipliers $\boldsymbol{\lambda}$ restores the constraints set (1)
- the variation $\delta \boldsymbol{v}$ shows that the multipliers $\boldsymbol{\mu}$ have the meaning of generalized momenta

$$\boldsymbol{\mu} = \boldsymbol{M} \boldsymbol{v} \tag{5}$$

- the variation of the generalized displacements $\boldsymbol{q}$ yields

$$\int_{t_1}^{t_2} \left\{ \delta \boldsymbol{q}^T \left(-\frac{\partial \mathcal{V}}{\partial \boldsymbol{q}} - \frac{\partial \boldsymbol{\Phi}^T}{\partial \boldsymbol{q}} \boldsymbol{\lambda} + \frac{\partial}{\partial \boldsymbol{q}} \left[(\boldsymbol{L} \dot{\boldsymbol{q}})^T \boldsymbol{\mu} \right] \right) + \delta \dot{\boldsymbol{q}}^T \boldsymbol{L}^T \boldsymbol{\mu} \right\} dt = 0 \tag{6}$$

  from which the equilibrium equations will be obtained.

The integration by parts of equation (6) yields

$$\left[\delta \boldsymbol{q}^T \boldsymbol{L}^T \boldsymbol{\mu} \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \delta \boldsymbol{q}^T \left\{ -\frac{\partial \mathcal{V}}{\partial \boldsymbol{q}} - \frac{\partial \boldsymbol{\Phi}^T}{\partial \boldsymbol{q}} \boldsymbol{\lambda} + \frac{\partial}{\partial \boldsymbol{q}} \left[(\boldsymbol{L} \dot{\boldsymbol{q}})^T \boldsymbol{\mu} \right] - \frac{d}{dt} \left(\boldsymbol{L}^T \boldsymbol{\mu} \right) \right\} dt = 0 \tag{7}$$

and the combination of (5) and (3) yields

$$\boldsymbol{\mu} = \boldsymbol{M} \boldsymbol{L}(\boldsymbol{q}) \dot{\boldsymbol{q}} \tag{8}$$

The equations of motion thus result in the form of a first-order differential-algebraic system of equations in $\boldsymbol{q}$, $\boldsymbol{\mu}$, and $\boldsymbol{\lambda}$:

$$\left\{ \begin{aligned} \boldsymbol{L}^T \dot{\boldsymbol{\mu}} + \frac{\partial \mathcal{V}}{\partial \boldsymbol{q}} + \boldsymbol{B}^T \boldsymbol{\lambda} + \dot{\boldsymbol{L}}^T \boldsymbol{\mu} - \frac{\partial}{\partial \boldsymbol{q}} \left[(\boldsymbol{L} \dot{\boldsymbol{q}})^T \boldsymbol{\mu} \right] &= \boldsymbol{0} \\ \boldsymbol{\mu} - \boldsymbol{M} \boldsymbol{L} \dot{\boldsymbol{q}} &= \boldsymbol{0} \\ \boldsymbol{\Phi} &= \boldsymbol{0} \end{aligned} \right. \tag{9}$$

where the matrix $\boldsymbol{B} = \partial \boldsymbol{\Phi} / \partial \boldsymbol{q}$ in equation (9-a) is the $R \times N$ Jacobian matrix of constraints. It is worth noting at this stage that the latter two terms in equation (9-a) can be put in the form

$$\dot{\boldsymbol{L}}^T \boldsymbol{\mu} - \frac{\partial}{\partial \boldsymbol{q}} \left[(\boldsymbol{L}\dot{\boldsymbol{q}})^T \boldsymbol{\mu} \right] = \boldsymbol{G}(\boldsymbol{\mu}, \boldsymbol{q}) \dot{\boldsymbol{q}} \tag{10}$$

where the skew-symmetric matrix $\boldsymbol{G}(\boldsymbol{\mu}, \boldsymbol{q})$ has the following components:

$$G_{jp} = \sum_i \mu_i \left(\frac{\partial L_{ij}}{\partial q_p} - \frac{\partial L_{ip}}{\partial q_j} \right) \tag{11}$$

The skew-symmetry of (11) follows from its very definition. The final form of the equations of motion is thus

$$\boxed{\begin{cases} \boldsymbol{L}^T \dot{\boldsymbol{\mu}} + \dfrac{\partial \mathcal{V}}{\partial \boldsymbol{q}} + \boldsymbol{B}^T \boldsymbol{\lambda} + \boldsymbol{G} \dot{\boldsymbol{q}} = \boldsymbol{0} \\ \boldsymbol{\mu} - \boldsymbol{M} \boldsymbol{L} \dot{\boldsymbol{q}} = \boldsymbol{0} \\ \boldsymbol{\Phi} = \boldsymbol{0} \end{cases}} \tag{12}$$

3 TIME DISCRETIZATION APPLYING THE MIDPOINT RULE

The mid-point integration rule is based on the application of the mean value theorem which states that, for any continuous and differentiable function $y(t)$, there is a scalar $\alpha \in [0, 1]$ such that $y(t + h)$ can be expressed in the form

$$y(t+h) = y(t) + h \left. \frac{dy}{dt} \right|_{t+\alpha h} \tag{13}$$

When applied to the solution of a first-order non linear differential equation

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, t) \tag{14}$$

and imposing $\alpha = \frac{1}{2}$, it yields the second-order accurate difference formula known as the mid-point rule

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h \boldsymbol{f}(\boldsymbol{y}_{n+\frac{1}{2}}, t_{n+\frac{1}{2}}) + O(h^2) \quad \text{where} \quad t_{n+\frac{1}{2}} = \frac{1}{2}(t_n + t_{n+1}) \tag{15}$$

which is equivalent to the trapezoidal rule in the linear case. The application of the mid-point rule to the momenta and to the generalized coordinates time derivatives yields

$$\begin{aligned} \dot{\boldsymbol{\mu}}_{n+\frac{1}{2}} &= \frac{1}{h}(\boldsymbol{\mu}_{n+1} - \boldsymbol{\mu}_n) \\ \dot{\boldsymbol{q}}_{n+\frac{1}{2}} &= \frac{1}{h}(\boldsymbol{q}_{n+1} - \boldsymbol{q}_n) \end{aligned} \tag{16}$$

By making use of equations (16) the equilibrium equation (12-a) is thus discretized in the form

$$\frac{1}{h}\boldsymbol{L}^T_{n+\frac{1}{2}}(\boldsymbol{\mu}_{n+1}-\boldsymbol{\mu}_n)+\left(\frac{\partial\mathcal{V}}{\partial\boldsymbol{q}}+\boldsymbol{B}^T\boldsymbol{\lambda}\right)_{n+\frac{1}{2}}+\frac{1}{h}\boldsymbol{G}_{n+\frac{1}{2}}(\boldsymbol{q}_{n+1}-\boldsymbol{q}_n)=\boldsymbol{0} \tag{17}$$

We also discretize the relationship (8) between momenta and time derivatives of the generalized coordinates expressed at the time $t_{n+\frac{1}{2}}$ in the form

$$\boldsymbol{\mu}_{n+\frac{1}{2}}=\frac{1}{2}(\boldsymbol{\mu}_{n+1}+\boldsymbol{\mu}_n)=\frac{1}{h}\boldsymbol{M}\boldsymbol{L}_{n+\frac{1}{2}}(\boldsymbol{q}_{n+1}-\boldsymbol{q}_n) \tag{18}$$

Combining (17) and (18) yields the final form of discretized equilibrium:

$$\frac{2}{h^2}\left(\boldsymbol{L}^T\boldsymbol{M}\boldsymbol{L}\right)_{n+\frac{1}{2}}(\boldsymbol{q}_{n+1}-\boldsymbol{q}_n)-\frac{2}{h}\boldsymbol{L}^T_{n+\frac{1}{2}}\boldsymbol{\mu}_n+\left(\frac{\partial\mathcal{V}}{\partial\boldsymbol{q}}+\boldsymbol{B}^T\boldsymbol{\lambda}\right)_{n+\frac{1}{2}}+\frac{1}{h}\boldsymbol{G}_{n+\frac{1}{2}}(\boldsymbol{q}_{n+1}-\boldsymbol{q}_n)=\boldsymbol{0} \tag{19}$$

Equation (19) and the constraint equation (1) can both be solved in an iterative form to obtain $\boldsymbol{q}_{n+1}$ and $\boldsymbol{\lambda}_{n+\frac{1}{2}}$.

The properties of this integration scheme in order to achieve the preservation of the total energy have been discussed elsewhere.[10] Let us recall that by multiplying equation (17) by the jump of displacements $\overline{\Delta}\boldsymbol{q}=(\boldsymbol{q}_{n+1}-\boldsymbol{q}_n)$ over the time step, we arrive to a series of conditions to be verified for energy preservation. Concerning the potential energy term, it can be shown that energy is preserved by replacing the midpoint derivative $(\partial\mathcal{V}/\partial\boldsymbol{q})_{n+\frac{1}{2}}$ by the approximation $(\partial\mathcal{V}/\partial\boldsymbol{q})^*_{n+\frac{1}{2}}$ (*discrete directional derivative*[21]) such that:

$$\boxed{(\boldsymbol{q}_{n+1}-\boldsymbol{q}_n)^T\left.\frac{\partial\mathcal{V}}{\partial\boldsymbol{q}}\right|^*_{n+\frac{1}{2}}=\overline{\Delta}\boldsymbol{q}^T\boldsymbol{f}^*_{int\ n+\frac{1}{2}}=\mathcal{V}_{n+1}-\mathcal{V}_n} \tag{20}$$

The term $\boldsymbol{f}^*_{int\ n+\frac{1}{2}}$ is the vector of internal forces of elastic origin, and depends on the particular finite element model. Note also that we use the notation

$$\overline{\Delta}(\bullet)=(\bullet)_{n+1}-(\bullet)_n \tag{21}$$

to indicate the difference from quantities at time t_{n+1} with respect to quantities at time t_n.

4 SPHERICAL MOTION AND ROTATIONS PARAMETERIZATION

Spherical motion corresponds to the rotation of a rigid body about a fixed point in space. The length of the position vector of a given point P attached to the body is not affected by the pure rotation and the relative angle between any two directions attached to the body remains constant under the transformation. If we describe with $\boldsymbol{X}$ the position vector of the point P in the reference configuration and with $\boldsymbol{x}$ the position vector of point P after transformation, the pure rotation can be expressed as the following linear transformation

$$\boldsymbol{x}=\boldsymbol{R}\boldsymbol{X} \tag{22}$$

being $\boldsymbol{R}$ proper orthogonal: $\boldsymbol{R}^T = \boldsymbol{R}^{-1}$ and $\det(\boldsymbol{R}) = 1$. The absolute velocity vector of point P is computed as

$$\boldsymbol{v} = \dot{\boldsymbol{R}}\boldsymbol{X} = \boldsymbol{R}\widetilde{\boldsymbol{\Omega}}\boldsymbol{X} \tag{23}$$

where $\tilde{\boldsymbol{\Omega}}$ is the skew-symmetric matrix of angular velocities, defined by

$$\widetilde{\boldsymbol{\Omega}} = \boldsymbol{R}^T\dot{\boldsymbol{R}} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} \tag{24}$$

The angular velocity vector can then be computed in terms of the rotation operator $\boldsymbol{R}$ as

$$\boldsymbol{\Omega} = \text{vect}(\widetilde{\boldsymbol{\Omega}}) = \text{vect}(\boldsymbol{R}^T\dot{\boldsymbol{R}}) \tag{25}$$

If we apply the midpoint rule to the time incrementation of the finite rotations, the discrete approximation of the angular velocities can be computed as

$$\widetilde{\boldsymbol{\Omega}}_{n+\frac{1}{2}} = \boldsymbol{R}^T_{n+\frac{1}{2}}\dot{\boldsymbol{R}}_{n+\frac{1}{2}} \simeq \frac{1}{h}\boldsymbol{R}^T_{n+\frac{1}{2}}(\boldsymbol{R}_{n+1} - \boldsymbol{R}_n) \tag{26}$$

In order to define the configuration that is half-way between $\boldsymbol{R}_n$ and $\boldsymbol{R}_{n+1}$ let us decompose the rotation increment from $\boldsymbol{R}_n$ to $\boldsymbol{R}_{n+1}$ in the form

$$\boldsymbol{R}_n^T\boldsymbol{R}_{n+1} = \boldsymbol{F}^2 \tag{27}$$

The resulting operator $\boldsymbol{F}$ is such that

$$\boldsymbol{R}_{n+\frac{1}{2}} = \boldsymbol{R}_n\boldsymbol{F} = \boldsymbol{R}_{n+1}\boldsymbol{F}^T \tag{28}$$

and verifies the properties of orthonormality $\boldsymbol{F}\boldsymbol{F}^T = \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{I}$. After replacing equation (28) into (26), the angular velocity can be computed as

$$\widetilde{\boldsymbol{\Omega}}_{n+\frac{1}{2}} = \frac{1}{h}(\boldsymbol{F} - \boldsymbol{F}^T) \tag{29}$$

and the material rotational increment takes the form

$$\Delta\widetilde{\boldsymbol{\Theta}} = (\boldsymbol{F} - \boldsymbol{F}^T) \tag{30}$$

The operator $\boldsymbol{F}$ can be described in terms of the invariants $(\boldsymbol{n},\, \Delta\phi)$ of the relative rotation in the form

$$\boldsymbol{F} = \boldsymbol{R}(\boldsymbol{n}, \frac{1}{2}\Delta\phi) \tag{31}$$

If we choose for instance the Euler parameters, we have

$$\text{vect}(\boldsymbol{F}) = \boldsymbol{n}\sin\frac{1}{2}\Delta\phi = \boldsymbol{e} \tag{32}$$

where $\boldsymbol{e}$ is the vectorial part of the Euler parameters of the relative rotation. The operator $\boldsymbol{F}$ thus has the following explicit form

$$\boldsymbol{F} = e_0\boldsymbol{I} + \frac{1}{1+e_0}\boldsymbol{e}\boldsymbol{e}^T + \widetilde{\boldsymbol{e}} \tag{33}$$

and we get the following simplified approximations

$$\boldsymbol{\Omega}_{n+\frac{1}{2}} \simeq \frac{2}{h}\boldsymbol{e} \qquad\qquad \Delta\boldsymbol{\Theta} \simeq 2\boldsymbol{e} \tag{34}$$

After identification with equation (3), we may see that for this parameterization the matrix $\boldsymbol{L}_{n+\frac{1}{2}} = 2\boldsymbol{I}$ is constant.

5 FORMULATION OF A NONLINEAR BEAM

5.1 Kinematics of a beam

Let $(\boldsymbol{X}_0(s), \boldsymbol{R}_E)$ be the reference configuration of the beam such that

$$\boldsymbol{X}_0(s) = \frac{1}{L}(\boldsymbol{X}_B s + \boldsymbol{X}_A(L-s)) \qquad s \in [0,\ L] \tag{35}$$

where $\boldsymbol{X}_A,\ \boldsymbol{X}_B$ are the reference positions of the beam element end nodes A and B, and

$$\boldsymbol{E}_i = \boldsymbol{R}_E\boldsymbol{i}_i \qquad (i=1,2,3) \qquad \rightarrow \qquad \boldsymbol{R}_E = [\boldsymbol{E}_1\ \boldsymbol{E}_2\ \boldsymbol{E}_3] \tag{36}$$

The reference configuration of point P of relative coordinates $\boldsymbol{Y} = (s, Y_2, Y_3)$ in local beam axes is

$$\boldsymbol{X}(s) = \boldsymbol{X}_0(s) + \boldsymbol{R}_E\boldsymbol{Y} \tag{37}$$

and the current configuration can be described as

$$\boldsymbol{x}(s) = \boldsymbol{X}_0(s) + \boldsymbol{u}(s) + \boldsymbol{R}(s)\boldsymbol{R}_E\boldsymbol{Y} \tag{38}$$

where $\boldsymbol{u}(s)$ is the displacement field of the centerline along the beam axis. It is interpolated linearly as usual:

$$\boldsymbol{u}(s) = N_A(s)\boldsymbol{u}_A + N_B(s)\boldsymbol{u}_B \tag{39}$$

where $N_A(s),\ N_B(s)$ are the linear shape functions, and $\boldsymbol{u}_A,\ \boldsymbol{u}_B$ are the displacements at nodes A and B of the beam. The displacements at the middle of the beam write

$$\boldsymbol{u}_{0.5} = \frac{1}{2}(\boldsymbol{u}_A + \boldsymbol{u}_B) \tag{40}$$

The rotation $\boldsymbol{R}(s)$ in (38) describes the finite rotation of the cross-section at point s from reference to actual configuration. Let $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ be the finite rotations (3×3 orthogonal matrices) at nodes A and B of the beam. Finite rotations are objects that lie on a curved manifold (the so called special orthogonal group $SO3$) and do not form vector space. For this reason, we are not able to interpolate the rotations field as we did for the displacements (note that $\frac{1}{2}(\boldsymbol{R}_A + \boldsymbol{R}_B)$ is not orthogonal unless both rotations have a common axis).

We will express instead the rotation at the mid-point of the beam in the form:

$$\boldsymbol{R}_{0.5} = \boldsymbol{R}_A \boldsymbol{H} = \boldsymbol{R}_B \boldsymbol{H}^T \tag{41}$$

where the rotation increment $\boldsymbol{H}$ is written:

$$\boldsymbol{H} = \sqrt{\boldsymbol{R}_A^T \boldsymbol{R}_B} = \exp(\widetilde{\boldsymbol{\phi}}) \tag{42}$$

Here, $\boldsymbol{\phi}$ is the axial vector of three rotation parameters corresponding to the rotation $\boldsymbol{H}$.

The deformation of the beam in material frame is computed from:[3]

$$\boldsymbol{D}(s) = \boldsymbol{R}_E^T \left(\boldsymbol{R}(s)^T \frac{d\boldsymbol{x}}{ds} - \frac{d\boldsymbol{X}}{ds} \right) \tag{43}$$

which, owing to (35-38), can be put in the form

$$\boldsymbol{D}(s) = \boldsymbol{\Gamma}(s) + \tilde{\boldsymbol{K}}(s)\boldsymbol{Y} \tag{44}$$

where

– $\boldsymbol{\Gamma}(s)$ is the deformation of the neutral axis

$$\boldsymbol{\Gamma}(s) = \boldsymbol{R}_E^T \boldsymbol{R}(s)^T \left(\boldsymbol{E}_1 + \frac{d\boldsymbol{u}}{ds}(s) \right) - \boldsymbol{i}_1 \tag{45}$$

with $\boldsymbol{i}_1^T = (1\ 0\ 0)$. The deformation at the middle of the beam is thus interpolated in the form

$$\boxed{\boldsymbol{\Gamma}_{0.5} = \boldsymbol{R}_E^T \boldsymbol{R}_{0.5}^T \left(\boldsymbol{E}_1 + \frac{\boldsymbol{u}_B - \boldsymbol{u}_A}{L} \right) - \boldsymbol{i}_1} \tag{46}$$

– $\boldsymbol{K}(s)$ is the curvature vector of the neutral axis extracted from the skew-symmetric matrix

$$\tilde{\boldsymbol{K}}(s) = \boldsymbol{R}_E^T \left(\boldsymbol{R}(s)^T \frac{d\boldsymbol{R}}{ds}(s) \right) \boldsymbol{R}_E \tag{47}$$

By approximating the derivative of $\boldsymbol{R}$ as the difference between rotations at the beam nodes over the beam length, we may write the curvature tensor at the mid-point of the beam in the form:

$$\widetilde{\boldsymbol{K}}_{0.5} = \boldsymbol{R}_E^T \frac{\boldsymbol{H} - \boldsymbol{H}^T}{L} \boldsymbol{R}_E \tag{48}$$

We remark that this form of computing the curvature is not fully consistent with the particular character of rotations. However, this expression may be acceptable whenever rotations at nodes A and B do not differ too much between them.

The corresponding axial curvature vector is then expressed:

$$\boxed{\boldsymbol{K}_{0.5} = \boldsymbol{R}_E^T \, \frac{2\,\mathrm{vect}(\boldsymbol{H})}{L}} \tag{49}$$

where the *vector part* of a matrix $\boldsymbol{A}$ is defined : $[\mathrm{vect}(\boldsymbol{A})]_i = \varepsilon_{ijk} A_{kj}/2$.

5.2 Rotations and derivatives of rotations at the middle of the beam

In order to compute the different terms of the beam formulation, we will need to differentiate the expressions of deformations (equations (46) and (49)). Therefore, we will necessitate derivatives of the rotation operator in terms of rotation parameters at the nodes.

The middle rotation has been expressed as :

$$\boldsymbol{R}_{0.5} = \boldsymbol{R}_A \boldsymbol{H} = \boldsymbol{R}_A \exp(\widetilde{\boldsymbol{\phi}}) \tag{50}$$

The variation of rotations at the middle of the beam then results:

$$\delta \boldsymbol{R}_{0.5} = \boldsymbol{R}_A \delta \widetilde{\boldsymbol{\Theta}}_A \boldsymbol{H} + \boldsymbol{R}_A \boldsymbol{H} \delta \widetilde{\boldsymbol{\phi}} = \boldsymbol{R}_{0.5} \delta \widetilde{\boldsymbol{\Theta}}_{0.5} \tag{51}$$

with $\boldsymbol{\Theta}_A, \boldsymbol{\Theta}_{0.5}$ the axial rotation vectors corresponding to $\boldsymbol{R}_A, \boldsymbol{R}_{0.5}$. Therefore:

$$\delta \widetilde{\boldsymbol{\Theta}}_{0.5} = \boldsymbol{R}_{0.5}^T \boldsymbol{R}_A \delta \widetilde{\boldsymbol{\Theta}}_A \boldsymbol{H} + \delta \widetilde{\boldsymbol{\phi}} = \delta \widetilde{\boldsymbol{\phi}} + \boldsymbol{H}^T \delta \widetilde{\boldsymbol{\Theta}}_A \boldsymbol{H} \tag{52}$$

and then, in terms of axial vectors we write:

$$\delta \boldsymbol{\Theta}_{0.5} = \delta \boldsymbol{\phi} + \boldsymbol{H}^T \delta \boldsymbol{\Theta}_A \tag{53}$$

From the definition (eq (42)), $\boldsymbol{H}^2 = \boldsymbol{R}_A^T \boldsymbol{R}_B$, and therefore, the variation of $\boldsymbol{H}$ may be computed by taking variations giving:

$$\boldsymbol{H} \delta \boldsymbol{H} + \delta \boldsymbol{H} \boldsymbol{H} = \boldsymbol{R}_A^T \boldsymbol{R}_B \delta \widetilde{\boldsymbol{\Theta}}_B - \delta \widetilde{\boldsymbol{\Theta}}_A \boldsymbol{R}_A^T \boldsymbol{R}_B \tag{54}$$

Since $\delta \boldsymbol{H} = \boldsymbol{H} \delta \boldsymbol{\phi}$, then:

$$\boldsymbol{H}^2 \delta \widetilde{\boldsymbol{\phi}} + \boldsymbol{H} \delta \widetilde{\boldsymbol{\phi}} \boldsymbol{H} = \boldsymbol{H}^2 \delta \widetilde{\boldsymbol{\Theta}}_B - \delta \widetilde{\boldsymbol{\Theta}}_A \boldsymbol{H}^2 \quad , \tag{55}$$

and in terms of axial vectors, we write

$$\delta \boldsymbol{\phi} + \boldsymbol{H}^T \delta \boldsymbol{\phi} = \delta \boldsymbol{\Theta}_B - {\boldsymbol{H}^T}^2 \delta \boldsymbol{\Theta}_A \tag{56}$$

Finally, the variation of ϕ may be computed as follows:

$$\boxed{\delta\boldsymbol{\phi} = [\boldsymbol{I} + \boldsymbol{H}^T]^{-1}\left(\delta\boldsymbol{\Theta}_B - \boldsymbol{H}^{T^2}\delta\boldsymbol{\Theta}_A\right)} \tag{57}$$

By replacing the latter equation into equation (53), we get:

$$\delta\boldsymbol{\Theta}_{0.5} = [\boldsymbol{I} + \boldsymbol{H}^T]^{-1}\delta\boldsymbol{\Theta}_B - \left([\boldsymbol{I} + \boldsymbol{H}^T]^{-1}\boldsymbol{H}^T - \boldsymbol{I}\right)\boldsymbol{H}^T\delta\boldsymbol{\Theta}_A \tag{58}$$

By noting that the following identity holds for any orthogonal matrix $\boldsymbol{H}$:

$$\left([\boldsymbol{I} + \boldsymbol{H}^T]^{-1}\boldsymbol{H}^T - \boldsymbol{I}\right)\boldsymbol{H}^T = [\boldsymbol{I} + \boldsymbol{H}^T]^{-1} - \boldsymbol{I} \tag{59}$$

we finally get the expression of variations of the rotational vector at the middle of the beam:

$$\boxed{\delta\boldsymbol{\Theta}_{0.5} = [\boldsymbol{I} + \boldsymbol{H}^T]^{-1}\delta\boldsymbol{\Theta}_B - \left([\boldsymbol{I} + \boldsymbol{H}^T]^{-1} - \boldsymbol{I}\right)\delta\boldsymbol{\Theta}_A} \tag{60}$$

Note that equations (57) and (60) are *exact* and do not make any approximation for the evaluation of the derivatives.

Even if the beam suffers large finite rotations, we may consider that rotations at both extreme nodes are not very different between them (i.e. $\boldsymbol{R}_A \simeq \boldsymbol{R}_B$). Note that the approximation for the computation of the curvature tensor (eq. (48)) is based on this fact. Then, by retaining first order terms in the Taylor series expansion :

$$[\boldsymbol{I} + \boldsymbol{H}^T]^{-1} = \frac{1}{2}\boldsymbol{I} + \frac{1}{4}\left(1 + \frac{1}{12}\|\boldsymbol{\phi}\|^2 + \frac{1}{120}\|\boldsymbol{\phi}\|^4 + \frac{17}{20160}\|\boldsymbol{\phi}\|^6 + \dots\right)\widetilde{\boldsymbol{\phi}} \quad , \tag{61}$$

we may write:

$$\boxed{\delta\boldsymbol{\Theta}_{0.5} \simeq \left[\frac{\boldsymbol{I}}{2} + \frac{\widetilde{\boldsymbol{\phi}}}{4}\right]\delta\boldsymbol{\Theta}_B + \left[\frac{\boldsymbol{I}}{2} - \frac{\widetilde{\boldsymbol{\phi}}}{4}\right]\delta\boldsymbol{\Theta}_A} \tag{62}$$

By a similar reasoning, we get

$$\boxed{\delta\boldsymbol{\phi} \simeq \left[\frac{\boldsymbol{I}}{2} + \frac{\widetilde{\boldsymbol{\phi}}}{4}\right]\delta\boldsymbol{\Theta}_B - \left[\frac{\boldsymbol{I}}{2} - \frac{3\widetilde{\boldsymbol{\phi}}}{4}\right]\delta\boldsymbol{\Theta}_A} \tag{63}$$

5.3 Deformations variations

In order to compute the beam internal forces, we need to compute the expressions of the derivatives of the axial strains and curvatures.

The variation of the material axial deformation $\boldsymbol{\Gamma}_{0.5}$, equation (46), is written:

$$\delta\boldsymbol{\Gamma}_{0.5} = -\boldsymbol{R}_E^T\delta\widetilde{\boldsymbol{\Theta}}_{0.5}\boldsymbol{R}_{0.5}^T\left(\boldsymbol{E}_1 + \frac{\boldsymbol{u}_B - \boldsymbol{u}_A}{L}\right) + \boldsymbol{R}_E^T\boldsymbol{R}_{0.5}^T\left(\frac{\delta\boldsymbol{u}_B - \delta\boldsymbol{u}_A}{L}\right) \tag{64}$$

If we now replace the approximation obtained for the variation of the rotation parameters at the middle of the beam, equation (62), we get:

$$\delta\boldsymbol{\Gamma}_{0.5} = \left(\widetilde{\boldsymbol{\Gamma}_{0.5} + \boldsymbol{i}_1}\right) \boldsymbol{R}_E^T \left[\frac{\boldsymbol{I}}{2} + \frac{\widetilde{\boldsymbol{\phi}}}{4}\right] \delta\boldsymbol{\Theta}_B + \left(\widetilde{\boldsymbol{\Gamma}_{0.5} + \boldsymbol{i}_1}\right) \boldsymbol{R}_E^T \left[\frac{\boldsymbol{I}}{2} - \frac{\widetilde{\boldsymbol{\phi}}}{4}\right] \delta\boldsymbol{\Theta}_A + \boldsymbol{R}_E^T \boldsymbol{R}^T \left(\frac{\delta\boldsymbol{u}_B - \delta\boldsymbol{u}_A}{L}\right) \quad (65)$$

By differentiation of equation (49), the variation of the curvatures axial vector at the beam mid-point may be written as follows:

$$\delta\boldsymbol{K}_{0.5} = \boldsymbol{R}_E^T \left[\frac{\boldsymbol{H}^T - \mathrm{tr}(\boldsymbol{H})\boldsymbol{I}}{L}\right] \delta\boldsymbol{\phi} = \frac{\boldsymbol{R}_E^T}{L} \overline{\mathrm{dev}}[\boldsymbol{H}^T]\delta\boldsymbol{\phi} \quad (66)$$

where $\overline{\mathrm{dev}}[\boldsymbol{H}] = \boldsymbol{H} - \mathrm{tr}(\boldsymbol{H})\boldsymbol{I}$. By using Taylor series developments, we may express $\boldsymbol{H}^T$ and $\mathrm{tr}(\boldsymbol{H})$ in the form:

$$\boldsymbol{H}^T = \boldsymbol{I} - \widetilde{\boldsymbol{\phi}} + \frac{1}{2!}\widetilde{\boldsymbol{\phi}}^2 - \frac{1}{3!}\widetilde{\boldsymbol{\phi}}^3 + \dots \qquad \mathrm{tr}(\boldsymbol{H}) = 3 - \|\boldsymbol{\phi}\|^2 + \frac{1}{6}\|\boldsymbol{\phi}\|^4 + \dots \quad (67)$$

Then, after replacement of these two equations into equation (66), and by using equation (57), we get:

$$\delta\boldsymbol{K}_{0.5} = -\frac{2\boldsymbol{R}_E^T}{L} \left[\boldsymbol{I} + \frac{\widetilde{\boldsymbol{\phi}}}{2} + \dots\right] \delta\boldsymbol{\phi} = \frac{\boldsymbol{R}_E^T}{L} \left\{-\left[\boldsymbol{I} - \widetilde{\boldsymbol{\phi}}\right] \delta\boldsymbol{\Theta}_A + \left[\boldsymbol{I} + \widetilde{\boldsymbol{\phi}}\right] \delta\boldsymbol{\Theta}_B\right\} \quad (68)$$

From now on, we will omit the index 0.5 and notate directly $\boldsymbol{\Gamma}, \boldsymbol{K}, \dots$ for quantities evaluated at the middle of the beam.

Note that $\boldsymbol{K} = 2\boldsymbol{R}_E^T\boldsymbol{\phi}/L$, and therefore

$$\boldsymbol{R}_E^T\boldsymbol{\phi} = \frac{L}{2}\boldsymbol{K} \quad (69)$$

By replacing the latter equation into equations (65) and (68), we get:

$$\delta\boldsymbol{\Gamma} = \left(\frac{\widetilde{\boldsymbol{\Gamma} + \boldsymbol{i}_1}}{2}\right) \left[\boldsymbol{I} + \frac{L\widetilde{\boldsymbol{K}}}{4}\right] \boldsymbol{R}_E^T\delta\boldsymbol{\Theta}_B + \left(\frac{\widetilde{\boldsymbol{\Gamma} + \boldsymbol{i}_1}}{2}\right) \left[\boldsymbol{I} - \frac{L\widetilde{\boldsymbol{K}}}{4}\right] \boldsymbol{R}_E^T\delta\boldsymbol{\Theta}_A + \boldsymbol{R}_E^T\boldsymbol{R}^T \left(\frac{\delta\boldsymbol{u}_B - \delta\boldsymbol{u}_A}{L}\right) \quad (70)$$

$$\delta\boldsymbol{K} = -\left[\frac{\boldsymbol{I}}{L} - \frac{\widetilde{\boldsymbol{K}}}{2}\right] \boldsymbol{R}_E^T\delta\boldsymbol{\Theta}_A + \left[\frac{\boldsymbol{I}}{L} + \frac{\widetilde{\boldsymbol{K}}}{2}\right] \boldsymbol{R}_E^T\delta\boldsymbol{\Theta}_B \quad (71)$$

If we now define matrix $\boldsymbol{B}$ which relates strains variations at the midpoint with variations of nodal displacements and rotations:

$$\boxed{\boldsymbol{B} = \begin{pmatrix} -\frac{\boldsymbol{R}_E^T\boldsymbol{R}^T}{L} & \frac{1}{2}\left(\widetilde{\boldsymbol{\Gamma}+\boldsymbol{i}_1}\right)\left[\boldsymbol{I}-\frac{L}{4}\widetilde{\boldsymbol{K}}\right]\boldsymbol{R}_E^T & \frac{\boldsymbol{R}_E^T\boldsymbol{R}^T}{L} & \frac{1}{2}\left(\widetilde{\boldsymbol{\Gamma}+\boldsymbol{i}_1}\right)\left[\boldsymbol{I}+\frac{L}{4}\widetilde{\boldsymbol{K}}\right]\boldsymbol{R}_E^T \\ \boldsymbol{0} & -\frac{1}{L}\left[\boldsymbol{I}-\frac{L}{2}\widetilde{\boldsymbol{K}}\right]\boldsymbol{R}_E^T & \boldsymbol{0} & \frac{1}{L}\left[\boldsymbol{I}+\frac{L}{2}\widetilde{\boldsymbol{K}}\right]\boldsymbol{R}_E^T \end{pmatrix}}\;, \tag{72}$$

we may write

$$\boxed{\begin{pmatrix} \delta\boldsymbol{\Gamma} \\ \delta\boldsymbol{K} \end{pmatrix} = \boldsymbol{B}\begin{pmatrix} \delta\boldsymbol{u}_A \\ \delta\boldsymbol{\Theta}_A \\ \delta\boldsymbol{u}_B \\ \delta\boldsymbol{\Theta}_B \end{pmatrix}} \tag{73}$$

5.4 Deformation energy

The strain energy is computed by integrating the density of strain energy along the beam length. This integral may be approximated using one Gauss point at the middle of the beam, giving:

$$\mathcal{V} = \frac{1}{2}\int_0^L \begin{pmatrix} \boldsymbol{\Gamma}(s) \\ \boldsymbol{K}(s) \end{pmatrix} \cdot \boldsymbol{C} \begin{pmatrix} \boldsymbol{\Gamma}(s) \\ \boldsymbol{K}(s) \end{pmatrix} ds \simeq \frac{1}{2}\begin{pmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{K} \end{pmatrix} \cdot \boldsymbol{C} \begin{pmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{K} \end{pmatrix} L \tag{74}$$

where $\boldsymbol{C}$ is the matrix of elastic coefficients, which may takes usually the form:

$$\boldsymbol{C} = \text{diag}\left(EA \quad GA_2 \quad GA_3 \quad GJ \quad EI_2 \quad EI_3\right) \tag{75}$$

Here, EA is the axial stiffness, GA_2 and GA_3 are the shear bending stiffnesses along the transverse axes, GJ is the torsional stiffness and EI_2 and EI_3 are the bending stiffnesses.

The internal forces are evaluated by differentiation of the strain energy of the element:

$$\delta\mathcal{V} = \begin{pmatrix} \delta\boldsymbol{\Gamma} \\ \delta\boldsymbol{K} \end{pmatrix} \cdot \boldsymbol{C}L \begin{pmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{K} \end{pmatrix} = \begin{pmatrix} \delta\boldsymbol{u}_A \\ \delta\boldsymbol{\Theta}_A \\ \delta\boldsymbol{u}_B \\ \delta\boldsymbol{\Theta}_B \end{pmatrix} \cdot \boldsymbol{f}_{int} \tag{76}$$

After replacing equation (73) in the latter, we get the expression of the vector of internal forces of the beam:

$$\boxed{\boldsymbol{f}_{int} = \boldsymbol{B}^T \begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix} L = \begin{pmatrix} -\boldsymbol{R}\boldsymbol{R}_E\boldsymbol{N} \\ -\frac{1}{2}\boldsymbol{R}_E\left[\boldsymbol{I}+\frac{L}{4}\widetilde{\boldsymbol{K}}\right]\left(\widetilde{\boldsymbol{\Gamma}+\boldsymbol{i}_1}\right)\boldsymbol{N}L - \boldsymbol{R}_E\left[\boldsymbol{I}+\frac{L}{2}\widetilde{\boldsymbol{K}}\right]\boldsymbol{M} \\ \boldsymbol{R}\boldsymbol{R}_E\boldsymbol{N} \\ -\frac{1}{2}\boldsymbol{R}_E\left[\boldsymbol{I}-\frac{L}{4}\widetilde{\boldsymbol{K}}\right]\left(\widetilde{\boldsymbol{\Gamma}+\boldsymbol{i}_1}\right)\boldsymbol{N}L + \boldsymbol{R}_E\left[\boldsymbol{I}-\frac{L}{2}\widetilde{\boldsymbol{K}}\right]\boldsymbol{M} \end{pmatrix}} \tag{77}$$

where $\begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix} = \boldsymbol{C} \begin{pmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{K} \end{pmatrix}$ is the vector of internal efforts and moments at the middle point evaluated in material axes.

After differentiating the internal forces vector, we get the tangent stiffness $\boldsymbol{S}$:

$$D\boldsymbol{f}_{int} \cdot \Delta\boldsymbol{q} = \boldsymbol{S}\Delta\boldsymbol{q} = \boldsymbol{B}^T\boldsymbol{C}L\boldsymbol{B} \begin{pmatrix} \Delta\boldsymbol{u}_A \\ \Delta\boldsymbol{\Theta}_A \\ \Delta\boldsymbol{u}_B \\ \Delta\boldsymbol{\Theta}_B \end{pmatrix} + (D\boldsymbol{B}^T\Delta\boldsymbol{q}) \cdot L \begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix} \tag{78}$$

The first term on the RHS is the so-called material tangent matrix:

$$\boldsymbol{S}_{mat} = \boldsymbol{B}^T\boldsymbol{C}L\boldsymbol{B} \tag{79}$$

The second term on the RHS of equation (78) is the geometric stiffness matrix, which may be written in the form:

$$\boldsymbol{S}_{geo}\Delta\boldsymbol{q} = D\Big|_{N,M} \left(\boldsymbol{B}^T L \begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix}\right) \cdot \Delta\boldsymbol{q} \tag{80}$$

where $D\Big|_{N,M}\{\} \cdot \Delta\boldsymbol{q}$ denotes the Frechet derivative along the direction $\Delta\boldsymbol{q}$ with $\boldsymbol{N}, \boldsymbol{M}$ held constant. These derivatives can be computed straightforwardly and are not given here for conciseness. Both matrices $\boldsymbol{S}_{mat}$ and $\boldsymbol{S}_{geo}$ are symmetric.

6 FORMULATION OF A BEAM WITH ENERGY CONSERVATION

The key aspect in achieving energy conservation is the computation of internal stresses at the (time) mid-point. Indeed, when the mid-point rule is applied for time integration the work produced by the discrete internal forces takes the form $\boldsymbol{f}^{T\,*}_{int,n+\frac{1}{2}}(\boldsymbol{q}_{n+1} - \boldsymbol{q}_n)$, as shown in section 3.

The strain energy of the beam has the expression:

$$\mathcal{V} = \frac{1}{2} \begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{K} \end{pmatrix} L \tag{81}$$

and therefore:

$$\begin{aligned} \mathcal{V}_{n+1} - \mathcal{V}_n &= \frac{L}{2} \left[\begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix}_{n+1} \cdot \begin{pmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{K} \end{pmatrix}_{n+1} - \begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix}_{n} \cdot \begin{pmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{K} \end{pmatrix}_{n} \right] \\ &= \begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix}_{n+\frac{1}{2}} \cdot \left[\begin{pmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{K} \end{pmatrix}_{n+1} - \begin{pmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{K} \end{pmatrix}_{n} \right] L \end{aligned} \tag{82}$$

Then, if we build a strain matrix $\boldsymbol{B}^*_{n+\frac{1}{2}}$ (discrete directional derivatives) such that:

$$\begin{pmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{K} \end{pmatrix}_{n+1} - \begin{pmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{K} \end{pmatrix}_{n} = \boldsymbol{B}^*_{n+\frac{1}{2}}(\boldsymbol{q}_{n+1} - \boldsymbol{q}_n) \quad , \tag{83}$$

energy will be preserved provided internal forces at the mid-point are computed in the averaged form:

$$\boldsymbol{f}^*_{int,n+\frac{1}{2}} = \boldsymbol{B}^{*\,T}_{n+\frac{1}{2}} \begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix}_{n+\frac{1}{2}} L \tag{84}$$

6.1 Computation of the strains/displacements discrete directional derivatives matrix

In order to arrive to the expression of $\boldsymbol{B}^*_{n+\frac{1}{2}}$, we will use the approximation $\boldsymbol{R}_{0.5} \simeq \frac{\boldsymbol{R}_A + \boldsymbol{R}_B}{2}$ in the computation of the axial strains giving:

$$\boxed{\boldsymbol{\Gamma}_{0.5} = \boldsymbol{R}_E^T \frac{\boldsymbol{R}_A^T + \boldsymbol{R}_B^T}{2} \left(\boldsymbol{E}_1 + \frac{\boldsymbol{u}_B - \boldsymbol{u}_A}{L} \right) - \boldsymbol{i}_i} \tag{85}$$

Note that the *increment* in average rotations between the initial and final time instants may be written:

$$\begin{aligned} 2\overline{\Delta}\boldsymbol{R} = 2\left(\boldsymbol{R}_{n+1} - \boldsymbol{R}_n\right) = \left(\boldsymbol{R}_{A\,n+1} - \boldsymbol{R}_{A\,n}\right) + \left(\boldsymbol{R}_{B\,n+1} - \boldsymbol{R}_{B\,n}\right) = \\ \boldsymbol{R}_{A\,n+\frac{1}{2}}\left(\boldsymbol{F}_A - \boldsymbol{F}_A^T\right) + \boldsymbol{R}_{B\,n+\frac{1}{2}}\left(\boldsymbol{F}_B - \boldsymbol{F}_B^T\right) = \boldsymbol{R}_{A\,n}\boldsymbol{F}_A\widetilde{\overline{\Delta}\boldsymbol{\Theta}_A} + \boldsymbol{R}_{B\,n}\boldsymbol{F}_B\widetilde{\overline{\Delta}\boldsymbol{\Theta}_B} \end{aligned} \tag{86}$$

Note also that the *mean* average rotation between the initial and final time instants results:

$$\frac{\boldsymbol{R}_{n+1} + \boldsymbol{R}_n}{2} = \frac{1}{4}(\boldsymbol{R}_{A\,n+1} + \boldsymbol{R}_{A\,n} + \boldsymbol{R}_{B\,n+1} + \boldsymbol{R}_{B\,n}) = \frac{1}{4}\boldsymbol{R}_{A\,n}(\boldsymbol{F}_A^2 + \boldsymbol{I}) + \frac{1}{4}\boldsymbol{R}_{B\,n}(\boldsymbol{F}_B^2 + \boldsymbol{I}) \tag{87}$$

Finally, using equations (85-87), we get the *identity*:

$$\begin{aligned} \boldsymbol{\Gamma}_{n+1} - \boldsymbol{\Gamma}_n = \frac{\boldsymbol{R}_E^T}{2L}\left(-\widetilde{\overline{\Delta}\boldsymbol{\Theta}_A}\boldsymbol{F}_A^T\boldsymbol{R}_{A\,n}^T - \widetilde{\overline{\Delta}\boldsymbol{\Theta}_B}\boldsymbol{F}_B^T\boldsymbol{R}_{B\,n}^T\right)(L\boldsymbol{E}_1 + \boldsymbol{u}_{B\,n+\frac{1}{2}} - \boldsymbol{u}_{A\,n+\frac{1}{2}}) + \\ \frac{\boldsymbol{R}_E^T}{4L}\left[(\boldsymbol{F}_A^{T\,2} + \boldsymbol{I})\boldsymbol{R}_{A\,n}^T + (\boldsymbol{F}_B^{T\,2} + \boldsymbol{I})\boldsymbol{R}_{B\,n}^T\right](\overline{\Delta}\boldsymbol{u}_B - \overline{\Delta}\boldsymbol{u}_A) = \\ = -\frac{\boldsymbol{R}_E^T}{4L}\left[(\boldsymbol{F}_A^{T\,2} + \boldsymbol{I})\boldsymbol{R}_{A\,n}^T + (\boldsymbol{F}_B^{T\,2} + \boldsymbol{I})\boldsymbol{R}_{B\,n}^T\right]\overline{\Delta}\boldsymbol{u}_A \\ + \frac{\boldsymbol{R}_E^T}{2L}[\boldsymbol{F}_A^T\boldsymbol{R}_{A\,n}^T(L\boldsymbol{E}_1 \widetilde{+ \boldsymbol{u}}_{B\,n+\frac{1}{2}} - \boldsymbol{u}_{A\,n+\frac{1}{2}})]\overline{\Delta}\boldsymbol{\Theta}_A \\ + \frac{\boldsymbol{R}_E^T}{4L}\left[(\boldsymbol{F}_A^{T\,2} + \boldsymbol{I})\boldsymbol{R}_{A\,n}^T + (\boldsymbol{F}_B^{T\,2} + \boldsymbol{I})\boldsymbol{R}_{B\,n}^T\right]\overline{\Delta}\boldsymbol{u}_B \\ + \frac{\boldsymbol{R}_E^T}{2L}[\boldsymbol{F}_B^T\boldsymbol{R}_{B\,n}^T(L\boldsymbol{E}_1 \widetilde{+ \boldsymbol{u}}_{B\,n+\frac{1}{2}} - \boldsymbol{u}_{A\,n+\frac{1}{2}})]\overline{\Delta}\boldsymbol{\Theta}_B \end{aligned} \tag{88}$$

After truncation of the exponential series $\boldsymbol{H} = \exp(\widetilde{\boldsymbol{\phi}}) = \boldsymbol{I} + \widetilde{\boldsymbol{\phi}} + \ldots$ to first order, we may approximate the skew of $\boldsymbol{\phi}$ in the form:

$$\widetilde{\boldsymbol{\phi}} \simeq \sqrt{\boldsymbol{R}_A^T \boldsymbol{R}_B} - \boldsymbol{I} \simeq \frac{1}{2}\left(\boldsymbol{R}_A^T \boldsymbol{R}_B - \boldsymbol{I}\right) \tag{89}$$

By using the latter expression into equation (48), we get the following approximation for the curvatures vector at the mid-point of the beam which is suitable for implementation of the energy conserving time integration algorithm:

$$\boxed{\boldsymbol{K}_{0.5} = \frac{\boldsymbol{R}_E^T}{L}\,\text{vect}(\boldsymbol{R}_A^T \boldsymbol{R}_B - \boldsymbol{I})} \tag{90}$$

Indeed, the time increment of the curvatures at the mid-point of the beam may be written:

$$\boldsymbol{K}_{n+1} - \boldsymbol{K}_n = \frac{\boldsymbol{R}_E^T}{L}\,\text{vect}\left(\boldsymbol{R}_{A\,n+1}^T \boldsymbol{R}_{B\,n+1} - \boldsymbol{R}_{A\,n}^T \boldsymbol{R}_{B\,n}\right) \tag{91}$$

The term between parentheses on the RHS may be transformed giving:

$$\begin{aligned}\boldsymbol{R}_{A\,n+1}^T \boldsymbol{R}_{B\,n+1} - \boldsymbol{R}_{A\,n}^T \boldsymbol{R}_{B\,n} = \boldsymbol{R}_{A\,n+1}^T \boldsymbol{R}_{B\,n+1} - \boldsymbol{R}_{A\,n}^T \boldsymbol{R}_{B\,n+1} + \boldsymbol{R}_{A\,n}^T \boldsymbol{R}_{B\,n+1} - \boldsymbol{R}_{A\,n}^T \boldsymbol{R}_{B\,n} = \\ (\boldsymbol{F}_A^T - \boldsymbol{F}_A)\boldsymbol{R}_{A\,n+\frac{1}{2}}^T \boldsymbol{R}_{B\,n+\frac{1}{2}} \boldsymbol{F}_B - \boldsymbol{F}_A \boldsymbol{R}_{A\,n+\frac{1}{2}}^T \boldsymbol{R}_{B\,n+\frac{1}{2}} (\boldsymbol{F}_B^T - \boldsymbol{F}_B) = \\ -\overline{\Delta}\widetilde{\boldsymbol{\Theta}}_A \boldsymbol{R}_{A\,n+\frac{1}{2}}^T \boldsymbol{R}_{B\,n+1} + \boldsymbol{R}_{A\,n}^T \boldsymbol{R}_{B\,n+\frac{1}{2}} \overline{\Delta}\widetilde{\boldsymbol{\Theta}}_B = -\overline{\Delta}\widetilde{\boldsymbol{\Theta}}_A \boldsymbol{P}_A + \boldsymbol{P}_B \overline{\Delta}\widetilde{\boldsymbol{\Theta}}_B\end{aligned} \tag{92}$$

with $\boldsymbol{P}_A = \boldsymbol{R}_{A\,n+\frac{1}{2}}^T \boldsymbol{R}_{B\,n+\frac{1}{2}} \boldsymbol{F}_B$ and $\boldsymbol{P}_B = \boldsymbol{F}_A \boldsymbol{R}_{A\,n+\frac{1}{2}}^T \boldsymbol{R}_{B\,n+\frac{1}{2}}$. After some algebraic steps, we can verify that:

$$\text{vect}(-\overline{\Delta}\widetilde{\boldsymbol{\Theta}}_A \boldsymbol{P}_A + \boldsymbol{P}_B \overline{\Delta}\widetilde{\boldsymbol{\Theta}}_B) = \frac{1}{2}(\boldsymbol{P}_B^T - \text{tr}(\boldsymbol{P}_B^T)\boldsymbol{I})\overline{\Delta}\boldsymbol{\Theta}_B - \frac{1}{2}(\boldsymbol{P}_A - \text{tr}(\boldsymbol{P}_A)\boldsymbol{I})\overline{\Delta}\boldsymbol{\Theta}_A \quad , \tag{93}$$

and therefore, the curvatures time increment is written:

$$\boldsymbol{K}_{n+1} - \boldsymbol{K}_n = \frac{\boldsymbol{R}_E^T}{2L}\overline{\text{dev}}[\boldsymbol{P}_B^T]\overline{\Delta}\boldsymbol{\Theta}_B - \frac{\boldsymbol{R}_E^T}{2L}\overline{\text{dev}}[\boldsymbol{P}_A]\overline{\Delta}\boldsymbol{\Theta}_A \tag{94}$$

Finally, using equations (88) and (94), we get:

$$\boxed{\begin{pmatrix}\overline{\Delta}\boldsymbol{\Gamma} \\ \overline{\Delta}\boldsymbol{K}\end{pmatrix} = \boldsymbol{B}_{n+\frac{1}{2}}^* \begin{pmatrix}\overline{\Delta}\boldsymbol{u}_A \\ \overline{\Delta}\boldsymbol{\Theta}_A \\ \overline{\Delta}\boldsymbol{u}_B \\ \overline{\Delta}\boldsymbol{\Theta}_B\end{pmatrix}} \tag{95}$$

where

$$\boldsymbol{B}_{n+\frac{1}{2}}^* = \begin{pmatrix}\boldsymbol{B}_{11}^* & \boldsymbol{B}_{12}^* & \boldsymbol{B}_{13}^* & \boldsymbol{B}_{14}^* \\ \boldsymbol{B}_{21}^* & \boldsymbol{B}_{22}^* & \boldsymbol{B}_{23}^* & \boldsymbol{B}_{24}^*\end{pmatrix} \tag{96}$$

and:

$$
\begin{aligned}
\boldsymbol{B}^*_{11} &= -\frac{\boldsymbol{R}_E^T}{4L}\left[(\boldsymbol{F}_A^{T\,2}+\boldsymbol{I})\boldsymbol{R}^T_{A\,n} + (\boldsymbol{F}_B^{T\,2}+\boldsymbol{I})\boldsymbol{R}^T_{B\,n}\right] \\
\boldsymbol{B}^*_{12} &= \frac{\boldsymbol{R}_E^T}{2L}[\boldsymbol{F}_A^T\boldsymbol{R}^T_{A\,n}\widetilde{(L\boldsymbol{E}_1 + \boldsymbol{u}_{B\,n+\frac{1}{2}} - \boldsymbol{u}_{A\,n+\frac{1}{2}})}] \\
\boldsymbol{B}^*_{13} &= \frac{\boldsymbol{R}_E^T}{4L}\left[(\boldsymbol{F}_A^{T\,2}+\boldsymbol{I})\boldsymbol{R}^T_{A\,n} + (\boldsymbol{F}_B^{T\,2}+\boldsymbol{I})\boldsymbol{R}^T_{B\,n}\right] \\
\boldsymbol{B}^*_{14} &= \frac{\boldsymbol{R}_E^T}{2L}[\boldsymbol{F}_B^T\boldsymbol{R}^T_{B\,n}\widetilde{(L\boldsymbol{E}_1 + \boldsymbol{u}_{B\,n+\frac{1}{2}} - \boldsymbol{u}_{A\,n+\frac{1}{2}})}] \\
\boldsymbol{B}^*_{21} &= \boldsymbol{0} \\
\boldsymbol{B}^*_{22} &= -\frac{\boldsymbol{R}_E^T}{2L}\overline{\mathrm{dev}}[\boldsymbol{F}_A^T\boldsymbol{R}^T_{A\,n}\boldsymbol{R}_{B\,n}\boldsymbol{F}_B^2] \\
\boldsymbol{B}^*_{23} &= \boldsymbol{0} \\
\boldsymbol{B}^*_{24} &= \frac{\boldsymbol{R}_E^T}{2L}\overline{\mathrm{dev}}[\boldsymbol{F}_B^T\boldsymbol{R}^T_{B\,n}\boldsymbol{R}_{A\,n}]
\end{aligned}
$$

After replacement into equation (97), we obtain the expression of the averaged internal forces:

$$
\boxed{\boldsymbol{f}^*_{int,n+\frac{1}{2}} = \begin{pmatrix}
-\frac{1}{4}\left[\boldsymbol{R}_{A\,n}(\boldsymbol{I}+\boldsymbol{F}_A^2) + \boldsymbol{R}_{B\,n}(\boldsymbol{I}+\boldsymbol{F}_B^2)\right]\boldsymbol{R}_E\boldsymbol{N}_{n+\frac{1}{2}} \\
-\frac{1}{2}[\boldsymbol{F}_A^T\boldsymbol{R}^T_{A\,n}\widetilde{(L\boldsymbol{E}_1 + \boldsymbol{u}_{B\,n+\frac{1}{2}} - \boldsymbol{u}_{A\,n+\frac{1}{2}})}]\boldsymbol{R}_E\boldsymbol{N}_{n+\frac{1}{2}} - \frac{1}{2}\overline{\mathrm{dev}}[\boldsymbol{P}_A^T]\boldsymbol{R}_E\boldsymbol{M}_{n+\frac{1}{2}} \\
\frac{1}{4}\left[\boldsymbol{R}_{A\,n}(\boldsymbol{I}+\boldsymbol{F}_A^2) + \boldsymbol{R}_{B\,n}(\boldsymbol{I}+\boldsymbol{F}_B^2)\right]\boldsymbol{R}_E\boldsymbol{N}_{n+\frac{1}{2}} \\
-\frac{1}{2}[\boldsymbol{F}_B^T\boldsymbol{R}^T_{B\,n}\widetilde{(L\boldsymbol{E}_1 + \boldsymbol{u}_{B\,n+\frac{1}{2}} - \boldsymbol{u}_{A\,n+\frac{1}{2}})}]\boldsymbol{R}_E\boldsymbol{N}_{n+\frac{1}{2}} + \frac{1}{2}\overline{\mathrm{dev}}[\boldsymbol{P}_B]\boldsymbol{R}_E\boldsymbol{M}_{n+\frac{1}{2}}
\end{pmatrix}} \tag{97}
$$

We may see, by comparing with equation (77), the time averaged character of the internal forces vector $\boldsymbol{f}^*_{int,n+\frac{1}{2}}$. The tangent stiffness matrix $\boldsymbol{S}^*$ is obtained by differentiation of the internal forces vector, as we did previously in section 5.4. It is worth noting that in this case the stiffness is not symmetric, which is characteristic of the energy conserving algorithm. We do not present here the final expression of this matrix for brevity.

7 CONCLUSION

A large rotations nonlinear beam finite element model was developed, in both a classical formulation and an energy conserving formulation. The element makes several simplifications that lead to compact expressions and simple to be adapted to the energy conserving algorithm.

The discussion was centered on deriving the expressions of the internal forces vectors for both formulations. The stiffness matrices may be straightforwardly computed by differentiation of these terms.

The inertia terms, although not presented here, can be computed by following a corotational approach. The mass matrix and gyroscopic and inertia forces have a similar pattern as that of the rigid body, which was presented elsewhere.[10] Interested readers are referred to this work to find also more details on implementation of the time integration algorithm. Examples of application will be presented at the oral presentation.

ACKNOWLEDGMENTS

Financial support from Agencia Nacional de Promoción Científica y Tecnológica, Argentina, through grant PID 99-76, from Conicet, through grant PIP 266, and from Universidad Nacional del Litoral, through grant CAI+D PE 214, is gratefully acknowledged.

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