APPLICATION OF A HIERARCHIC FINITE ELEMENT TO THE ANALYSIS OF SKELETAL STRUCTURES

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RESUMEN

Para el análisis de estructuras planas de secciones e inercias variables, se propone un nuevo elemento finito basado sobre el concepto jerárquico. El orden variable de este elemento permite un análisis y modelación rápido y eficaz de estructuras.

Se presenta el desarrollo teórico del método junto con dos ejemplos de aplicación.

ABSTRACT

A new finite element methodology based on the hierarchic concept is proposed for the static and dynamic analysis of tapered skeletal structures. The variable order of the hierarchic element allows for fast and efficient modeling and analysis.

Theoretical basis of the method together with two illustrative examples are presented.

1.- INTRODUCTION

To satisfy architectural and functional requirements as well as a better distribution of weight and strength, nonprismatic beams are often used in civil engineering structures. This paper deals with the static and dynamic analysis of linear elastic structures composed of beams with variable cross sectional area.

The importance of the analysis of tapered beams was first stressed by Amirkian [1] who used detailed tables and by the Portland Cement Association [2] with the introduction of a variation of the moment distribution method. Newmark [3] presented an approximate numerical method to determine static deflection and moments in nonuniform beams. More recently general purpose finite element programs, such as a SAP4 [4] have allowed the analysis of tapered members by breaking them into a number of uniform beam elements which are then superimposed to produce the desired effect of tapering.

Gallagher and Lee [5] introduced a general nonuniform beam element by computing its steffnen & consistent man matrices using cubri displacement functions. This element proved to give accurate results in fue vibration analysis.

Resende & Doyle [6] presented later an approximate analysis of tapered beams using a 3 Node line element.

Karabalis and Beskos [7] proposed the static and dynamic analisis of tapered planar beams which experience only axial and flexural deformations their method yielded the exact stiffenes matrices for rectangular, box and I-sections.

In this paper a new finite element methodology based on the hierarchic concept, is proposed for the static and dynamic analysis of tapered skeletal structures. Axial, flexural and shear deformations are considered. The variable order of the hierarchic element allows for fast and efficient re-analysis, this, in order to improve accuracy and assure convergence.

Most Civil engineering skeletal structures can be modeled with the present element which is proven to be more efficient than general purpose programs of structural analysis since some afficiency is lost in the generalization.

Theoretical basis for the analysis is presented in section 2, two exampler demostrating the capability, accuracy and efficiency of the method are presented in section 3 and finally conclusions are drawn in section 4.

2.- Theoretical basis

It is anumed that the total rotation of a plane section, originally normal to the neutral axis of the beam elements, is due to the rotation of the tangent to the neutral axis and to the shear deformation γ

$$\beta = \frac{dv}{dx} - \gamma \tag{1}$$

The total potential energy ref 3 of the beam is given by

$$\pi = \frac{1}{2}\int_0^L EI\left(\frac{d\beta}{dx}\right)^2 dx + \frac{1}{2}\int_0^L GAk\left(\frac{dv}{dx}-\beta\right)^2 dx + \frac{1}{2}\int_0^L EA\left(\frac{dv}{dx}\right)^2 dx$$

$$- \int_0^L pv\,dx - \int_0^P m\beta dx - \int_0^L n\,u\,dx - \sum_{i=1}^{S} P_i v_i - \sum_{i=1}^{S} M_i\beta_i - \sum_{i=1}^{S} N_i v_i \tag{2}$$

The applied distributed and concentraded loads are shown in figure 1.

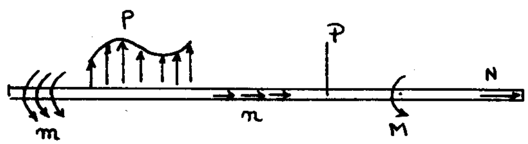


Fig. 1.- Applied distributed and Concentraded loads on the beam element.

A bases of polynomial shape functions is formed such that the bases form a hierarchy, i.e. The bases of degree k contains explicitly polynomial bases of degree 1, 2, ... , k-1. This property is later exploited to achieve efficiency in the re-analysis of structural problems to improve accuracy.

For $k = 1$ we define any of our nodal unknowns u, v, and β as

$$\hat{\phi}(\xi) = \phi_1 N_1 + \phi_2 N_2 \tag{3}$$

when ϕ_i is any nodal unknown and

$$N_1^{(1)}(\xi) = \frac{1}{2}(1-\xi) \qquad N_2^{(1)}(\xi) = \frac{1}{2}(1+\xi) \tag{4}$$

For $k = 2$ we write the nodal unknowns as

$$\hat{\phi}(\xi) = \phi_1 N_1^{(1)} + \phi_2 N_2^{(1)} + a_3 N_3^{(2)} \tag{5}$$

here $N_3^{(2)}$ is a quadratic function of the form

$$N_3^{(2)} = c_o + c_1 \xi + c_2 \xi^2 \tag{6}$$

with coefficients c_i chosen so as to give $N_3^{(2)} = 0$ at $\xi = \pm 1$, this in order to preserve the required $C°$-continuity of $\hat{\phi}$ between elements. This yields

$$N_3^{(2)} = -\frac{1}{2}(1 - \xi^2) \tag{7}$$

replacing the above expression in (5) we find

$$a_3 = \left. \frac{d^2 \hat{\phi}}{d \xi^2} \right|_{\xi=o} \tag{8}$$

Thus the new unknown can be interpreted as the curvature at the mid-point of the element. The above procedure is easily generalized to obtain

$$\hat{\phi}(\xi) = \phi_1 N_1 + \phi_2 N_2 + \sum_{i=2}^{k} \phi_3^{(i)} N_{(i+1)}$$

and the hierarchic bases polynomials for $k = 4$ will be defined as

$$N_3(\xi) = \frac{-1}{2}(1 - \xi^2)$$

$$N_4(\xi) = \frac{-1}{6}(1 - \xi^2)\xi$$

$$N_5(\xi) = \frac{-1}{24}(1 - \xi^4)$$

The nodal desplacements u, v and β will be approximated by

$$\begin{aligned} \hat{u}(\xi) &= u_1 N_1 + u_2 N_2 \\ \hat{v}(\xi) &= v_1 N_1 + v_2 N_2 + \sum_{i=2}^{4} v_3^{(i)} N_{i+1} \\ \hat{\beta}(\xi) &= \beta_1 N_1 + \beta_2 N_2 + \sum_{i=1}^{4} \beta_3^{i} N_{i+1} \end{aligned} \tag{10}$$

where a linear variation for the axial deformation is thought to be sufficient.

To make full use of the computational efficiency of the hierarchic element the generalized desplacement vector is reordered in the following way

$$\{U\}^T = \{u_1 \; u_2 \; v_1 \; \beta_1 \; v_2 \; \beta_2 \quad u_3^{(2)} \; \beta_3^{(2)} \; v_3^{(3)} \; \beta_3^{(3)} \; v_3^{(4)} \; \beta_3^{(4)} \;\} \tag{11}$$

and for simplicity the condensed notation is introduced for the above expression

$$\{U\}^T = \{U^{(p)} \qquad U^{(h)} \;\}^T \tag{12}$$

and

$$\begin{aligned} u(\xi) &= N_u \; \{U\} \\ v(\xi) &= N_v \; \{V\} \\ \beta(\xi) &= N_\beta \; \{\beta\} \end{aligned} \tag{13}$$

The same partition is performed on the shape functions associated with the two extreme nodal points and those associated with the hierarchic shape functions.

$$\begin{aligned} N_u &= \left[N_u{}^p \;\vdots\; N_u{}^h \right] \\ N_v &= \left[N_v{}^{(p)} \;\vdots\; N_v{}^h \right] \\ N_\beta &= \left[N_\beta{}^{(p)} \;\vdots\; N_\beta{}^h \right] \end{aligned} \tag{14}$$

The potential energy is now minimezed as in the standard finite element method, and therms are rearranged to give a steffness and load matrix of the following condensed form.

$$\begin{bmatrix} K_{pp} & K_{ph} \\ K_{hp} & K_{hn} \end{bmatrix} \begin{Bmatrix} U^p \\ U^h \end{Bmatrix} = \begin{Bmatrix} R^p \\ R^h \end{Bmatrix} \tag{15}$$

detailled expression of these submatrices can be found in reference [8]

Submatrix K_{pp} relating the two external nodes each with three degrees of freedom will always be a 6 x 6 matrix. Submatrices K_{ph}, K_{hp}, and K_{nn} will however have variable dimensions depending on the degree of approximation. Each added degree will increase the K matrix by two rows and two columns. Since only the nodal unknowns v and β are approximated by the hierarchic functions.

Therefore on a fixed mesh several analysis with polynomials of successively higher degrees are made by means of the hierarchic nesting of the basis functions. Computation of the previous analysis is fully utilized in the successive more accurate analysis.

Since all of the $\{U^{(h)}\}$ unknowns are associated with a single internal node they can be condensed out, at the element level.

An important advantage of the hierarchic element is that, moments and shear variations at all points in the structure can be accuratly computed using the unknown associated with the hierarchic shape function.

$$\frac{d^2v}{dx^2} = \frac{M}{EI} + \frac{q}{GAK} \tag{16}$$

also

$$\beta = \frac{dv}{dx} - \gamma \qquad \text{and} \qquad \gamma = \frac{Q}{GAh} \tag{17}$$

therefore

$$\frac{d\beta}{dx} = \frac{d^2v}{dx^2} - \frac{d}{dx}\left(\frac{Q}{GAh}\right) \tag{18}$$

Thus

$$M = EI \frac{d\beta}{dx} \tag{19}$$

and

$$Q = \left(\frac{dI}{dx} + \frac{dE}{dx} I\right)\frac{d\beta}{dx} + EI\frac{d^2\beta}{dx^2}$$

changing to local coordinates and using for β its approximation

$$\hat{\beta} = \beta_1 N_1 + \beta_2 N_2 + \beta_3^{(2)} N_3 + \beta_3^{(3)} N_4 + \beta_3^{(4)} N_5$$

Yields

$$M = EI \quad - \frac{\beta_1}{2} + \frac{\beta_2}{2} + \beta_3^{(2)} \cdot \xi + \beta_3^{(3)} \left(-\frac{1}{6} + \frac{1}{2}\xi^2\right) + \beta_3^{(4)} \left(\frac{1}{6}\xi^3\right) \frac{2}{p} \tag{20}$$

and

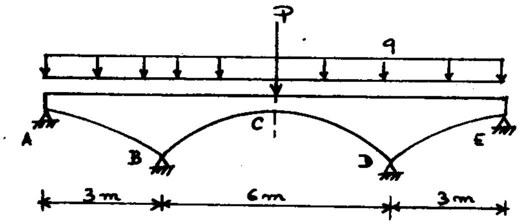
$$Q = \left(E\frac{dI}{d\xi} + \frac{dE}{d\xi} \cdot I\right) \cdot \frac{4}{L}\left(-\frac{\beta_1}{2} + \frac{\beta_2}{2} + \beta_3^{(2)}\xi + \beta_3^{(3)}\left(-\frac{1}{6} + \frac{\xi^2}{2}\right)\right.$$
$$\left. + \beta_3^{(4)}\frac{\xi^3}{6}\right) + EI\left(\beta_3^{(2)} + \beta_3^{(3)}\xi + \beta_3^{(4)}\frac{\xi^2}{2}\right) \cdot \frac{4}{L} \tag{21}$$

We notice again that the computation of the unknown associated with the higher order hierarchic shape function allows a series type of approximation for both M and Q. Inertia I and modulus of elasticity E, are assumed to be known and variable along the element.

3.- EXAMPLES

3.1 Three span continuous beam

The structure shown in figure 2 is analyzed using program EJV. For which was developped according to the theory explained in section 2.



$E = 2.1 \times 10^6 \ T/m^2$

$\gamma = 0.2$

$p = 20 \ T$

$q = 2 \ T/m$

Figure 2.- Continuous beam of variable section

Four elements where found to be sufficient to accurately represent cross section and inertia variation in the beam elements. Table I shows the influence of the hierarchic unknowns in the reactions and moments at different points in the structure, as one, two and three unknown one selected successively.

Table I - Accuracy as more hierarchic unknowns are selected

N° of Hierarchic unknowns	0	1	2	3
R_A (tons)	-2.981	-3.9122	-5.371	-5.459
R_B (tons)	24.981	27.9122	27.371	27.459
M_c (t - m)	21.057	18.263	13.886	13.623
δ_c (mm)	-0.3928	-2.0843	-3.7417	-4.1942

As mode of comparaison it was necessary to use 52 elements with the regular stiffness method to obtain an acceptable value for the deflection at point C of 4.181 mm.

3.2. Static and dynamic analysis of a four story building

The Frame of Figure 3 is next analyzed, notice that in every case two nodes define each element independent of the degree of approximation sought.

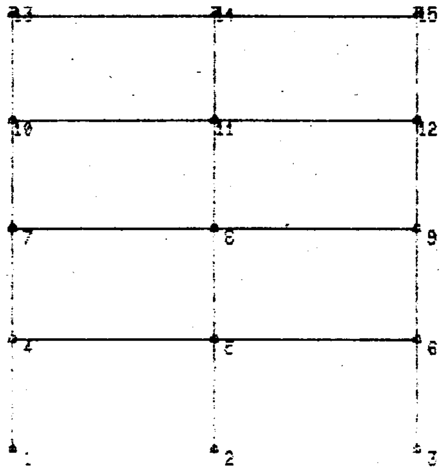
First, second and third floor have a distributed load of 50 kg/cm, the last floor is loaded with 35 kg/cm. Shear and moment diagrams are easily found using equations (20) and (21). The dynamic analysis is carried out using modal superposition. Concentrated masses are used and eigenvalues and eigenvector are found using the subspace iteration algorithim.

Figure 4 shows moment & shear diagram, & Figure 5 the first, second and third mode of vibration.

4.- Conclusions

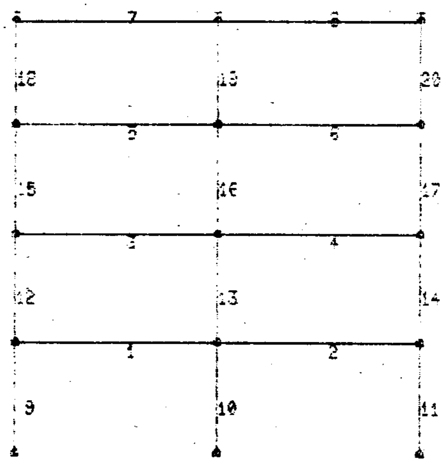
A new finite element specially well suited to the analysis of non-prismatic skeletal structures is introduced. The hierarchic base of shape functions, allows for repeted analysis with higher and higher degrees of approximations withouth modifying the original node and element numbering. Also each analysis utilizes fully all of the computations of the previous steps. Very few elements are required to model accurately cross sectional and inertia variations which are approximated by polynomials up to degree five.

A carefull comparaison with the stiffness method and the standard finite element method, showed this element to be more economic computationaly. Finally shear and moment values are accurately determined in all points of the structures using the primary variables, and are not found as secondary variables as in the standard finite element or stiffness method.



Node Numbers

EJV2

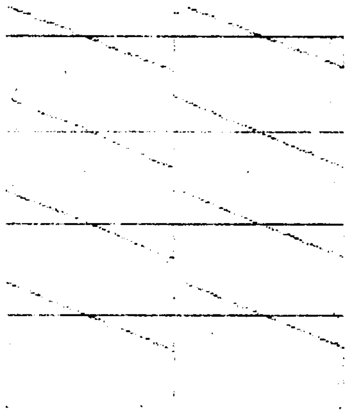


Element numbers

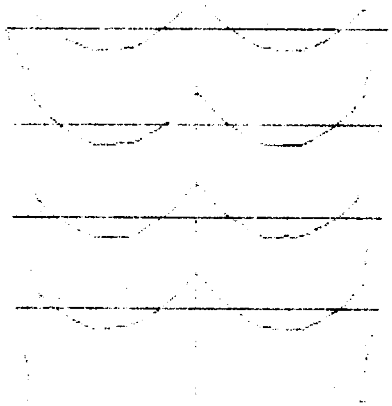
EJV2

Figure 3. **Node and element numbering**

Shear diagram



Moment diagram



Figur 4.- Shear and moment diagrams

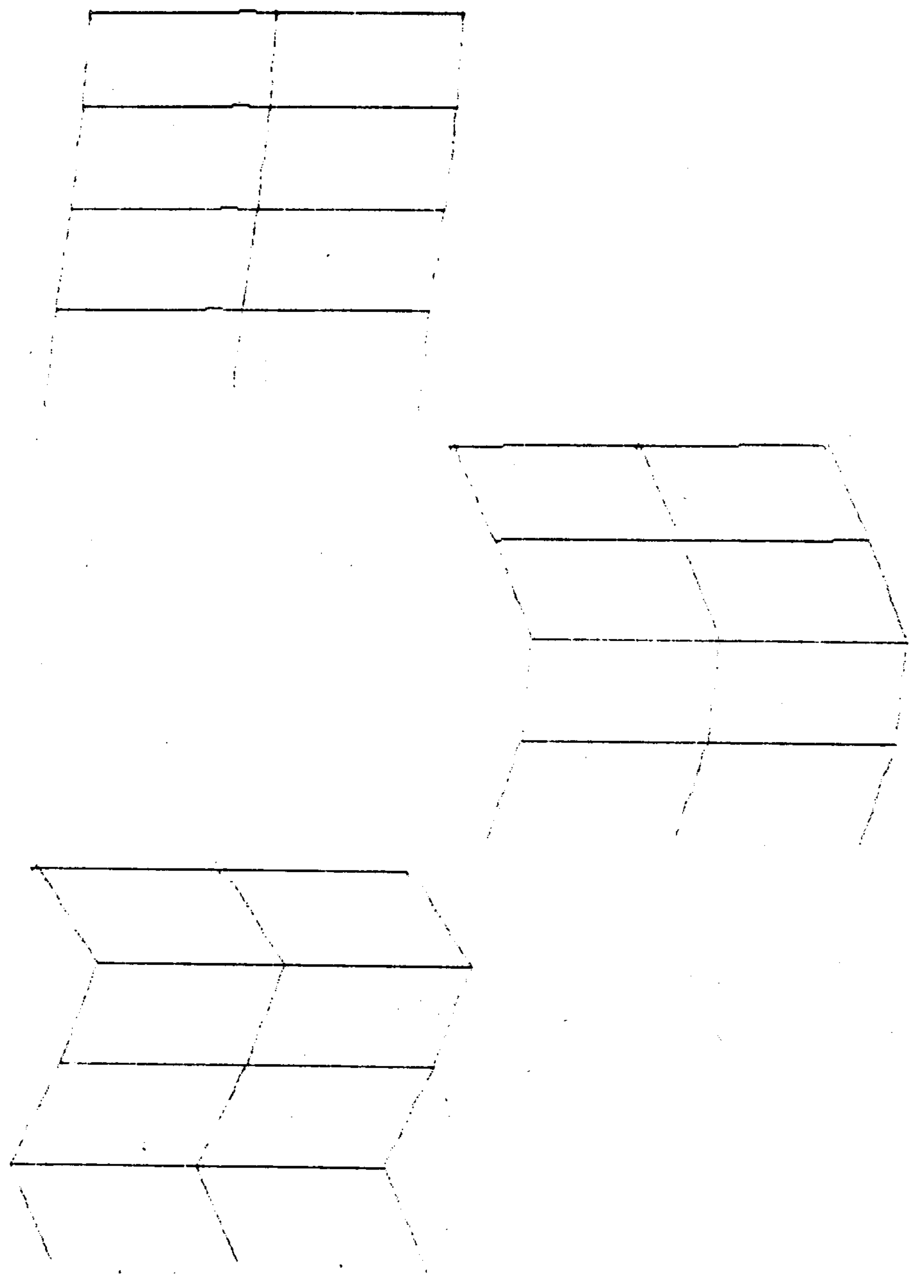


Figure 5.- First, second and third mode of vibration

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