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THE CLASSIC DYNAMIC THEORY OF THICK CURVED BEAMS IN THE CONTEXT OF FUNCTIONALLY GRADED MATERIALS

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Abstract. During the last two decades materials, that exhibit graded properties, left their essence of conceptual laboratory specimens to become a technological reality with a well established background. However structural applications of these materials are not a fulfilled research. Neither in what can be considered new uses of given structures, nor in the development of new theories to explain certain effects.

Models for straight and curved beams are normally reported in the scientific literature as the easiest way to understand certain existing aspects in mechanics of structures. Most of these models are formulated appealing to numerical approaches such as the finite element or differential quadrature methods among others without taking into account theoretical aspects that can be quite useful to reduce algebraic complexity.

In the present work the classical strength-of-materials theory for dynamic analysis of thick curved beams is deduced in the context functionally graded materials. The derivation process consists in the reduction of the algebraic handling by employing the concept of material neutral–axis shifting. This leads to the possibility to find analytical solutions of the governing differential system, even if the differential system has variable coefficients.

Parametric studies on natural frequencies are offered to show the versatility of the adopted formulation by means of solutions handled with the Power Series Method.

1 INTRODUCTION

In the last decade functionally graded materials (FGM) are increasingly recognized as a factual solution and potential answer to many challenging problems in a broad range of engineering applications. This kind of materials was reported in the middle eighties (Koizumi, 1993) as a potential way to cope with the problem of failure and presence cracks in the interfaces of sandwich structures or laminated structures due to, for example, high thermo-mechanical stress gradients. Laminated composite structures and sandwich structures differ from FGM in what these last ones have mass, elastic and thermo-mechanical properties changing smoothly and continuously in prescribed directions. Structural models for functionally graded material were introduced for different geometric configurations and scales covering a broad area from 3D solids through shells and plates to finish in beams or bars.

The development of models for curved beams has been the topic of interest of many researchers during the last thirty years. Those investigations were oriented to a wide range of engineering problems, such as instability, vibration analysis, etc (Chidamparam and Leissa, 1993). Many curved beam models were introduced to account for linear and non-linear behavior in structures made of both isotropic (Cortínez et al., 1999; Piovan et al., 2000) and/or composite materials and arranged for thin-walled (Piovan and Cortínez, 2007) or solid cross-sections (Matsunaga, 2004; Tufekci and Yasar Dogruer, 2006). A number of different methodologies and principles such as principle of virtual work, Hellinger-Reissner principle, Hu-Washizu principle among others were employed to develop a variety of models. The classical theory of strength of materials, although considered as old fashioned, proved to be a conceptually easy way to derive generalized or more complex beam and bar models for curved or straight axis (Filipich, 1991; Filipich et al., 2003).

It has to be noted that, despite its technological interest, very few studies on the dynamics of curved beams made of FGM have been performed in the past years according to the knowledge of the authors. In fact, Piovan et al. (2008a) developed a basic model for free vibration analysis of arcs under the presence of initial stresses. This model was derived employing the principle of Hellinger-Reissner and numerically implemented with the finite element method. Piovan and Sampaio (2009) introduced a model for rotating curved beams made of functionally graded materials. Malekzadeh (2009) and Lim et al. (2009) carried out numerical approaches for in-plane vibrations of arches in the context of bi-dimensional formulations.

The present article intends to be a contribution to current state-of-the-art in dynamics of arches. The bi-dimensional approaches are normally time consuming and allow analytical solutions in a couple of the simpler cases. Three-dimensional model of these structures should be analyzed with numerical formulations like finite element method among others. On the other hand the one-dimensional theories can reach an acceptable degree of approximation that could be nearly the same of 2D and 3D formulation but with a reasonable computational cost. Also one dimensional models offer an easy conceptual understanding of the dynamic phenomena in slender structures. Thus, in the present article the classical strength-of-materials theory for dynamic analysis of thick curved beams is deduced in the context functionally graded materials. The deduction process appeals to the concept of material neutral–axis shifting with the aim to reduce the algebraic handling. Thus, the equations of motion obtained under this conception are formally identical to the ones for isotropic materials, leading to the possibility to provide analytical solutions of the governing differential system.

The solution of the free vibration problem of thick curved beams made of functionally graded materials is performed by means of the power series method. A recurrence scheme is employed in power series handling in order to reduce the number of unknown to only few unknown coefficients that can be selected according to the boundary equations. Some comparisons are performed with other beam approaches and 3D finite element approximations. An especial analysis of particular features in certain boundary conditions is offered as well.

2 MODEL DEVELOPMENT

2.1 Hypotheses and definitions

In Figure 1 a sketch of the structural element analyzed in this work is shown. As one can see, there are two relevant points in the cross-section: Point ***G*** corresponds to the centroid of the section, whereas point ***D*** is a point belonging to the neutral axis. In this model the following material properties are assumed as:

$$E = E_0\varphi(r),\;\; G = G_0\mu(r),\;\; \rho = \rho_0 f(r), \tag{1}$$

where:

$$\varphi(r) = \mu(r) = f(r) = \left[k_j + (1-k_j)\left(\frac{1}{2} - \frac{1}{n} + \frac{r}{h}\right)\right], \text{ with } k_1 = \frac{E_i}{E_0}, k_2 = \frac{G_i}{G_0}, k_3 = \frac{\rho_i}{\rho_0} \tag{2}$$

In Eq. (2) $n = h/R_G$ and k_1, k_2 and k_3 correspond to $\varphi(r)$, $\mu(r)$ and $f(r)$, respectively; whereas E_i, G_i and ρ_i intend for the material properties at $r = r_i$ and E_0, G_0 and ρ_0 are the properties at $r = r_e$.

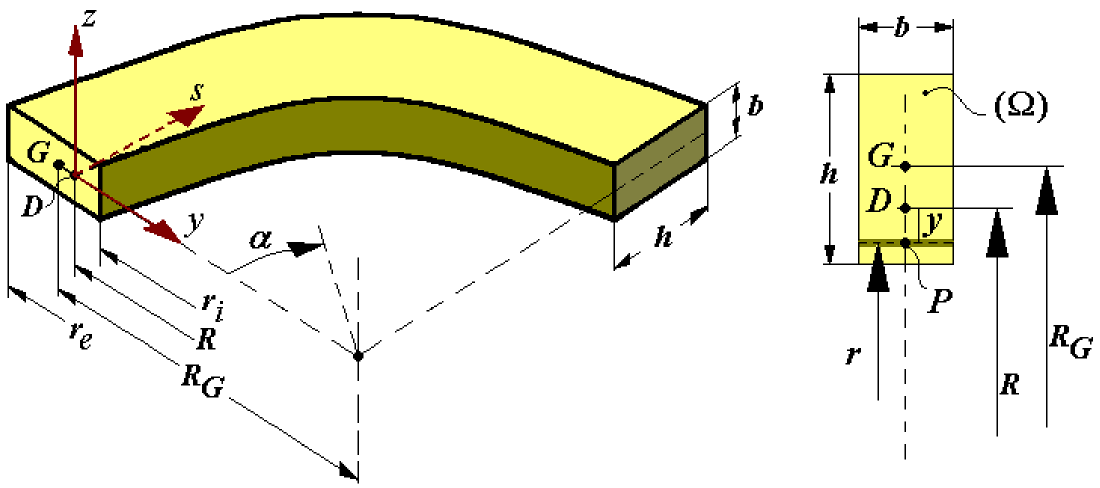

Figure 1: Structural model: a circumferential thick arc.

In order to derive the governing equations the following hypotheses are performed:

(a) The present study is confined within the context of strength of materials theory.

(b) The cross-section shape is not affected by the deformation: although the following developments are carried out for a rectangular cross section, it still being valid for any section with symmetry respect to a plane containing points ***G*** and ***D***.

(c) The following displacements are defined as:

$$u = u(\alpha, t), \quad w = w(\alpha, t), \quad \theta = \theta(\alpha, t), \tag{3}$$

where u is the transverse displacement, w is the tangential displacement and θ is the bending rotation all of them measured with respect to point ***D***, t is the time. Then:

$$\begin{aligned} u_P &= u, \\ w_P &= w + \theta\, y, \quad \text{with} \quad y = R - r, \end{aligned} \tag{4}$$

$$\begin{aligned} d\Omega &= b dy = -b dr, \\ s_P &= r\alpha, \\ s &= R\alpha. \end{aligned} \tag{5}$$

In Eq. (5), $d\Omega$ is the element of area and s and s_P are the circumferential co-ordinates of the arc containing point ***D*** and a generic point ***P***, respectively.

2.2 Kinematical Relations. Strains and Stresses

According to the assumptions taken into account, the kinematical relationships are defined in polar coordinates as:

$$\begin{aligned} \varepsilon_\alpha &= \varepsilon = \frac{w'_P}{r} - \frac{u}{r}, \\ \gamma_{r\alpha} &= \gamma_{\alpha r} = \gamma^* = \frac{u'}{r} + \frac{\partial w_P}{\partial y} + \frac{w_P}{r}. \end{aligned} \tag{6}$$

Now, substituting Eq. (4) into Eq. (6) one obtains:

$$\begin{aligned} \varepsilon &= \frac{1}{r}\left(w' - u + \theta' y\right), \\ \gamma^* &= \frac{1}{r}\left(w + u' + R\theta\right), \end{aligned} \tag{7}$$

where, primes mean derivation with respect to variable α.

For a functionally graded material, the classical linear constitutive law can be written as:

$$\sigma = E_0 \frac{\varphi(r)}{r}\left(w' - u + \theta' y\right), \tag{8a}$$

$$\left\{\tau = G_0 \frac{\mu(r)}{r}\left(w + u' + R\theta\right)\right\}. \tag{8b}$$

Notice that Eq (8b) is not employed in the context of strength of materials theory.

2.3 Axial force and bending moment. Neutral Axis.

The axial force N and bending moment M are defined in terms of the normal stress by means of the following expressions:

$$\begin{aligned} N &= \iint \sigma d\Omega, \\ M_D &= M = \iint \sigma y d\Omega. \end{aligned} \tag{9}$$

Then employing the previous definitions one obtains:

$$N = bE_0\left[\alpha_0\left(w' - u\right) + \alpha_1\theta'\right],$$
$$M = bE_0\left[\alpha_1\left(w' - u\right) + \alpha_2\theta'\right]. \tag{10}$$

In Eq. (10), the coefficients α_m, $m= 0,1,2$, are:

$$\alpha_m = \int_{r_i}^{r_e} y^m \frac{\varphi(r)}{r} dr. \tag{11}$$

In order to obtain the expression for the neutral axis (for $N \neq 0$ and $M \neq 0$) one should guarantee $\varepsilon = \sigma = 0$, thus from Eq (8a):

$$\left(w' - u + \theta' a\right) = 0, \tag{12}$$

Where $y = a$ is the location of the neutral axis, i.e.

$$a = -\frac{w' - u}{\theta'}, \tag{13}$$

or, taking into account Eq. (10), one gets:

$$a = \frac{\alpha_2 N - \alpha_1 M}{\alpha_1 N - \alpha_0 M}. \tag{14}$$

Once the neutral axis has been generically defined, it is possible to rearrange the origin location (Point ***D***) with the scope to substantially reduce the algebraic manipulation, under the following condition:

$$a = 0 \;\text{ when }\; N = 0. \tag{15}$$

Then, according to Eq. (14), the condition given in Eq. (15) leads to:

$$\alpha_1 = 0 \tag{16}$$

Then, from Eq. (16) and Eq. (11) one deduces the neutral axis radius as:

$$R = \frac{q_0 h + q_1 R_G}{q_0 Ln\left[r_e / r_i\right] + q_1}, \tag{17}$$

where in the present case q_0 and q_1 are

$$q_0 = k_1 + \left(1 - k_1\right)\left(\frac{1}{2} - \frac{1}{n}\right),$$
$$q_1 = \left(1 - k_1\right). \tag{18}$$

Remark: If the material is homogeneous ($q_1=0$), then

$$R = \frac{h}{Ln\left[r_e / r_i\right]}$$

which is the neutral radius used in Strength of material theory for thick curved beam.

Finally, with Eq. (16), the coefficients α_0 and α_2 are automatically defined as:

$$\alpha_0 = q_0 Ln[r_e / r_i] + q_1 ,\\ \alpha_2 = -R^2\alpha_0 + q_0 hR_G + \frac{q_1(r_e^3 - r_i^3)}{3h} , \tag{19}$$

with these definitions it is possible to introduce the following entities:

$$A = bR\alpha_0 ,\\ J = bR\alpha_2 , \tag{20}$$

that mean material area and material moment of inertia, respectively. In the case of a homogeneous material and straight beam, $A = \Omega$ and $J = J_G$, i.e. the area and centroidal moment of inertia.

2.4 Constitutive equations. Shear coefficient

Under the assumption of $\alpha_1=0$ applied to Eq. (10), we can write:

$$N = bE_0\alpha_0(w' - u) = \frac{E_0 A}{R}(w' - u),\\ M = bE_0\alpha_2\theta' = \frac{E_0 J}{R}\theta'. \tag{21}$$

The derivation of the shear force Q expression in terms of the displacements is a more difficult task than the derivation of Eq. (21). In order to deduce the shear force expression the following steps have to be performed:

(a) *From the expression of internal equilibrium in polar coordinates*:

$$r\frac{\partial\tau}{\partial r} + 2\tau - \sigma' = 0 \tag{22}$$

one can find the solution as:

$$\tau = \frac{B}{r^2} + \frac{C}{r} + \frac{\beta_0}{2} + \frac{\beta_1}{3} r \tag{23}$$

where, B an arbitrary constant, while C and β_j are:

$$C = \frac{QR^2}{J}\left(R + \frac{\alpha_2}{\alpha_0 R}\right)q_0 ,\ \beta_0 = \frac{QR^2}{J}\left[\left(R + \frac{\alpha_2}{\alpha_0 R}\right)q_1 - q_0\right],\ \beta_1 = -\frac{QR^2}{J}q_1 \tag{24}$$

Notice that Eq. (23) is a generalization of the classical Colignon-Jouravsky formulae, although for curved bar and for non-homogeneous and graded materials.

To determine B two equivalent conditions can be employed:

$$Q = \iint_{\Omega} \tau\, d\Omega \quad \text{or} \quad \tau(r_e) = \tau(r_i) = 0 \tag{25}$$

Both lead to:

$$hB = r_i\left(\frac{\beta_0 r_e^2}{2} + \frac{\beta_1 r_e^3}{3}\right) - r_e\left(\frac{\beta_0 r_i^2}{2} + \frac{\beta_1 r_i^3}{3}\right) \tag{26}$$

(b) *Derivation of the shear coefficient expression*

Appealing to the deformation energy *U* in terms of the stresses:

$$2U = \iiint_V \left[\frac{\sigma^2}{E_0 \varphi(r)} + \frac{\tau^2}{G_0 \mu(r)} \right] dV \tag{27}$$

Then integrating Eq. (27), with σ taken from Eq. (8a) and τ taken from Eq (23), one obtains:

$$2U = \frac{R}{E_0} \int_\alpha \left(\frac{M^2}{J} + \frac{N^2}{A} \right) d\alpha + \frac{mR}{G_0} \int_\alpha \left(\frac{Q^2}{A} \right) d\alpha \tag{28}$$

Where the shear coefficient is defined as:

$$m = \frac{bAR^3}{J^2} \int_{r_i}^{r_e} \frac{\left(B^* + C^* r + \beta_0^* r^2/2 + \beta_1^* r^3/3 \right)^2}{r^3 \mu(r)} dr \tag{29}$$

In Eq (29) the following constants have been employed:

$$B^* = B\frac{J}{QR^2},\ C^* = C\frac{J}{QR^2},\ \beta_0^* = \beta_0 \frac{J}{QR^2},\ \beta_1^* = \beta_1 \frac{J}{QR^2} \tag{30}$$

Remark: If $\varphi(r) = \mu(r) = 1$ then $A = \Omega = bh$, $J = AR(R_G - R)$ and

$$m = \frac{hR^3}{J^2} \int_{r_i}^{r_e} \frac{S_G^2}{r^3} dr, \text{ such that } S_G \equiv \frac{b}{2}\left[\frac{h^2}{4} - (R_G - r)^2 \right]$$

If $R_G \to \infty$ the straight beam case is obtained: $J = J_G$ and $m=6/5$.

(c) *Shear force*:

To obtain the expression of the shear force in terms of the displacements one should come across that the deformation energy due to shear effects can be written as:

$$2U_S = \iiint_V \left[\frac{\tau^2}{G_0 \mu(r)} \right] dV = \frac{mR}{G_0} \int_\alpha \left(\frac{Q^2}{A} \right) d\alpha = \iiint_V \tau\gamma^*\, dV \tag{31}$$

Where, the last one is the classical definition of the deformation energy, and can be expressed as:

$$\iiint_V \tau\gamma^*\, dV = \int_\alpha \gamma^* d\alpha \iint_\Omega \tau d\Omega = \int_\alpha Q\gamma^* d\alpha \tag{32}$$

Comparing Eq. (32) with the second expression of Eq. (31) and taking into account Eq. (7), one finally gets:

$$Q = \frac{G_0 A}{mR}\left(w + u' + R\theta \right) \tag{33}$$

Eq. (33) and Eq. (21) fulfill the constitutive equations for a problem in the plane.

2.5 Equations of motion

In order to deduce the equations of motions the principle of Hamilton is employed. Then, the kinetic energy K is (recall Eq. (4)):

$$2K = \rho_0 \iiint_V \left[\dot{u}^2 + \left(\dot{w} + \dot{\theta} y \right)^2 \right] f(r) \, d\Omega \, d\alpha , \tag{34}$$

where, points indicate derivation with respect time t.

Now introducing:

$$\gamma_m = \int_{r_i}^{r_e} y^m r f(r) dr , \tag{35}$$

Then Eq. (34) can be written as:

$$2K = \rho_0 b \int_\alpha \left[\gamma_0 \left(\dot{u}^2 + \dot{w}^2 \right) + 2\gamma_1 \dot{\theta} \dot{w} + \gamma_2 \dot{\theta}^2 \right] d\alpha . \tag{36}$$

On the other hand, the potential energy P due to external loads is written as:

$$P = -R \int_\alpha \left[p_r u + p_t w + \mu_\theta \theta \right] d\alpha , \tag{37}$$

where, p_r, p_t and μ_θ are the distributed radial force on the neutral axis, the distributed tangential force on the neutral axis and the distributed moment applied on the same axis, respectively.

The deformation energy is taken from Eq. (28), and remembering the constitutive expressions given in Eq. (21) and Eq. (33), it can be then rewritten in terms of displacements:

$$2U = bE_0 \int_\alpha \left[\alpha_2 \theta'^2 + \alpha_0 \left(w' - u \right)^2 \right] d\alpha + \frac{bG_0 \alpha_0}{m} \int_\alpha \left(w + u' + R\theta \right)^2 d\alpha . \tag{38}$$

Finally performing the conventional variational steps in the Hamilton's principle one gets the following equations of motions in terms of displacements:

$$\begin{aligned} &C_{33} u'' + \left(C_{11} + C_{33} \right) w' + RC_{33} \theta' - C_{11} u - D_{11} \ddot{u} = -p_r R, \\ &C_{11} w'' - \left(C_{11} + C_{33} \right) u' - C_{33} w - RC_{33} \theta - \left(D_{11} \ddot{w} + D_{22} \ddot{\theta} \right) = -p_t R, \\ &C_{22} \theta'' - RC_{33} \left(u' + w + R\theta \right) - \left(D_{22} \ddot{w} + D_{33} \ddot{\theta} \right) = -\mu_\theta R, \end{aligned} \tag{39}$$

where,

$$\begin{aligned} &C_{11} = bE_0 \alpha_0 , \quad C_{22} = bE_0 \alpha_2 , \quad C_{33} = bG\alpha_0 / m, \\ &D_{11} = b\rho_0 \gamma_0 , \quad D_{22} = b\rho_0 \gamma_1 , \quad D_{33} = b\rho_0 \gamma_2 . \end{aligned} \tag{40}$$

It is interesting to remark that Eq. (39) are a generalization of the Timoshenko straight beam approach in the context of functionally graded material for curved beam. Besides Eq. (39) is formally the same of the problem corresponding to the homogeneous case developed by Filipich (1991).

Notice that the differential problem given by Eq. (39) is a completely coupled system, but if the structure is such that $R_G \to \infty$, the model is reduced to the case of a straight beam where the longitudinal motion is decoupled from the shear and bending motion.

3 NUMERICAL ANALYSIS AND COMPARISONS

3.1 Power series method for the natural frequencies problem

In order to study vibratory patterns of this type of structures, the motion equations are solved with a power series solution. The exact solution of the eigenvalue problem can be carried out by means of a generalization of the power series scheme developed originally by Filipich et al. (2003) and Rosales and Filipich (2006) for structural problems involving isotropic materials. The methodology requires a previous non-dimensional re-definition of the differential equations, which implies that $x = R\alpha / L \in [0,1] \quad \forall \alpha \in [0,\Delta_\alpha]$, being L the circumferential length of the neutral axis of the curved beam, and Δ_α is the subtended angle of the curved beam.

The displacement variables have the common harmonic motion:

$$\{u,w,\theta\} = \{\overline{u},\overline{w},\overline{\theta}\} e^{i\omega t} \tag{41}$$

where ω is the circular frequency of the curved beam measured in rad/seg and $i = \sqrt{-1}$, and $\{\overline{u},\overline{w},\overline{\theta}\}$ are the corresponding modal shapes. Now working without the presence of external loads and accepting Eq. (41) the differential equations system given by Eq. (39) can be re-arranged in the following form:

$$\begin{aligned}
&\overline{u}'' + \frac{L}{\Phi R}(\Phi + m)\overline{w}' + L\overline{\theta}' - \frac{mL^2}{\Phi R^2}\overline{u} + \frac{m\alpha_2\gamma_0}{L^2\Phi R^2\alpha_0^2}\lambda^2\overline{u} = 0 \\
&\overline{w}'' - \frac{L}{mR}(\Phi + m)\overline{u}' - \frac{\Phi L^2}{mR^2}\left(\overline{w} + R\overline{\theta}\right) + \frac{\alpha_2}{L^2R^2\alpha_0^2}\lambda^2\left(\gamma_0\overline{w} + \gamma_1\overline{\theta}\right) = 0 \\
&\overline{\theta}'' - \frac{L\Phi\alpha_0}{m\alpha_2}\overline{u}' - \frac{L^2\Phi\alpha_0}{m\alpha_2 R}\left(\overline{w} + R\overline{\theta}\right) + \frac{\lambda^2}{L^2R^2\alpha_0}\left(\gamma_1\overline{w} + \gamma_2\overline{\theta}\right) = 0
\end{aligned} \tag{42}$$

Where:

$$\lambda^2 = \frac{\rho_0 A}{E_0 J}\omega^2 L^4 , \quad \Phi = \frac{G_0}{E_0} , \tag{43}$$

In Eq (42) the primes mean derivation with respect to x.

The displacements are expanded with the following power series:

$$\{\overline{u},\overline{w},\overline{\theta}\} = \sum_{k=0}^{Z} \{U_k, W_k, \Theta_k\} x^k . \tag{44}$$

Theoretically $Z \to \infty$, however for practical purposes Z may be an arbitrary large integer.

Applying the boundary conditions (see Appendix II) in non-dimensional form and appealing to a recurrence scheme of the power series (Filipich et al, 2003; Piovan et al., 2008b) one can represent the solution system in the following form:

$$\begin{bmatrix} \varepsilon_{11}(\lambda) & \varepsilon_{12}(\lambda) & \varepsilon_{13}(\lambda) \\ \varepsilon_{21}(\lambda) & \varepsilon_{22}(\lambda) & \varepsilon_{23}(\lambda) \\ \varepsilon_{31}(\lambda) & \varepsilon_{32}(\lambda) & \varepsilon_{33}(\lambda) \end{bmatrix} \begin{Bmatrix} U^* \\ W^* \\ \Theta^* \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{45}$$

One should note, the differential system given in Eq (42) can be rearranged as a one differential equation of sixth order; so one has six arbitrary integration constants. Three of them are imposed in one end and the remaining three symbolically expressed as $\{U^*, W^*, \Theta^*\}$ in Eq. (45) mean the three free coefficients after the substitution of the power series in the problem (for further explanations see Piovan et al., 2008; Filipich et al, 2003 and Rosales and Filipich, 2006). Thus, from Eq (45) where one can obtain the solution for the eigenvalue problem through out this characteristic equation:

$$Det[\varepsilon(\lambda)] = 0 \tag{46}$$

The aforementioned recurrence scheme (Filipich et al, 2003; Piovan et al., 2008b) allows to shrink the algebraic problem from 3($M+1$) unknowns to only 3 unknown coefficients that can be selected according to the boundary equations.

It should be stressed that the solution expressed in Eq. (46) lead to an arbitrary precision value for the frequency by selecting appropriately the limit Z of the power series. The orthogonality conditions among the modal shapes for the present problem are offered in the Appendix I.

Once the eigenvalues λ_j with $j=1,2,3$ are calculated by means of Eq. (46), the eigenvector related to a given eigenvalue λ_j should be calculated. As one can see Eq. (45) is not of the canonical form (e.g. $\boldsymbol{Aq} - \mu \boldsymbol{Iq} = \boldsymbol{0}$, where μ is the eigenvalue, and $\boldsymbol{I}$ the unit matrix), consequently it can not be solved with the common functions of many programs like Matlab, Maple or Mathematica. As the set of eigenvalues is known, one can define a matrix $\boldsymbol{A}=\varepsilon(\lambda_i)$. Now, implementing $\boldsymbol{Aq} - \mu \boldsymbol{Iq} = \boldsymbol{0}$ and remembering that $Det[\boldsymbol{A}] = \mu_1\ \mu_2\ \mu_3$, from Eq.(46) one should note that, at least one $\mu_j = 0$. Thus, one can calculate (with Matlab, Maple or Mathematica) the eigenvector $\boldsymbol{q}_j$ for $\mu_j = 0$ and according to Eq. (45) one obtains $\{U^*, W^*, \Theta^*\}^T = \{\boldsymbol{q}_{j1}, \boldsymbol{q}_{j2}, \boldsymbol{q}_{j3}\}^T$. Knowing this last vector and appealing to the recurrence scheme implemented within the power series method (Filipich et al, 2003; Piovan et al, 2008b), by means of Eq. (44) the modal shapes for a given eigenvalue λ_j are obtained.

3.2 Numerical analysis

In this section some numerical studies are performed in order to show certain features of the dynamics of curved beam associated with the imposition of the boundary conditions (see Appendix II). In Table 1 the material properties of a ceramic and a metal are shown. All the following examples hold the same ceramic/metal distribution. In fact, the properties are graded according to Eq. (2) from a full metallic inner surface (at $r = r_i$) to a full ceramic outer surface (at $r = r_e$).

Properties of materials	Steel	Alumina (Al_2O_3)
Young's Modulus E (GPa)	214.00	390.00
Shear modulus G (GPa)	82.20	137.00
Material Density ρ (Kg/m^3)	7800.00	3200.00

Table 1: Properties of metallic and ceramic materials.

The first example corresponds to a comparison of the present model with other approaches. The present strength of materials approach for the curved beam is compared with the response of a flexible 3D general solver (called FlexPDE) of partial differential equations within the context of the finite element method. In this solver one can easily cope with the complex material laws to be included in the structural model as well as the model itself (see http://www.pdesolutions.com for further explanations and illustrative examples of the program, also see the book of Backstrom, 1998). Also another one dimensional model of a FGM curved beam (Piovan et al, 2008a) derived according to the Hellinger-Reissner principle (HR) is employed for comparison purposes. The boundary conditions of the curved beam can be clamped at both ends or clamped in one end and free to move in the remaining. The geometric features of the curved beam are the following: $b = 20\ mm$, $h = 50\ mm$, $\Delta\alpha = 1$ rad and $R_G = 500\ mm$, the shear coefficient is $m = 1.1619$ and the ratio $R/R_G = 1.004046$. Note that the neutral axis is greater than the R_G radius, conversely to the homogeneous classical approach. In the following Table 2 the comparison of the three models and numerical approaches is presented for the first four frequencies of the arch. PSM intend for power series method. The calculations were carried out with fifty terms in the PSM (or $Z = 50$), ten quadric curved beam elements (for HR model) and nearly 850 tetrahedral elements in FlexPDE.

As one can see in Table 2, differences between the approaches are negligible; however it should be mentioned that the 3D approach demanded more than 20 minutes to reach the desired precision on a 3.7 GHz Pentium IV computer (this is due to the quite fine mesh employed in order to cope with the non-homogeneity of the material). On the other hand both 1D numerical approaches demanded just two or three seconds.

Boundary Conditio n	Model [approach]	Frequencies [Hz]			
		First	Second	Third	Fourth
Clamped Clamped	1D Present model [PSM]	2364.97	3388.57	6417.37	7657.81
	1D HR model [FEM]	2364.97	3388.57	6417.37	7657.81
	3D [FEM]	2366.59	3431.47	6497.00	7702.20
Clamped Free	1D Present model [PSM]	240.28	1237.25	3394.78	4397.86
	1D HR model [FEM]	240.28	1237.24	3394.78	4397.86
	3D [FEM]	241.39	1244.95	3418.79	4396.93

Table 2: Comparison of frequencies of different models and numerical approaches.

The second example corresponds to a study of the vibratory behavior for the generalized simply supported boundary condition (see Appendix II). The curved beam is such that $R_G = 500\ mm.$, the height of the section is $b = 20\ mm.$, the subtended angle is $\Delta\alpha = 1$ rad., the depth of the cross-section $h = 150\ mm$. The arch has simply supported boundary conditions in both ends; however the restrictions can be applied at the points corresponding to R_G, R, r_e or r_i. Table 3 shows the first five vibration frequencies for the mentioned varieties of the simply supported boundary condition. Notice the difference in the values of the frequencies when the supports are located at r_i and r_e in comparison to the ones when the supports are located at R or R_G. The shear coefficient $m = 1.1601$ and the ratio $R/R_G = 0.9950$.

Location of the support	Frequency [Hz]				
	First	Second	Third	Fourth	Fifth
r_i	5972	14001	23609	27052	36759
R	11950	20419	23737	29473	36119
R_G	11833	20627	23734	29268	36264
r_e	8388	12386	23579	31868	32969

Table 3: Frequencies of different varieties of the simply supported boundary condition.

Figure 2 shows a curved beam defined by two parameters: the horizontal distance between the centroidal points of both ends '*a*' and the arch height parameter '*c*' which is measured between the horizontal level and the centroidal point (i.e. ***G***) of the mid cross-section. In the following example the influence of shallowness ratio *c/a* together with the position of the support at the ends is analyzed. The horizontal distance is fixed to $a = 1000$ mm. The cross-section is such that $b = 20$ mm and $h = 80$ mm.

Figure 3 shows the variations of the frequencies (related to a particular mode shape) with respect to the shallowness ratio *c/a*, for the case where the beam is supported at both ends in the point corresponding to R_G. Figure 4 shows the proper frequency variation with c/a but for the case where the supports are located at r_i.

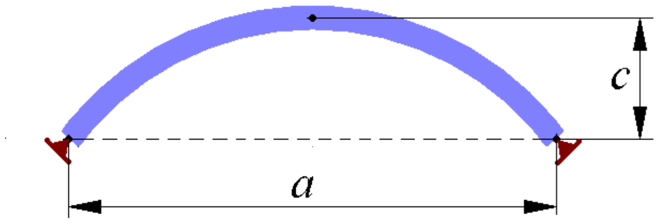

Figure 2: Sketch of a shallow arc.

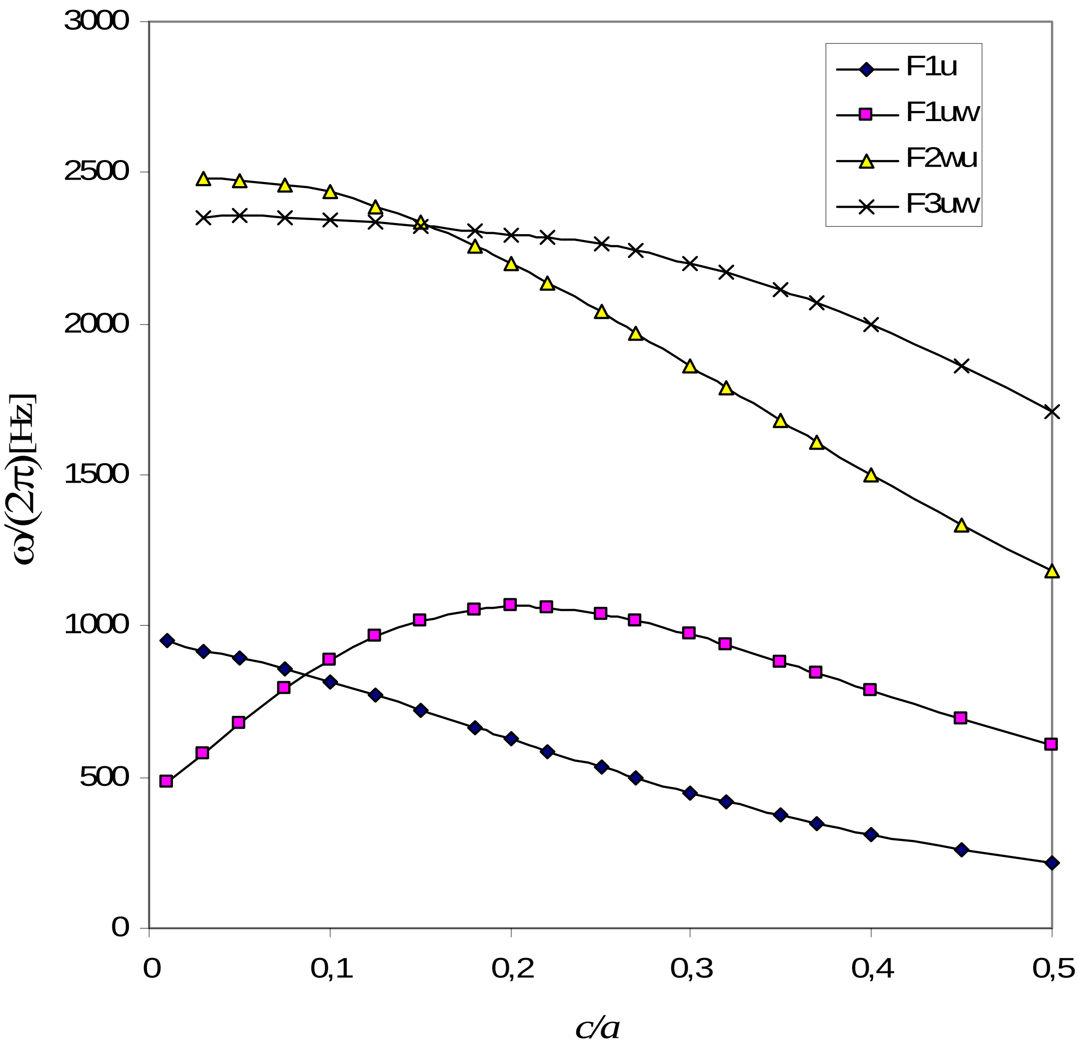

Figure 3: Variation of the frequencies with parameter *c/a* for arches supported at $r = R_G$.

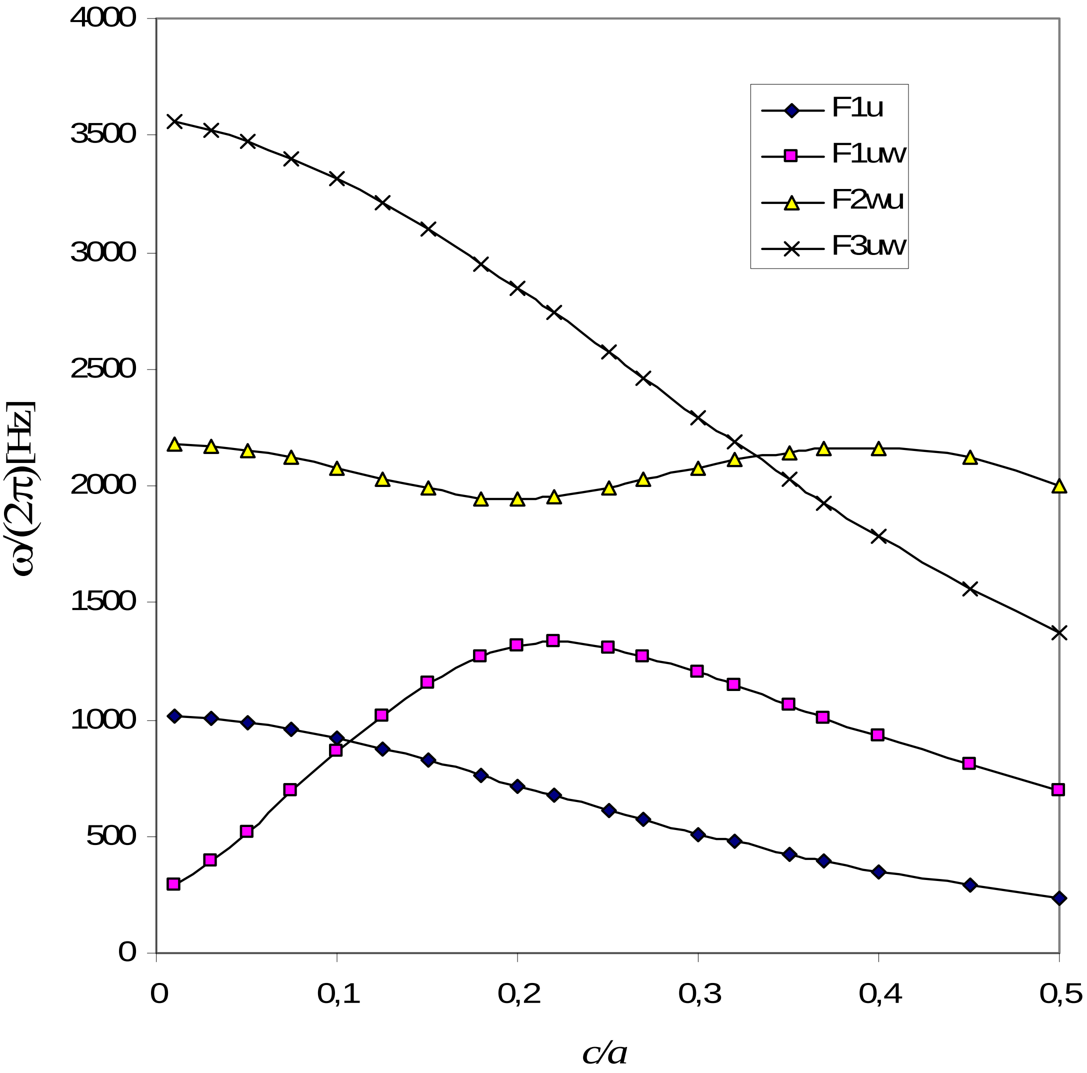

Figure 4: Variation of the frequencies with parameter *c/a* for arches supported at $r = r_i$.

In the previous two figures the nomenclature F1u intend for the first flexural dominant mode F1uw mean the first flexural dominant mode with membranal coupling; whereas F2wu, F3uw are second and third flexural – membranal coupled modes. The first letter after the number means the dominating motion, eg. F3uw means a coupled mode with bending dominant motion.

A difference between these two figures can be observed in Figure 5 but only for the first two modes. Note how sensible is the variation of frequency value with the change of the supporting point along the same cross section in the case of the coupled modes; whereas the bending dominant mode has a practically stable difference surrounding 10% along the shallowness ratio *c/a*.

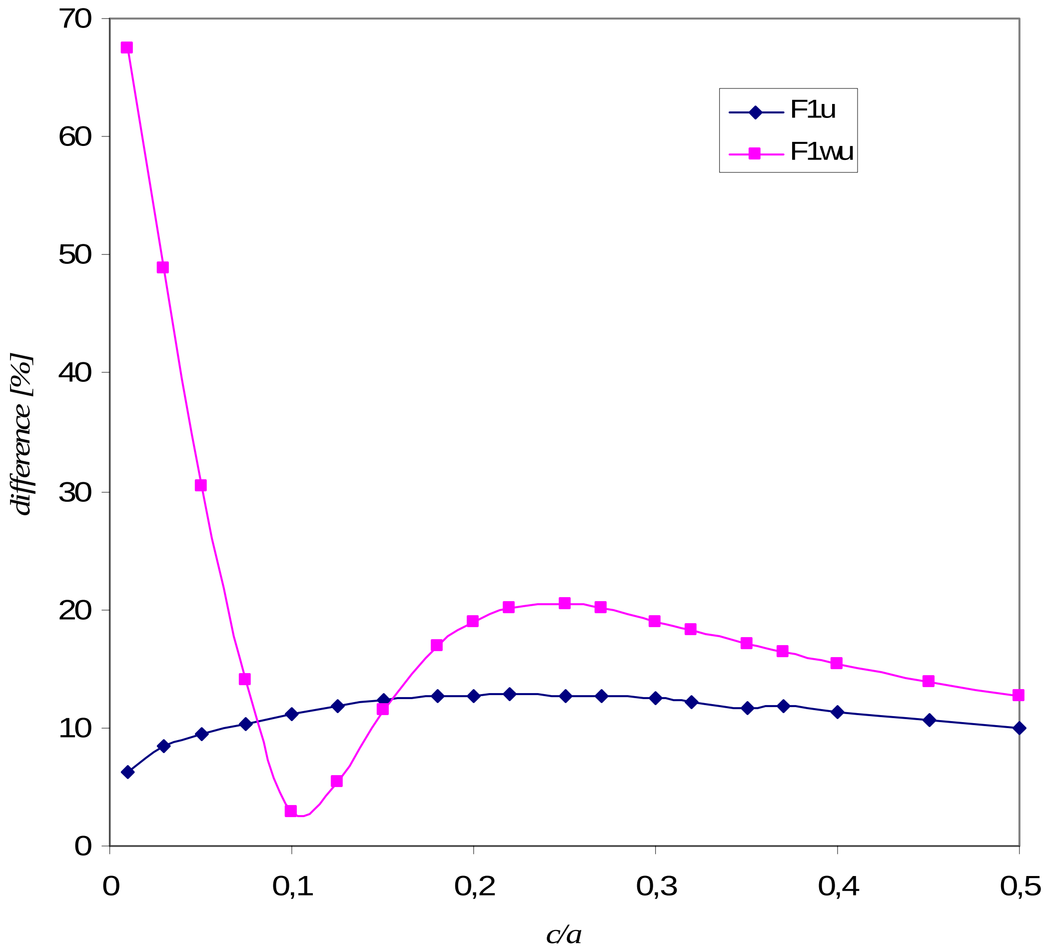

Figure 5: Difference in percentage of the first two modes.

4 CONCLUSIONS

In the present article a model for non-homogeneous thick curved beam settled in the context of the theory of strength of materials is developed. The structural non-homogeneity is confined in the context of functionally graded materials. The derivation process consisted in the employment of the concept of neutral–axis shifting, which allows the possibility to reduce the algebraic manipulation, leading to the possibility to find analytical solutions of the governing differential system, even if the differential system has variable coefficients. The motion equations are solved with a power series approach which gives arbitrary precision to the frequencies values. An especial analysis featuring simply supported conditions is performed. This analysis have shown the strong dependence of the position along the end of the simply support restrictions in the firsts in-plane frequencies for the graded material especially if the bending and axial motions are coupled.

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APPENDIX I: Sixth order Sturm-Liouville problem: orthogonality conditions

To find the solution to a forced vibrations separable problem, the algorithm of modal superposition is usually employed. Consequently one should know the orthogonality conditions in order to diagonalize the problem in terms of the time. By means of an appropriate and selective manipulation in the differential system given by Eq (42) one arrives to a non-classical sixth order Sturm-Liouville problem whose orthogonality conditions are given by the following expression:

$$N_{ij} = \int_0^1 \left[\bar{u}_i(x)\bar{u}_j(x) + \bar{w}_i(x)\bar{w}_j(x) + \frac{\gamma_2 R}{hR_G}\bar{\theta}_i(x)\bar{\theta}_j(x) + \frac{\gamma_1}{hR_G}\left(\bar{\theta}_i(x)\bar{w}_j + \bar{\theta}_j(x)\bar{w}_i\right) \right] dx, \quad \text{(I.1)}$$

where $\bar{u}_k(x)$, $\bar{w}_k(x)$ and $\bar{\theta}_k(x)$ are the modal shapes corresponding to *k-th* natural frequency ω_k of the countable set ($k=1,2,3,4,\ldots$).

Then:

$$N_{ij} = \begin{cases} 0 & \text{when } i \neq j \\ N_i^2 & \text{when } i = j \end{cases}. \quad \text{(I.2)}$$

If one divides each modal shape by N_i, the conditions of orthonormality are obtained.

APPENDIX II: Boundary conditions

In Figure A.1 one can see the sketch to describe a generalized simple support not necessarily located at the neutral axis.

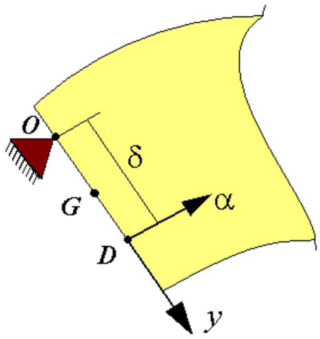

Figure A.1: Sketch of a generic end.

The boundary conditions at $\alpha = 0$ are:

$$\begin{aligned} u_0 &= 0 \\ w_0 &= 0 \\ M_0 &= 0 \end{aligned} \quad \text{(II.1)}$$

Consequently, Eq. (II.1) can be rewritten, according to Eq. (3) and (4), as:

$$\begin{aligned} u &= 0 \\ w - \theta\delta &= 0 \\ M + N\delta &= 0 \end{aligned} \quad \text{(II.2)}$$

According to Eq. (21), then Eq. (II.2) can be written in terms of displacements as:

$$\begin{aligned} u &= 0 \\ w - \theta\delta &= 0 \\ J\theta' + Aw' &= 0 \end{aligned} \quad \text{(II.3)}$$

On the other hand the boundary conditions corresponding to a clamped end are:

$$\begin{aligned} u &= 0 \\ w &= 0 \\ \theta &= 0 \end{aligned} \tag{II.4}$$

The boundary conditions corresponding to a free end are:

$$\begin{aligned} N &= 0 \\ M &= 0 \\ Q &= 0 \end{aligned} \tag{II.5}$$

Then, according to Eq (21) and Eq (33), Eq (II.5) writes as:

$$\begin{aligned} w' - u &= 0 \\ \theta' &= 0 \\ w + u' + R\theta &= 0 \end{aligned} \tag{II.6}$$

Evidently, the boundary conditions in the other end are treated in the same way.