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DYNAMIC BEHAVIOR OF A HARMONICALLY EXCITED PLATE WITH AN ELASTICALLY ATTACHED MASS

Mariano Febbo[a,b], **Diana V. Bambill**[b] **and Carlos A. Rossit**[b]

[a]*Instituto de Física del Sur, Universidad Nacional del Sur, Avda. Alem 1253, 8000 Bahía Blanca, Buenos Aires, Argentina, mfebbo@uns.edu.ar http://www.uns.edu.ar*

[b]*Instituto de Mecánica Aplicada, Universidad Nacional del Sur, Avda. Alem 1253, 8000 Bahía Blanca, Buenos Aires, Argentina, dbambill@criba.edu.ar http://www.uns.edu.ar*

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Abstract. In this work, we study the frequency response of a plate (host structure) with an elastically attached mass. The plate is harmonically excited with a sinusoidal force of variable frequency and its amplitude is then observed at a given predetermined point. The attached mass is modelled as a three degree of freedom (3-DOF) system and is elastically attached to the plate by four springs and dashpots in parallel with the plate. After solving the dynamics of the coupled system, we analyze the frequency response of the plate as the parameters of the 3-DOF system are changed. Then, it is reasonable to think the 3-DOF system as a dynamic vibration absorber to control the vibration amplitude of the plate for the set of values of the parameters that give the best results for the required performances. The range of frequencies under study includes the first three resonance frequencies of the plate.

1 INTRODUCTION

The study of the dynamic characteristics of discrete systems attached to continuum ones (host structures) is of great technological importance. This is because discrete systems, as mounted to other systems, provide a way to control undesirable high amplitudes of vibration that may be produced on the later. The idea is simple, the attached system, frequently called dynamic vibration absorber (DVA) (see Ormondroid and DenHartog (1928)) reduces the vibrational levels on the host structure by an increment of its displacement amplitude when properly tuned. A properly tuned DVA means that if the natural frequency of the DVA is equal or near the natural frequency of the host structure an important transfer of energy occurs, producing a considerable reduction of its vibration amplitude. To know the way this mechanism of energy transfer takes place, it is necessary to study the dynamic characteristics of the total system (host structure + DVA). In our case, the DVA is modelled as an attached mass (3-DOF system) elastically mounted on a plate.
The dynamic characteristics of plates with elastically attached masses has been the work of many researchers over the past thirty years. Examples of them are the works of Laura et al. (1977), Dowell (1979) and Rossit and Laura (2001), to cite a few of them. These works studied only 1 DOF systems attached to primary systems. Two DOF attached to structural elements are studied less frequently. An interesting approach in the case of beam vibrations was given by Jen and Magrab (1993), and more recently by Vera et al. (2005). In the case of 3-DOF systems attached to continuum elements, the authors are not aware of studies involving this type of complexity. Possibly one paper in this direction is the work of Febbo et al. (2010) which calculates the natural frequencies and normal modes of the title problem. In this sense, the present work increases substantially the scope of previously developed analysis and provides the frame for further studies.

The aim of this work is to observe how the dynamic characteristics of a plate with an attached 3-DOF system change as the parameters of the 3-DOF (mass, moment of inertia, damping, tuning ratio) are modified. In this sense, this provides a basis upon which a proper design of a DVA may rely on. The paper is organized as follows. In the first section, the mathematical formulation of the problem is presented. Section two is the main section of the work. There we present the frequency response curves of the plate as the parameters of the 3-DOF vary and discuss how the plate is affected in comparison with a bare plate (plate without attachments). Finally, the paper is concluded with a summary of the results in a last section.

2 MATHEMATICAL FORMULATION

Here, the derivation of the equations of motion of a plate carrying a three (3) degree-of-freedom system is derived. To this purpose, we select a Lagrangian formalism to model the plate-type structure and then we add the 3 DOF system as restrictions through Lagrange multiplier approach.

Figure 1 a) presents the system under study which consists in a plate (primary or main system) and a three degree-of-freedom system attached to it (subsidiary or secondary system). As stated in the introduction, the secondary system is used as a dynamic vibration absorber (DVA) to minimize the amplitude of vibration at a given point of the primary structure. The intervening parameters of the 3 DOF DVA, m_e, I_{ex} and I_{ey} are, respectively, the lumped mass and mass moments of inertia in a direction parallel to x and y axes; k_1, k_2, k_3, k_4, and c_1, c_2, c_3, c_4 are the spring and damper constants and a_1, a_2, a_3, a_4 are the distances between the center of mass and the sides of the rigid mass of the 3 DOF DVA (see Fig. 1 b)). The total kinetic and strain energies and the dissipation function of the system (plate + 3 DOF DVA) are:

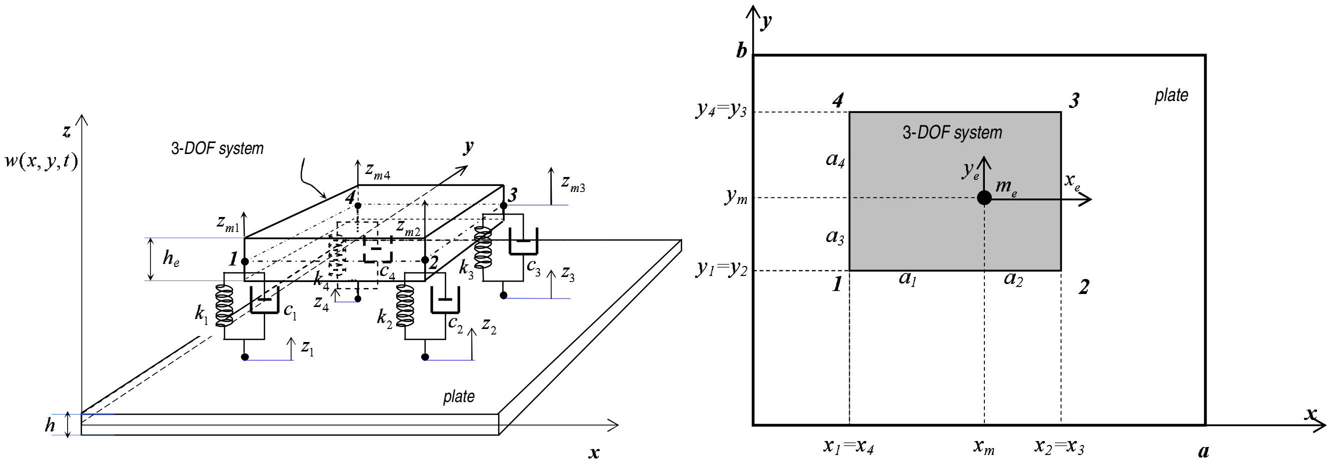

Figure 1: (a) Plate with a three degree-of-freedom DVA attached to it. (b) Nomenclature for the coordinates selected to describe the motion of the 3 DOF DVA.

$$T = \frac{1}{2}\sum_{i,j}^{n,n'} m_{ij}\dot{c}_{ij}^2 + \frac{1}{2}m_e\left(\frac{\dot{z}_{m1}a_2}{a_1+a_2} + \frac{\dot{z}_{m2}(a_4a_1 - a_3a_2)}{(a_1+a_2)(a_3+a_4)} + \frac{\dot{z}_{m3}a_3}{a_3+a_4}\right)^2 + \frac{1}{2}I_{ey}\left(\frac{\dot{z}_{m2}-\dot{z}_{m1}}{a_1+a_2}\right)^2 + \frac{1}{2}I_{ex}\left(\frac{\dot{z}_{m3}-\dot{z}_{m2}}{a_3+a_4}\right)^2 \quad (1)$$

$$V = \frac{1}{2}\sum_{i,j}^{n,n'} m_{ij}\omega_{ij}^2 c_{ij}^2 + \frac{1}{2}k_1(z_{m1}-z_1)^2 + \frac{1}{2}k_2(z_{m2}-z_2)^2 + \frac{1}{2}k_3(z_{m3}-z_3)^2 + \frac{1}{2}k_4(z_{m4}-z_4)^2 \quad (2)$$

$$D = \frac{1}{2}\sum_{i,j}^{n,n'} d_{ij}\dot{c}_{ij}^2 + \frac{1}{2}c_1(z_{m1}-z_1)^2 + \frac{1}{2}c_2(z_{m2}-z_2)^2 + \frac{1}{2}c_3(z_{m3}-z_3)^2 + \frac{1}{2}c_4(z_{m4}-z_4)^2 \quad (3)$$

where the c_{ij}'s are the normal mode amplitudes, ω_{ij} the eigenfrequencies of the primary system and the $m'_{ij}s$ are given by $\rho h \int_\Omega \phi_{ij}\phi_{mn} d\Omega = \delta_{ij} m_{ij}$ (δ_{ij} is Kronecker's delta, ρ is the plate's mass density and h its thickness, see Fig. 1 (a)). Plate's internal damping is assumed to be of viscous type with d_{ij} as the modal damping parameters. Additionally, the transverse displacement of the plate is represented by $w(x,y,t) = \sum_{i,j}^{n,n'} c_{ij}(t)\phi_{ij}(x,y)$ where the $\phi_{ij}(x,y)$ are the normal mode shapes of the selected plate. The summation is carried out up to the n, n' normal mode where the first $N = n \times n'$ modes are considered in increasing order of frequencies. Four restriction functions f_l's has to be imposed for the system of equations to be complete and solvable. This can be expressed by:

$$f_l = \sum_{i,j}^{n,n'} c_{ij}(t)\phi_{ij}(x_l, y_l) - z_l(t) = 0, \; l = 1, ..., 4; \quad (4)$$

which represent the connection of the 3 DOF DVA to the plate at the points x_l, y_l, $l = 1, 2, 3, 4$ (see Fig. 1 (b)). At the same time, a fifth restriction is needed, which has to be defined to satisfy the rigidity condition of the mass of the 3 DOF DVA (rigid solid):

$$f_5 = z_{m4} - (z_{m1} + z_{m3} - z_{m2}) \tag{5}$$

Then, the equations of motion can be obtained from Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{s}_k}\right) + \frac{\partial D}{\partial \dot{s}_k} + \frac{\partial V}{\partial s_k} = Q_k + \sum_{l=1}^{5} \lambda_l \frac{\partial f_l}{\partial s_k} \qquad k = 1, ...N + 8 \tag{6}$$

where Q_k represent the generalized forces applied at the point (x_f, y_f) and λ_l are Lagrange's multipliers. Finally, Lagrange's equations of motion yield, after the elimination of the λ_l's, a set of $N + 3$ coupled linear second order differential equations in terms of the new independent set of coordinates $\mathbf{q} \equiv [q_1, ...q_N, q_{N+1}, q_{N+2}, q_{N+3}] \equiv [c_{11}, ...c_{nn'}, z_{m1}, z_{m2}, z_{m3}]$, and the set of generalized forces $\mathbf{Q} \equiv [Q_1, ...Q_N, 0, 0, 0]$:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{Q} \tag{7}$$

Matrices $(N + 3 \times N + 3)$ $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are given by:

$$\mathbf{M}= \begin{bmatrix} \mathbf{M}_p & \mathbf{0} \\ \mathbf{0}^T & \mathbf{M}_{\text{3DOF}} \end{bmatrix}; \qquad \mathbf{C} = \begin{bmatrix} \mathbf{C}_p + \mathbf{C}_{\text{sub}} & \mathbf{C}_c \\ \mathbf{C}_c^T & \mathbf{C}_{\text{3DOF}} \end{bmatrix};$$
$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_p + \mathbf{K}_{\text{sub}} & \mathbf{K}_c \\ \mathbf{K}_c^T & \mathbf{K}_{\text{3DOF}} \end{bmatrix};$$

where the $(N \times N)$ matrices $\mathbf{M}_p$, $\mathbf{C}_p$ and $\mathbf{K}_p$ are diagonal matrices whose elements are m_k, $2\xi_k m_k \omega_k$ ($\xi_k = d_k/2m_k\omega_k$) and $m_k\omega_k^2$ ($k = 1, 2, ...N$), respectively. The rest of the matrices, $\mathbf{K}_{\text{sub}}, \mathbf{C}_{\text{sub}}$, $\mathbf{K}_c$, $\mathbf{C}_c$ and $\mathbf{M}_{\text{3DOF}}, \mathbf{C}_{\text{3DOF}}, \mathbf{K}_{\text{3DOF}}$ are given in appendix (A). In order to calculate the displacement amplitude $\mathbf{q}(t)$ of the coupled system, a simple harmonic motion of vector $\mathbf{q}(t) = \bar{\mathbf{q}}e^{i\omega t}$ is imposed. Then, vector $\bar{\mathbf{q}}$ is obtained by solving

$$\bar{\mathbf{q}} = [-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}]^{-1}\widetilde{\mathbf{\Phi}}(x_f, y_f) \tag{8}$$

where $\widetilde{\mathbf{\Phi}}(x_f, y_f) = [\phi_1(x_f, y_f)...\phi_N(x_f, y_f)\ 0\ 0\ 0]^T$. Finally, the displacement of the primary system at the point (x_a, y_a) is

$$w(x_a, y_a, t) = \sum_{k=1}^{N} \bar{q}_k e^{i\omega t}\phi_k(x_a, y_a)$$

3 EFFECTIVENESS OF A 3-DOF SYSTEM UNDER PARAMETER CHANGE

In this section, a thorough study on the dynamic behavior of a 3-DOF system mounted on a plate is carried out. The main concern is to observe how the variation of some of the parameters of the 3-DOF systems affects the dynamics of the plate.

A totally simply supported steel plate is considered for all cases whose physical constants are shown in table 1. Its first natural frequencies are: $\omega_{nat1} = 100.0720$ (15.927 Hz), $\omega_{nat2} = 160.1153$, (25.483 Hz), $\omega_{nat3} = 260.1873$ (41.41 Hz). The parameters of the 3-DOF system are selected in order to have approximately its natural frequencies equal to the first three natural frequencies of the plate. Given a mass of $m_e = m_p/20 = 3.9250$ kg (mass), and a height of $h_e = 0.356$ m then, being $a_1 = a_2 = 0.07$ m; $a_3 = a_4 = 0.47$ m, the natural frequencies of the 3 DOF system results:

- $\omega_{3DOF1} = 100.6518$; $\alpha_1 = 1.005$ (mainly rotation about y axis)
- $\omega_{3DOF2} = 160.3340$; $\alpha_2 = 1.001$ (mainly translation)
- $\omega_{3DOF3} = 260.0878$; $\alpha_3 = 0.999$ (mainly rotation about x axis)

where $\alpha_i = \frac{\omega_{3DOFi}}{\omega_{nati}}$ is the tuning ratio, i. e. the ratio between a natural frequency of the 3 DOF system and any natural frequency of the plate. With those values, the moments of inertia are $I_{ex} = \frac{m_e}{12}(h_e^2 + (a_3 + a_4)^2) = 0.330464066666667$ kgm^2, $I_{ey} = \frac{m_e}{12}(h_e^2 + (a_1 + a_2)^2) = 0.047864066666667$ kgm^2. The observation point $(x_a, y_a) = (0.8125, 0.3125)$ is selected in order to obtain a non-vanishing displacement amplitude for the first three modes of a totally simply supported rectangular plate. The plate is excited by a sinusoidal source located at $(x_f, y_f) = (0.3, 0.3)$.

Table 1: Physical constants of the considered plate.

ρ $[kg/m^3]$ density	a $[m]$ length	$E[N/m^2]$ Young modulus	h $[m]$ height	b $[m]$ width	m_p $[kg]$ mass	ξ [adim] damping coeff.
7.850×10^3	2	$2.051 \times 10^1 1$	0.005	1	78.5	0.01

3.1 Variation of the damping constants

In order to evaluate the influence of the dampers of the attached discrete system, its damping constants are varied. Figure 2 shows the typical behavior when there's no dampers in the 3-DOF system ($c_i = 0$), duplicating the first three resonances (see, for example DenHartog (1956)). It can be observed in each case that the new two natural frequencies differ slightly from the original. Increasing the damping constant of the dampers (see $c_i = 10$) cause, as it is expected, the missing of this duplication.

3.2 Variation of the location

The effect of the position of the 3 DOF system on the displacement amplitude of the plate is studied in this section. The 3-DOF system is located at the four different positions shown in Fig. 3(varying the orientation). The coordinates of the extreme points (x_1, y_1) and (x_3, y_3) are presented in table 2 for a better understanding of the results.

Clearly, it can be seen from Figs. 4, 5 and 6 that the best performance of the 3-DOF system thinking of it as an absorber is at the second frequency of the bare plate: the displacement amplitude is reduced significantly and the "new" two frequencies are well separated from the second resonance of the bare plate. One must remember that this second frequency tunes to the translational mode of the 3-DOF system. For the first and third natural frequencies of the bare plate, the reduction performed by the 3-DOF system is not significative. However, different locations of the 3-DOF system perform different levels of reduction. At this stage is important to point out that the rigidity constant of the four springs are different. Another point of interest results the performance of location B. It is not difficult to see that it has the lowest influence

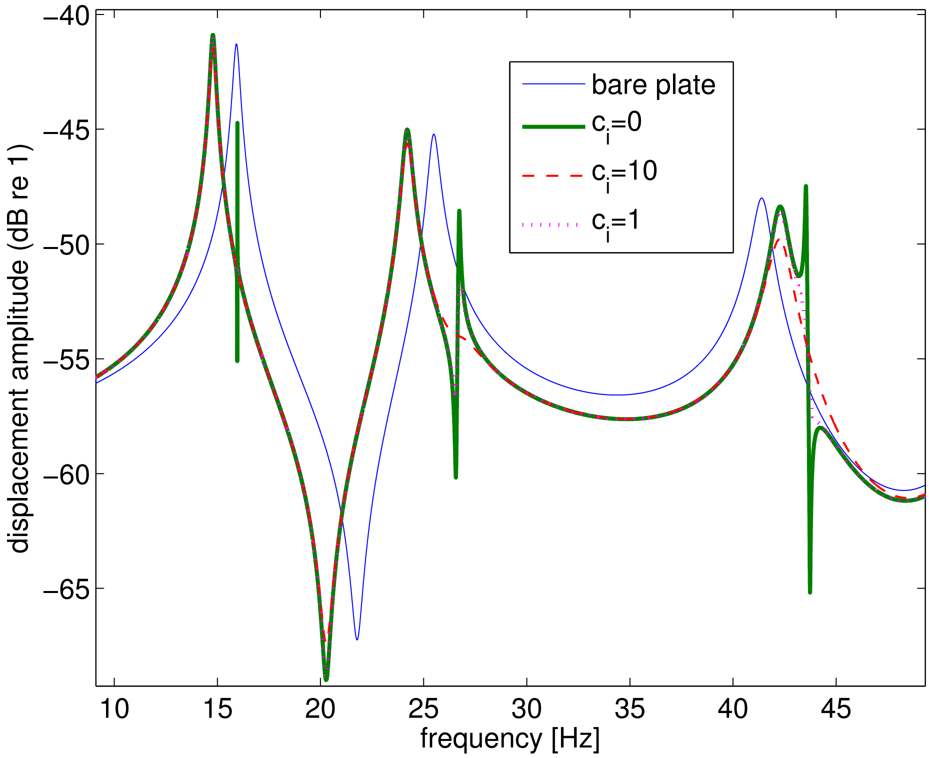

Figure 2: Displacement amplitude of a plate with a 3-DOF system for varying damping constant of the 3-DOF.

Table 2: Different considered locations of the 3-DOF.

Position	$(x_1; y_1)$	$(x_3; y_3)$
A	(0.53;0.57)	(1.47;0.43)
B	(0.93;0.03)	(1.07;0.97)
C	(0.5483;0.2272)	(1.4517;0.6476)
D	(0.6109;0.7728)	(1.3891;0.2272)

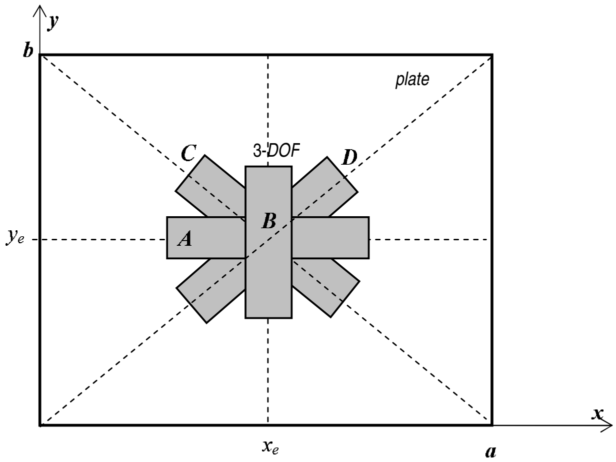

Figure 3: Different locations of the 3 DOF system on the plate to study the effect of its location on the dynamic behavior of the plate to which it is mounted.

on the dynamic behavior of the plate. This may be understood bearing in mind that, at the first resonance of the plate, the corresponding mode of the 3-DOF system is mainly a rotation about y axis. Next, at the second resonance of the plate (one nodal line in the y direction), the mode of the 3-DOF is pure translational and its location, almost totally over a nodal line, has no effect on the dynamics of the plate. Finally, at the third resonance of the plate (two nodal lines in the y direction) the 3-DOF pure rotational mode about x axis affects again poorly the dynamics of the plate.

3.3 Variation of the mass

Figures 7, 8 and 9 show the influence of the coefficient $\mu = m_e/m_p$ that represents the ratio of the attached mass to the mass of the bare plate. As the mass of the discrete system is varied, the remaining parameters of the 3-DOF system also vary in order to preserve the tuning to the natural frequencies of the bare plate. Clearly, the increment in the magnitude of the mass, increases the effect in the dynamic behavior of the plate for the three frequencies. This means that the separation between the peaks and the reduction at the frequency of maximum reduction is substantially modified as well. Again, for the second frequency, the effect is more important, acting as a well behaved vibration absorber at the tuned frequency.

3.4 Variation of the tuning ratio

Different values of α_i are considered. In every case, the value of the tuning ratio is the same for the three frequencies. The variation of the tuning ratio is attained modifying the moment of inertia and the stiffness constants of the 3-DOF system.

Figures 10, 11, and 12 show the results for frequencies near the first, second and third resonances of the plate respectively. It can be observed that the effect is rather wandering, varying for each frequency. Generally speaking, it can be mentioned that the “exact" tuning ($\alpha_i = 1$) does not mean advantage in suppression of vibration. Analyzing the first frequency Fig. 9, it is observed that the major separation between the peaks is attained for $\alpha_i = 0.85$ and also the best reduction. However, for the second frequency (Fig. 10) it seems that the best performance is for $\alpha_i = 1$. Remarkably, the effect of the 3-DOF system at the third frequency is rather poor.

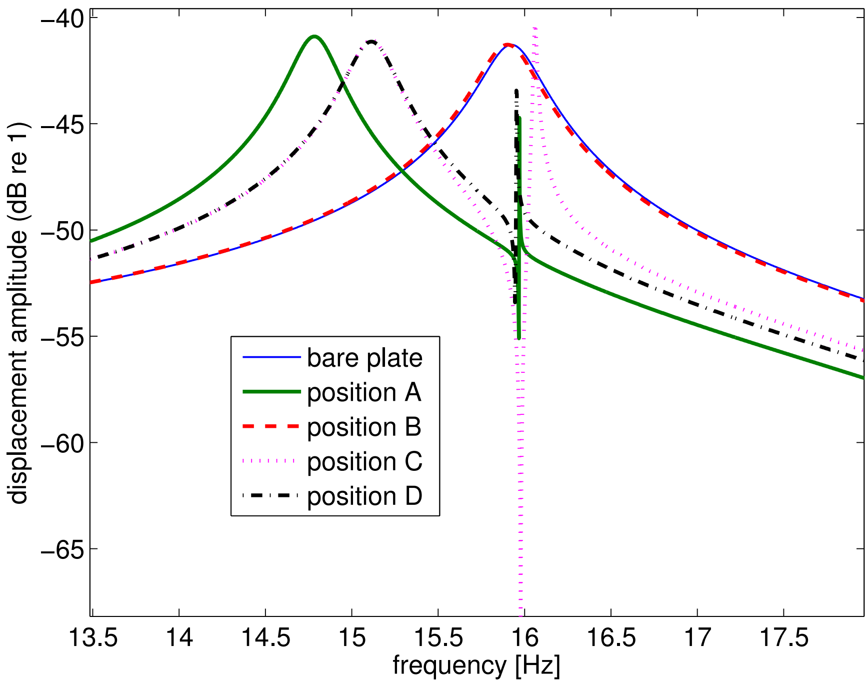

Figure 4: Displacement amplitude of a plate with a 3-DOF system for different locations of it. Frequencies near the first resonance of the plate.

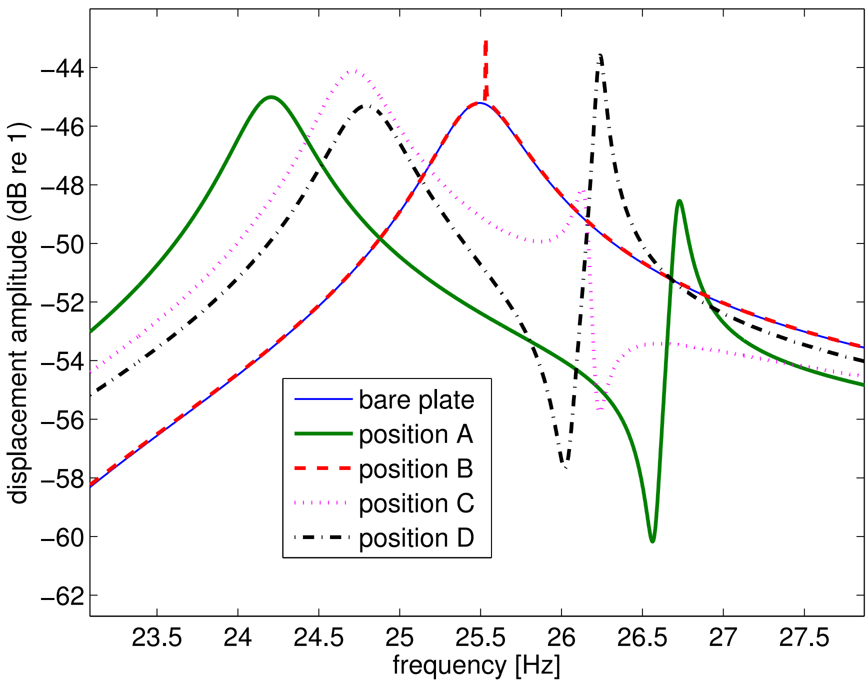

Figure 5: Idem Fig. 4. Frequencies near the second resonance of the plate.

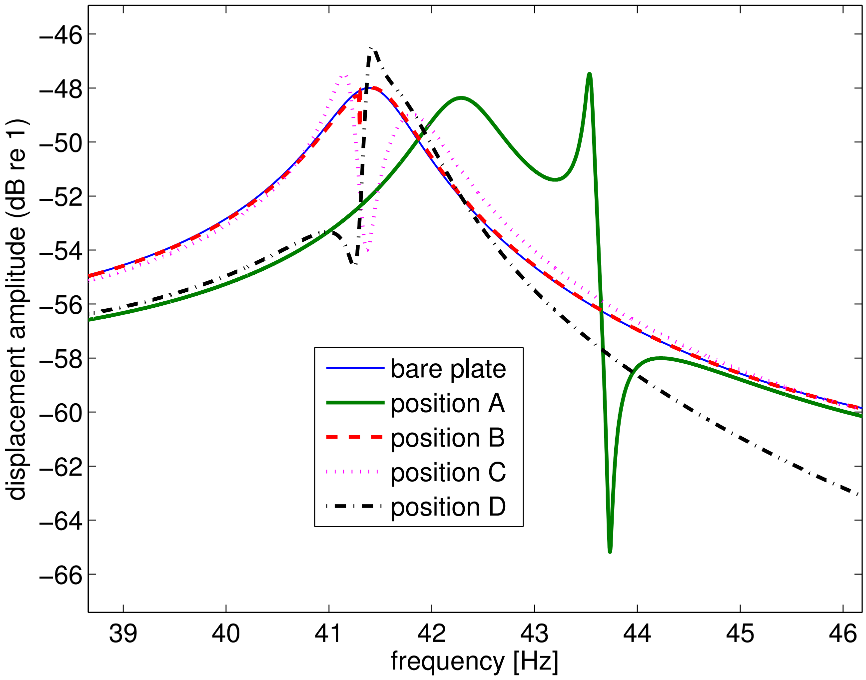

Figure 6: Idem Fig. 4. Frequencies near the third resonance of the plate.

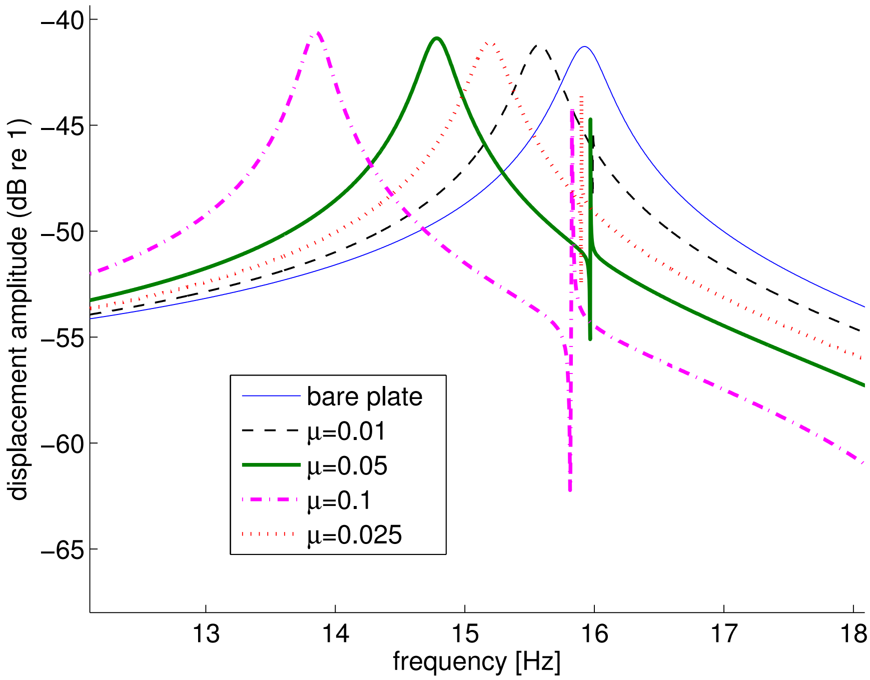

Figure 7: Displacement amplitude of a plate with a 3 DOF system for varying mass ratio. Frequencies near the first resonance of the plate.

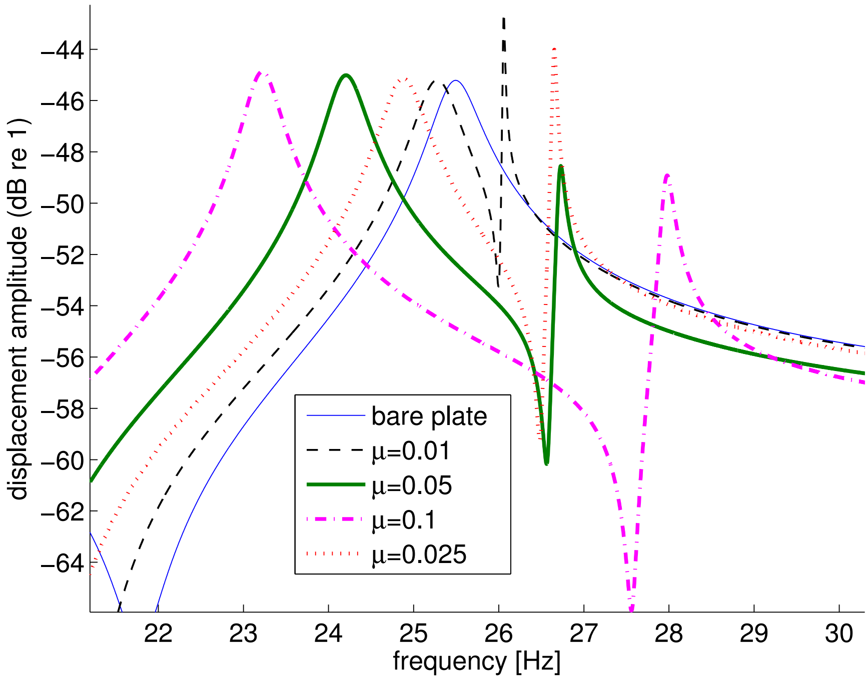

Figure 8: Idem Fig. 7. Frequencies near the second resonance of the plate.

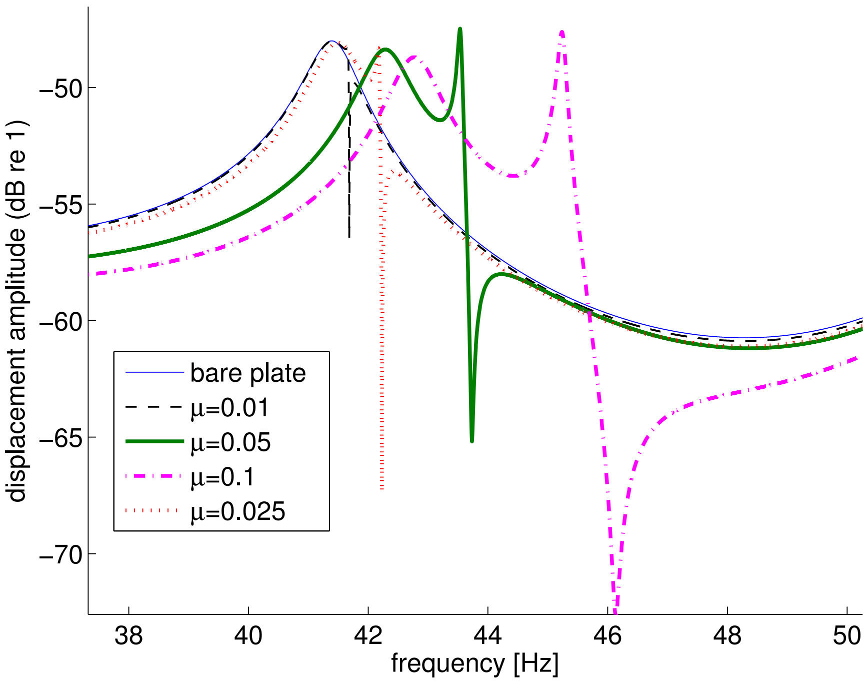

Figure 9: Idem Fig. 7. Frequencies near the third resonance of the plate.

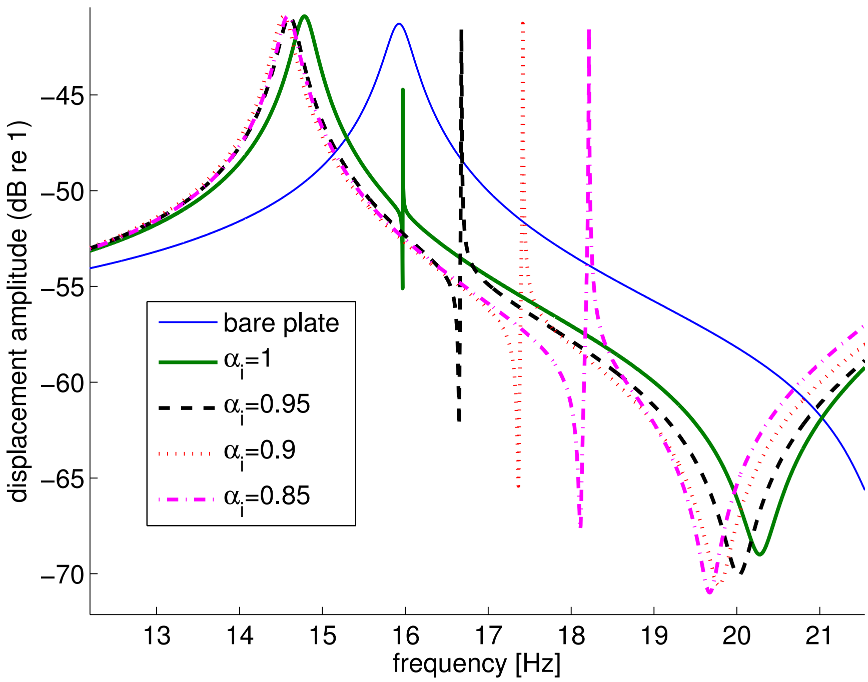

Figure 10: Displacement amplitude of a plate with a 3 DOF system for varying tuning ratio. Frequencies near the first resonance of the plate.

Neither the amplitude nor the peaks show a good reduction or separation.

4 CONCLUSIONS

In this paper, the forced vibration characteristics of a rectangular plate carrying a 3-DOF spring-mass system were calculated and analyzed by means of Lagrange multipliers method (analytical approach). In order to perform a systematic study of problem, we analyzed the influence of the parameters of the 3-DOF system (damping constant, mass, location, tuning ratio) in the dynamic behavior of the compound system (plate + 3-DOF). Summarizing the results it can be concluded that the strongest effect in the suppression of vibrations is obtained for the second frequency of the plate. As it was stated, this frequency tunes to the (mainly) translational mode of the discrete attached system. Consequently, one may suppose that the behavior of the 3-DOF system acting as a Dynamic Vibration Absorber (DVA) for translational modes is not so different from the behavior of a 1-DOF DVA. On the contrary, predominantly rotational modes of the discrete system seem to be not effective in order to control vibrations if properly tuned. Nevertheless, this is an introductory study in the subject and the quantity and variability of the geometric and mechanical parameters involved in the description of the dynamical behavior of these kinds of structures, does not allow drawing definitive conclusions. Additional and thorough studies must be performed in order to understand the behavior of the 3-DOF system as a Dynamic Vibration Absorber.

5 ACKNOWLEDGEMENTS

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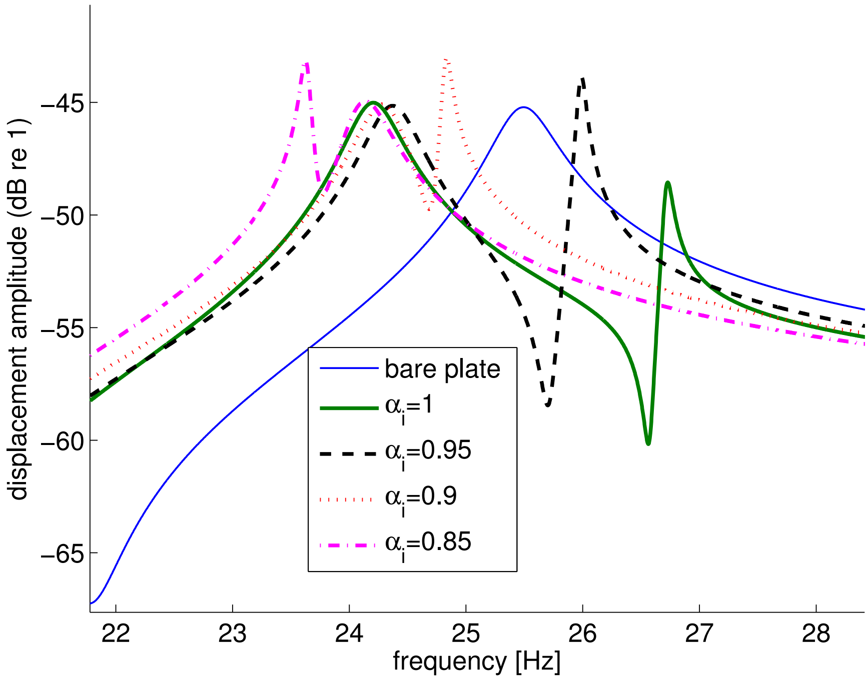

Figure 11: Idem Fig. 10. Frequencies near the second resonance of the plate

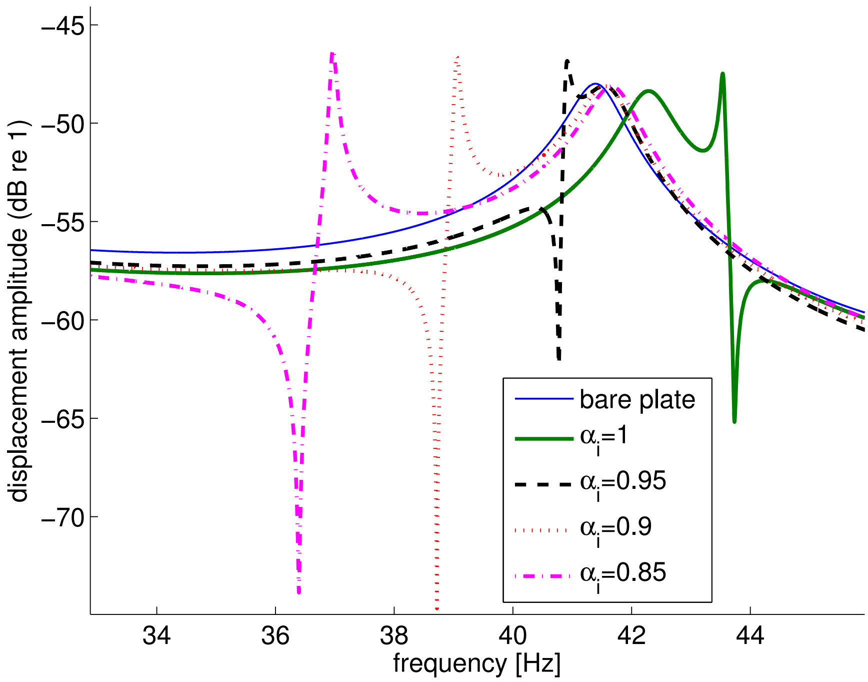

Figure 12: Idem Fig. 10. Frequencies near the third resonance of the plate

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A APPENDIX: MASS, DAMPING AND STIFFNESS MATRICES

The $(N \times N)$ $\mathbf{C}_{\text{sub}}$ and $\mathbf{K}_{\text{sub}}$ matrices and the $(N \times 3)$ $\mathbf{C}_c$ and $\mathbf{K}_c$ rectangular matrices are:

$$\begin{gathered}
\mathbf{\Phi}(x_l, y_l)=[\phi_1(x_l, y_l)\phi_2(x_l, y_l)...\phi_N(x_l, y_l)]^T \\
\mathbf{K}_{\text{sub}} = \textstyle\sum_{l=1}^4 k_l \mathbf{\Phi}(x_l, y_l)\mathbf{\Phi}^T(x_l, y_l); \qquad \mathbf{C}_{\text{sub}} = \textstyle\sum_{l=1}^4 c_l \mathbf{\Phi}(x_l, y_l)\mathbf{\Phi}^T(x_l, y_l); \\
\mathbf{K}_c = [-k_1\Phi(x_1, y_1) - k_4\Phi(x_4, y_4) \qquad - k_2\Phi(x_2, y_2) + k_4\Phi(x_4, y_4) \qquad - k_3\Phi(x_3, y_3) - k_4\Phi(x_4, y_4)] \\
\mathbf{C}_c = [-c_1\Phi(x_1, y_1) - c_4\Phi(x_4, y_4) \qquad - c_2\Phi(x_2, y_2) + c_4\Phi(x_4, y_4) \qquad - c_3\Phi(x_3, y_3) - c_4\Phi(x_4, y_4)]
\end{gathered}$$

and the (3×3) symmetric matrices $\mathbf{M}_{\text{3DOF}}$ $\mathbf{C}_{\text{3DOF}}$ and $\mathbf{K}_{\text{3DOF}}$ are:

$$\mathbf{M}_{\text{3DOF}} = \begin{bmatrix}
\frac{(m_e a_2^2 + I_{ey})}{(a_1+a_2)^2} & \frac{m_e a_2(a_4a_1-a_3a_2)-I_{ey}(a_3+a_4)}{(a_1+a_2)^2(a_3+a_4)} & \frac{(m_e a_2 a_3)}{(a_1+a_2)(a_3+a_4)} \\
\frac{m_e a_2(a_4a_1-a_3a_2)-I_{ey}(a_3+a_4)}{(a_1+a_2)^2(a_3+a_4)} & \frac{m_e(a_4a_1-a_3a_2)^2+I_{ey}(a_3+a_4)^2+I_{ex}(a_1+a_2)^2}{(a_1+a_2)^2(a_3+a_4)^2} & \frac{m_e a_3(a_4a_1-a_3a_2)-I_{ex}(a_1+a_2)}{(a_1+a_2)(a_3+a_4)^2} \\
\frac{(m_e a_2 a_3)}{(a_1+a_2)(a_3+a_4)} & \frac{m_e a_3(a_4a_1-a_3a_2)-I_{ex}(a_1+a_2)}{(a_1+a_2)(a_3+a_4)^2} & \frac{(m_e a_3^2+I_{ex})}{(a_3+a_4)^2}
\end{bmatrix}$$

$$\mathbf{C}_{\text{3DOF}} = \begin{bmatrix} c_1 + c_4 & -c_4 & c_4 \\ -c_4 & c_2 + c_4 & -c_4 \\ c_4 & -c_4 & c_3 + c_4 \end{bmatrix}$$

$$\mathbf{K}_{\text{3DOF}} = \begin{bmatrix} k_1 + k_4 & -k_4 & k_4 \\ -k_4 & k_2 + k_4 & -k_4 \\ k_4 & -k_4 & k_3 + k_4 \end{bmatrix}$$