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SIMPLE BENDING ANALYSIS OF BUILDING FLOOR STRUCTURES BY A BEM FORMULATION BASED ON REISSNER'S THEORY

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Abstract. In this work, the plate bending formulation of the boundary element method - BEM, based on the Reissner's hypothesis, is extended to the analysis of plates reinforced by beams. Equilibrium and compatibility conditions are automatically imposed by the integral equations, which treat this composed structure as a single body. In order to decrease the number of degrees of freedom, some approximations are considered for the displacements and tractions along the beam width. Therefore the problem values remain defined only on the beams axis and on the plate boundary without beams. The accuracy of the proposed model is showed by comparing the numerical results with a well-known finite element code.

1 INTRODUCTION

The boundary element method (BEM) has already proved to be a suitable numerical tool to deal with plate bending problems. The method is particularly recommended to evaluate internal force concentrations due to loads distributed over small regions that very often appear in practical problems. Moreover, the same order of errors is expected when computing deflections, slopes, moments and shear forces. Shear forces, for instance, are not obtained by differentiating approximation function as for other numerical techniques.

Bezine (1981) apparently was the first to use a boundary element to analyse building floor structures by considering plates with internal point supports. Recently, some authors have presented BEM formulations (without coupling BEM with FEM) to analyse stiffened plates (Sapountzakis and Katsikadelis (2000), Tanaka and Oida (2000), Paiva and Aliabadi (2004)). In Fernandes and Venturini (2002) a BEM formulation based on Kirchhoff's hypothesis to perform simple bending analysis of building floor structures is developed, which is modelled by a zoned plate where each sub-region defines a beam or a slab. Along the interfaces the tractions are eliminated and in order to reduce the degrees of freedom some Kinematic assumptions were made along the beam width. In Fernandes and Venturini (2005) the same authors have extended this previous formulation to take into account membrane effects.

In this work the BEM formulation developed in Fernandes and Venturini (2002) is modified to take into account the Reissner's hypothesis instead of the Kirchhoff's (see Fernandes and Konda (2008)). The inaccuracy of the classical theory (Kirchhoff's) turns out to be important for thick plates, especially in the edge zone of the plate and around holes whose diameter is not larger than the plate thickness. In the Reissner's theory [see Reissner (1947), Weën (1982), Palermo (2003)], which can be used either for thin or thick plate, takes into account the shear deformation effect and defines six boundary values. In the proposed model the tractions is no longer eliminated on the interfaces as occurred in the formulation presented in Fernandes and Venturini (2002). Therefore, in order to reduce the number of degrees of freedom, both traction and displacements must be approximated along the beam width, which leads to a model where the bending values are defined only on the beams axis and on the plate boundary without beams. The accuracy of the proposed model is illustrated by comparing the numerical results with a well-known finite element code.

2 BASIC EQUATIONS

Without loss of generality, let us consider the three sub-region plate depicted in Figure 1, where t_1, t_2 and t_3 are the sub-regions thickness. The plate sub-domains assumed as isolated plates are denoted by Ω_1, Ω_2 and Ω_3, with boundaries Γ_1, Γ_2 and Γ_3, respectively. Alternatively, when the whole solid is considered, Γ gives the total external boundary, while Γ_{jk} represents interfaces, for which the subscripts denote the adjacent sub-regions (see Figure 1). For a point placed at any of those plate sub-regions, the following equations are defined:

-The equilibrium equations in terms of internal forces:

$$M_{ij},_j - Q_i = 0 \qquad i, j = 1, 2 \tag{1}$$

$$Q_i,_i + g = 0 \tag{2}$$

where g is the distributed load acting on the plate middle surface, m_{ij} are bending and twisting moments and Q_i represents shear forces.

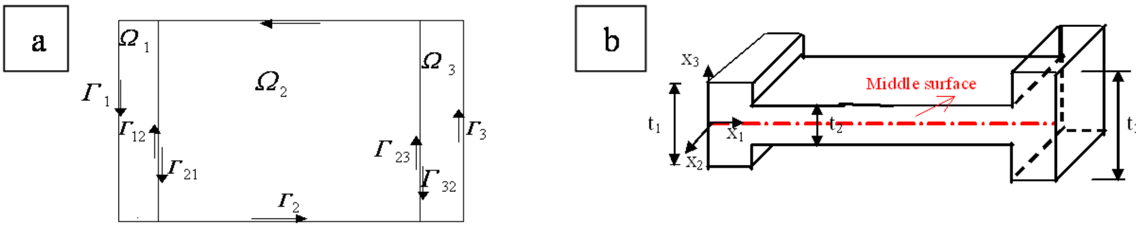

Figure 1: a) General zoned plate domain; b) Reference surface view.

-The generalised internal forces written in terms of displacement:

$$M_{ij} = \frac{D(1-\nu)}{2}\left(\phi_{i},_{j} + \phi_{j},_{i} + \frac{2\nu}{1-\nu}\phi_{k},_{k}\,\delta_{ij}\right) + \frac{\nu g}{(1-\nu)\lambda^2}\delta_{ij} \qquad \text{i, j, k =1, 2} \qquad (3)$$

$$Q_i = \frac{D(1-\nu)}{2}\lambda^2(\phi_i + w,_i) \qquad \text{i= 1,2} \qquad (4)$$

where $w,_i$ is the rotation in the i direction, w the deflection, $\phi_{k},_{l}$ the plate curvature, $\psi_{3l} = \phi_l + w,_l$ the shear deformation, $D = Eh^3/(1-\nu^2)$ the flexural rigidity, ν the Poisson's ration, λ a constant related to shear effect given by $\lambda = \sqrt{10}/h$ and δ_{ij} is the Kronecker delta.

-Finally, the plate bending differential equations are given by:

$$Q_i - \frac{1}{\lambda^2}\nabla^2 Q_i + \frac{1}{(1-\nu)\lambda^2}\frac{\partial g}{\partial x_i} = -D\frac{\partial}{\partial x_i}\nabla^2 w \qquad \text{i=1, 2} \qquad (5)$$

$$\nabla^4 w = \frac{1}{D}\left[g - \frac{(2-\nu)}{(1-\nu)\lambda^2}\nabla^2 g\right] \qquad (6)$$

where $w,_{iijj} = \nabla^4 w$, being ∇^4 the bi-harmonic operator; $w,_{ii} = \nabla^2 w$ being ∇^2 the bi-Laplacian operator.

Equations (5) and (6) result into the set of differential equations, being (5) and. (6) a second and fourth order equation, respectively, leading therefore to six independent boundary values: M_n; M_{ns}, Q_n, w, ϕ_n and ϕ_s, being (n, s) the local co-ordinate system, with n and s referred to the plate boundary normal and tangential directions, respectively. The problem definition is then completed by assuming the following boundary conditions over Γ: $U_i = \overline{U}_i$ on Γ_u (generalised displacements: deflections and rotations) and $P_i = \overline{P}_i$ on Γ_p (generalised tractions: bending and twisting moments and shear forces), where $\Gamma_u \cup \Gamma_p = \Gamma$.

3 INTEGRAL REPRESENTATIONS

Initially, the integral equations for zoned domain plate subject to simple bending will be derived, considering the case where the thickness may vary from one sub-region to another. The beams will be considered as small sub-regions with larger rigidities. Then some approximations will be adopted along the beam cross section in order to decrease the number of degrees of freedom. The equations are derived by applying the weighted residual method to each sub-region and summing them to obtain the equation for the whole body.

Considering the plate equilibrium equations (Eq. (1) and (2)) the following weighted residual equation can be obtained for a simple plate:

$$\int_{\Omega}\left[\phi_{ki}^{*}\left(M_{ij},_{j}-Q_{i}\right)+\left(Q_{i},_{i}+g\right)w_{k}^{*}\right]d\Omega=\int_{\Gamma_u}\left[\left(\bar{\phi}_{i}-\phi_{i}\right)M_{ki}^{*}+\left(\bar{w}-w\right)Q_{kn}^{*}\right]d\Gamma$$

$$-\int_{\Gamma_P}\left[\left(\overline{M}_{i}-M_{i}\right)\phi_{ki}^{*}+\left(\overline{Q}_{n}-Q_{n}\right)w_{k}^{*}\right]d\Gamma . \qquad \text{i, j =1, 2} \quad \text{k=1, 2, 3} \qquad (7)$$

where k = 1, 2 refers to unit moments applied in the x_1 and x_2 directions and k=3 refers to a unit load acting in the x_3 direction.

Integrating Eq. (7) by parts twice, considering Eqs. (3) and (4) and writing the values in terms of the local system of coordinates (n, s), the following integral equation of the generalised displacements can be obtained:

$$c(q)U_{k}(q)=\int_{\Omega_g} g\left[w_{k}^{*}(q,p)-\frac{\nu}{(1-\nu)\lambda^{2}}\phi_{ki}^{*},_{i}(q,p)\right]d\Omega$$

$$-\int_{\Gamma}\left[\phi_{n}(P)M_{kn}^{*}(q,P)+\phi_{s}(P)M_{kns}^{*}(q,P)+w(P)Q_{kn}^{*}(q,P)\right]d\Gamma$$

$$+\int_{\Gamma}\left[M_{n}(P)\phi_{kn}^{*}(q,P)+M_{ns}(P)\phi_{ks}^{*}(q,P)+Q_{n}(P)w_{k}^{*}(q,P)\right]d\Gamma \qquad \text{k=m,l,3} \quad \text{i=1,2} \qquad (8)$$

where Ω_g is the area where the load g is distributed, the free term value *c(q)* depends on the position of the point q: *c(q)=0* for external points *c(q)=1* for internal points and *c(Q) =0.5* for boundary points; $U_m=\phi_m$, $U_l=\phi_l$, and $U_3=w$, being *m* and *l* either the local system (n, s) for boundary points or any direction for internal points.

Let us now consider a zoned plate as the one depicted in the Figure 1 for example. In this case Eq. (8) is valid to each sub-region separately. Then, taking into account the equilibrium and compatibility conditions, writing Eq. (8) to all sub-regions and summing them the following integral equation for the zoned plate can be obtained:

$$U_{k}(q)=\sum_{j=1}^{N_s}\int_{\Omega_g} g\left[w_{k}^{*j}(q,p)-\frac{\nu}{(1-\nu)\lambda^{2}}\phi_{ki}^{*j},_{i}(q,p)\right]d\Omega$$

$$-\sum_{j=1}^{N_s}\int_{\Gamma_1}\left[\phi_{n}(P)M_{kn}^{*j}(q,P)+\phi_{s}(P)M_{kns}^{*j}(q,P)+w(P)Q_{kn}^{*j}(q,P)\right]d\Gamma$$

$$-\sum_{j=1}^{N_{\text{int}}}\int_{\Gamma_{ja}}\left\{\phi_{n}(P)\left[M_{kn}^{*j}(q,P)-M_{kn}^{*a}(q,P)\right]+\phi_{s}(P)\left[M_{kns}^{*j}(q,P)-M_{kns}^{*a}(q,P)\right]+w(P)\left[Q_{kn}^{*j}(q,P)-Q_{kn}^{*a}(q,P)\right]\right\}d\Gamma$$

$$+\sum_{j=1}^{N_s}\int_{\Gamma_1}\left[M_{n}(P)\phi_{kn}^{*j}(q,P)+M_{ns}(P)\phi_{ks}^{*j}(q,P)+Q_{n}(P)w_{k}^{*j}(q,P)\right]d\Gamma+Q_{n}(P)\left[w_{k}^{*1}(q,P)-w_{k}^{*a}(q,P)\right]\Big\}d\Gamma$$

$$+\sum_{j=1}^{N_{\text{int}}}\int_{\Gamma_{ja}}\left\{M_{n}(P)\left[\phi_{kn}^{*j}(q,P)-\phi_{kn}^{*a}(q,P)\right]+M_{ns}(P)\left[\phi_{ks}^{*j}(q,P)-\phi_{ks}^{*a}(q,P)\right]\right. \qquad (9)$$

where N_s is the sub-regions number, N_{int} the interfaces number, Γ_{ja} represents an interface for which the subscript a denotes the adjacent sub-region to Ω_j; $U_{ki}^{*(j)}$, $P_{ki}^{*(j)}$, U_i^j and P_i^j indicate their values in the sub-region Ω_j.

Note that in both integrals along the interface Γ_{ja} all values are related to the local system defined on Γ_{ja} and the fundamental values U_{ki}^{*a} and P_{ki}^{*a} are given in terms of the rigidity D and thickness t of the sub-region Ω_a.

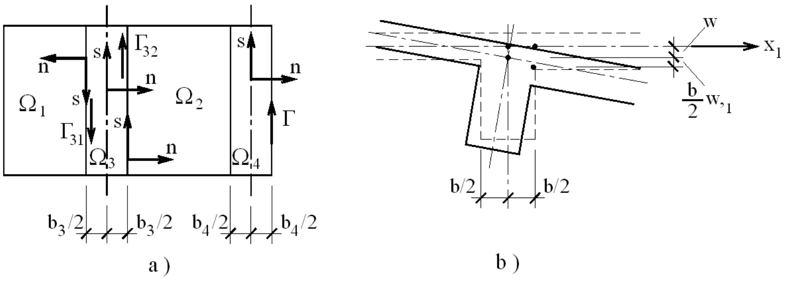

Figure 2: (a) Reinforced plate view; (b) Deflections approximations along interfaces

Let us now consider the beam B_3 represented in Figure 2a by the sub-region Ω_3. In order to reduce the number of degrees of freedom, the displacements w, ϕ_s and ϕ_n had been assumed to be linear along the beam width. Thus the interface displacement vector related to the beam interfaces are translated to the skeleton line, as follows:

$$\phi_k^{\Gamma_{32}} = \phi_k + \phi_{k,n} b_3 / 2 \qquad k=n,s \tag{10a}$$

$$\phi_k^{\Gamma_{31}} = -\left[\phi_k - \phi_{k,n} b_3 / 2\right] \tag{10b}$$

$$w^{\Gamma_{32}} = w + w_{,n} b_3 / 2 \tag{11a}$$

$$w^{\Gamma_{31}} = w - w_{,n} b_3 / 2 \tag{11b}$$

where b_3 is the beam width ,$\phi_k^{\Gamma_{ij}}$ and $w^{\Gamma_{ij}}$ are displacement components along the interface Γ_{ij} ; ϕ_k , w, $\phi_{k,n}$ and $w_{,n}$ are components along the skeleton line.

Observe that adopting these approximations (Eqs. (10) and (11)), new values are defined on the beam axis: the rotation $w_{,n}$ and the curvatures $\phi_{s,n}$ and $\phi_{n,n}$, being all of them considered constant along the beam width as well as the traction M_{ns} (see Eq. (12)). Different approximations for M_n and Q_n have been adopted, depending on the boundary conditions. In the case of having both internal beams ends free (part of the beam coincident to the external boundary) the interface tractions M_n and Q_n are written in terms of their values on the beam axis as follow:

$$M_{ns}^{\Gamma_{32}} = M_{ns}^{\Gamma_{31}} = M_{ns} \tag{12}$$

$$Q_n^{\Gamma_{32}} = -Q_n^{\Gamma_{31}} = Q_n \tag{13}$$

$$M_n^{\Gamma_{32}} = M_n + Q_n b_3 / 2 \tag{14a}$$

$$M_n^{\Gamma_{31}} = M_n - Q_n b_3 / 2 \tag{14b}$$

where M_n, M_{ns} and Q_n refers to the beam axis while the directions of $M_n^{\Gamma_{ij}}$, $M_{ns}^{\Gamma_{ij}}$ and $Q_n^{\Gamma_{ij}}$ are given by the local coordinate system defined on interfaces.

On the other hand, in the case of having one beam end fixed or simply supported, the following linear approximations have been considered:

$$Q_n^{\Gamma_{32}} = 0.5 Q_n \tag{15a}$$

$$Q_n^{\Gamma_{31}} = -1.5Q_n \tag{15b}$$

$$M_n^{\Gamma_{32}} = 1.5M_n \tag{16a}$$

$$M_n^{\Gamma_{31}} = 0.5M_n \tag{16b}$$

For external beams whose sides are free we have divided Q_n and M_n into two parts whose summation result into constant approximation across the beam width. Moreover, the linear part due to the shear forces has also been considered in the M_n expression. Thus for beam B_4 considered in Figure 2a as the sub-region Ω_4 we have assumed:

$$Q_n^{\Gamma_i} = \frac{1}{2}\Delta Q_n - 1.5Q_n, \tag{17a}$$

$$Q_n^{\Gamma} = \frac{1}{2}\Delta Q_n + \frac{1}{2}Q_n, \tag{17b}$$

$$M_n^{\Gamma_i} = -\frac{1}{2}\Delta M_n + 1.5M_n - Q_n b_4, \tag{18a}$$

$$M_n^{\Gamma} = \frac{1}{2}\Delta M_n + \frac{1}{2}M_n + Q_n b_4. \tag{18b}$$

where $M_n^{\Gamma_i}$ and $Q_n^{\Gamma_i}$ refer to the interface Γ_i while M_n^{Γ} and Q_n^{Γ} are related to the external boundary Γ; ΔQ_n and ΔM_n are written in terms of displacements by using Eqs. (3) and (4) .

Along simple supported or fixed sides the following approximations have been adopted for the moment M_n and shear force:

$$M_n^{\Gamma_i} = M_n - Q_n b_4 \tag{19a}$$

$$M_n^{\Gamma} = M_n + Q_n b_4 \tag{19b}$$

$$Q_n^{\Gamma_i} = -\frac{1}{2}Q_n \tag{20a}$$

$$Q_n^{\Gamma} = -1{,}5Q_n \tag{20b}$$

Note that the approximations along the beam width presented here (Eqs 10 to 20) are valid only for the case of beams with constant width b.

Then the remaining values on the external boundary without beams are six and on the beam axis are nine: ϕ_{n}, ϕ_{s}, $\phi_{\mathrm{s,n}}$; $\phi_{\mathrm{n,n}}$, w, $w_{,n}$, M_n, M_{ns} and Q_n requiring therefore nine algebraic representations for each internal beam axis node. Observe that the integral representations of w,m or $\phi_{\mathrm{k,m}}$ can be easily obtained by differentiating Eq. (9).

Note that despite of the values being defined along the beam axis, the integrals are still performed on the interfaces. Thus as the collocation points are adopted on the beam axis there is no problem of singularities.

4 ALGEBRAIC EQUATIONS

To obtain the problem solution, the integral representation (10) has to be transformed into algebraic expressions after discretizing the boundary and interfaces into elements. It has been adopted linear elements to approximate the problem geometry while the variables have been approximated by quadratic shape functions.

Along the external boundary without beams six values are defined: w, ϕ_n, ϕ_s, Q_n, M_n and M_{ns}, being three of them prescribed. Thus three algebraic equations have to be written for each boundary node. It has been adopted to write Eq. (9) related to the displacements w, ϕ_n and ϕ_s for an external collocation point very near to the boundary. On the other hand, for each external or internal beam node nine values are defined: ϕ_n, ϕ_s, $\phi_{s,n}$; $\phi_{n,n}$, w, $w_{,n}$, M_n, M_{ns} and Q_n. All these values remain as unknowns in the internal beams, requiring therefore nine algebraic equations. It has been chosen to write the corresponding unknowns equations for collocation points on the beam skeleton line. For external beams the displacements $\phi_{s,n}$; $\phi_{n,n}$ and $w_{,n}$ are problem unknowns while three of the six remaining values must be prescribed, leading to six unknowns for each external beam node. It has been adopted to write, for collocations points on the beam axis, the following algebraic equations: w, ϕ_n, ϕ_s, $\phi_{s,n}$; $\phi_{n,n}$ and $w_{,n}$. In both cases the collocations can be coincident with the chosen node or defined at element internal points when variable discontinuity is required at the element end.

After writing the recommended algebraic relations one obtains the set of equations defined bellow which can be solved after applying the boundary conditions.

$$\underset{\sim}{H}\,\underset{\sim}{U} = \underset{\sim}{G}\,\underset{\sim}{P} + \underset{\sim}{T} \tag{21}$$

In Eq. (16) {U} and {P} are displacements and tractions vectors; {T} is the vector due to the applied loads; [H] and [G] are matrices achieved by integrating all boundary and interfaces.

5 NUMERICAL APPLICATION

The building floor depicted in Figure 3a is now analysed. The Young's modulus, the Poisson's ratio, the plate and beams thicknesses are: E=25.0x10^{6}kN/m^2, ν=0.25, t_p=8.0cm and t_b=25cm. A distributed load of 20kN/m^2 is applied on the whole surface of the structure and all external beams axes have been assumed simply supported. The adopted mesh has 30 elements resulting into 77 nodes (see Figure 3b), including 24 nodes defined in the corners that are no represented in the figure because they are automatically generated by the code. The results are compared to a well-known finite element code (ANSYS, version 9), where shell elements (shell143) have been used to model both the beams and slabs.

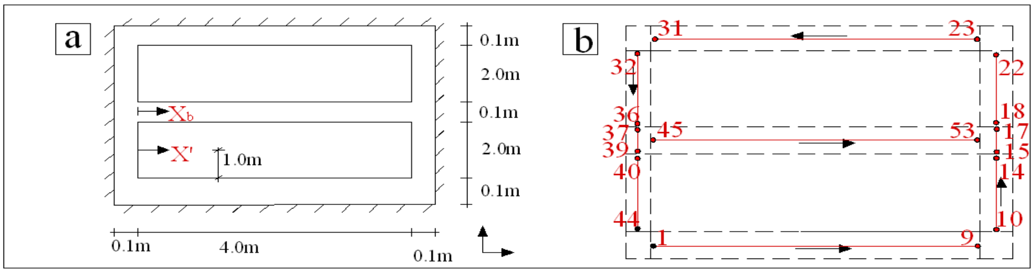

Figure 3– a) Plate geometry b) Plate discretization

The deflecitons and moments along the plate middle axis X' and the beam axis X_b as well are depicted in Figures (4) and (5) where can be observed that the values obtained with the proposed model are similar to the ones related to ANSYS and bigger than the ones obtained with the model proposed by Fernandes (2003), which takes into account the Kirchhoff's theory instead of Reissner's. This evidences the importance of considering the shear deformation in this numerical application.

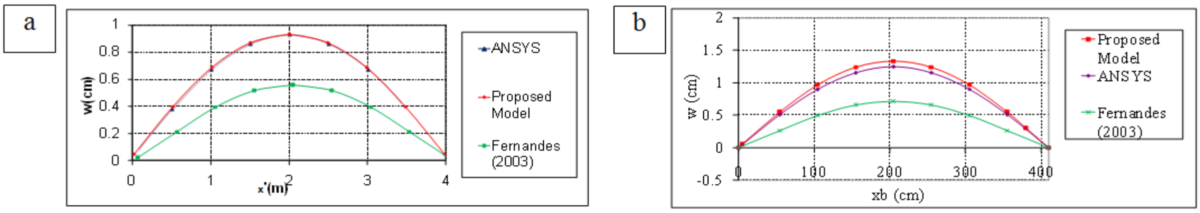

Figure 4 – a)Deflections along the plate axis X’ b)Deflections along the beam axis X_b

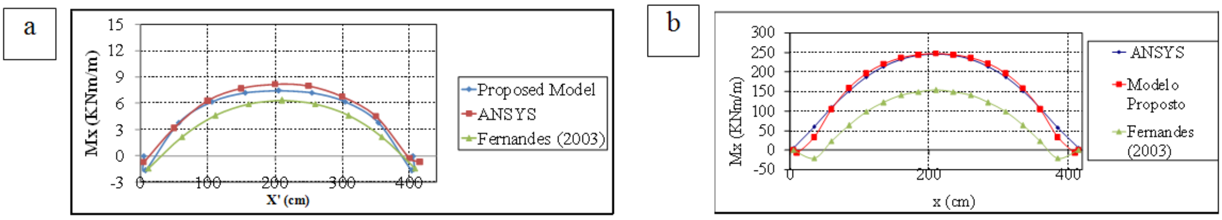

Figure 5 – a) Moments along the plate axis X’ b) Moments along the beam axis X_b

6 CONCLUSIONS

The BEM formulation based on Reissner’s hypothesis for analysing zoned plate-bending problem has been extended to deal with plate reinforced by beams. Beam rigidity is taken into account by assuming narrow sub-regions, without dividing the reinforced plate into beam and plate elements. Therefore this composed structure is treated as a single body, where equilibrium and compatibility conditions are automatically guaranteed by the global integral equations. In order to reduce the number of degrees of freedom some approximations are considered for both the displacements and tractions along the beam cross section, leading to a model where the problem values are defined on the beam axis and on the plate boundary without beams. The performance of the proposed formulation has been confirmed by comparing the results with a well-known finite element code.

7 ACKNOWLEDGEMENTS

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REFERENCES

Fernandes G. R. *Non Linear Analysis of Building Floor Structures by the Boundary Element Method*. PhD. Thesis, University of São Paulo (in Portuguese), 2003.

Fernandes, G.R and Venturini, W.S., Stiffened plate bending analysis by the boundary element method. *Computational Mechanics*, 28: 275-281, 2002.

Fernandes, G.R. and Venturini, W. S., Building floor analysis by the Boundary element method. *Computational Mechanics*, 35:277-291, 2005.

Fernandes, Gabriela R. ; Konda, Danilo H. . A BEM formulation based on Reissner s theory to perform simple bending analysis of plates reinforced by rectangular beams. *Computational Mechanics*, v. 42, p. 671-683, 2008

Paiva, J. B. and Aliabadi, M. H., Bending moments at interfaces of thin zoned plates with discrete thickness by the boundary element method. *Engineering Analysis with Boundary Elements*, 28: 747-751, 2004

Palermo Jr. L., Plate bending analysis using the classical or the Reissner-Mindlin models.

Engineering Analysis with Boundary Elements, 27: 603-609, 2003.

Reissner, E. (1947). On bending of elastic plates. *Quart. Appl. Math*.; 5(1); 55-68; 1947

Sapountzakis, E.J. & Katsikadelis, J.T., Analysis of plates reinforced with beams. *Computational Mechanics*, 26: 66-74, 2000.

Tanaka, M., Matsumoto, T. and Oida, S., A boundary element method applied to the elastostatic bending problem of beam-stiffened plate. *Engineering Analysis with Boundary Elements*, 24:751-758, 2000.

Weën, F. V. (1982). Application of boundary integral equation method to Reissner's plate model. *Int. J. Num. Meth. Eng*., v. 18 (1), p. 1-10.