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SPECTRAL ANALYSIS OF THE THREE-DIMENSIONAL LAPLACE TRANSFORM NODAL METHOD FOR TWO-GROUPS DISCRETE ORDINATES PROBLEMS IN CARTESIAN GEOMETRY

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Abstract. In this work, we describe a spectrum of the three-dimensional Laplace Transform Nodal method (LTS_N) in order to solve the transport problem in a parallelepiped domain with two energy groups.

We present the LTS_N nodal method to generate an analytical solution for discrete ordinates (S_N) problems in three-dimensional cartesian geometry and two energy groups. We first transverse integrate the S_N equations and then we apply the Laplace transform. The essence of this method is the diagonalization of the LTS_N transport matrices and the spectral analysis garantees this, because the eigenvalues can have multiplicity greater than one and corresponding linearly independent eigenvectors.

The transverse leakage terms that appear in the transverse integrated S_N equations are represented by exponential functions with decay constants that depend on the characteristics of the material of the medium of the particles leave behind. We use continuity conditions across the region interfaces, in order to obtain the approximated problem solution. The only approximation we use in the derivation of the present method is the exponential approximation for the transverse leakage terms.

1 INTRODUCTION

The linear Boltzmann equation is an integro-differential equation wich describes the angular, energy and spatial variations of the neutral particle transport. The complexity of the mathematical models associated with transport problems, mainly in multidimensional geometries, is always an important issue of investigations and developments, taking into account the wide range of applications for these problems. The discrete ordinate method (S_N) is a technique used for solving the linear Boltzmann equation (Lewis and Miller, 1991). The present research work present the Laplace transform nodal method (LTS_N), (Panta and Vilhena, 1999; Hauser et al., 2009), to generate an analytical solution for discrete ordinates problems in three-dimensional cartesian geometry and two energy groups. We first transverse integrate the SN equations and then we apply the Laplace transform.

The LTS_N nodal method is based on three transverse integrations across the three coordinate planes within a homogeneous region of the domain of solution. These transverse integrations lead to three one-dimensional equations coupled by the leakage terms, that we approximate by exponential functions and solve the resulting equations analytically by the Laplace transform technique in space The present LTS_N nodal method is based on the the spectral nodal methods for discrete ordinates problems (Barros and Larsen, 1990), wherein the only approximation involved is the approximation for the transverse leakage terms. we approximate the transverse leakage terms by exponential functions, that are chosen based on the physics of shielding problems, where the neutron flux attenuates exponentially with increasing distance from the source.

The essence of this method is the diagonalizability of the LTSN transport matrices and we developed the spectral analysis for to garant this, in a way that is very similar that was performed for the spectral Green's Function method by Barros and Larsen (1992).

An outline of the remainder of this paper follows. In Section 2 we describe the three-Dimensional two-Group LTS_N nodal method. Finally, we present spectral analysis and and we list some numerical results in Section 3.

2 THE TWO-GROUP $\mathbf{LTS_N}$ NODAL METHOD IN $\mathbf{X-Y-Z}$ GEOMETRY

We consider the two-group energy S_N equations with linearly isotropic scattering in a homogeneous $x-y-z$ geometry

$$\mu_m \frac{d}{dx}\Psi_{m,g}(x,y,z) + \eta_m \frac{d}{dy}\Psi_{m,g}(x,y,z) + \xi_m \frac{d}{dz}\Psi_{m,g}(x,y,z) + \sigma_{t,g}(x,y,z)\Psi_{m,g}(x,y,z) = \frac{1}{8}\left[\sigma_{s,1,g}\sum_{n=1}^{M} w_n \Psi_{n,1}(x,y,z) + \sigma_{s,2,g}\sum_{n=1}^{M} w_n \Psi_{n,2}(x,y,z)\right] + Q_{m,g}(x,y,z)\,, \tag{1}$$

where, for $g=1,2$, we have defined $\Psi_{m,g}(x,y,z) = \Psi_g(x,y,z,\mu_m,\eta_m,\xi_m)$ as the g^{th} group angular flux in the discrete direction (μ_m,η_m,ξ_m) , $m = 1 : M$, $M = N(N+2)$, ω_m the angular quadrature weights , $(x,y,z) \in [0,a] \times [0,b] \times [0,c]$, $\sigma_{t,g}, \sigma_{s,1,g}, \sigma_{s,2,g}$ are the cross section, and $Q_{m,g}(x,y)$ is the isotropic interior source.

By transverse-integrating Eq.(1) with respect to $y-z$ plane, we obtain

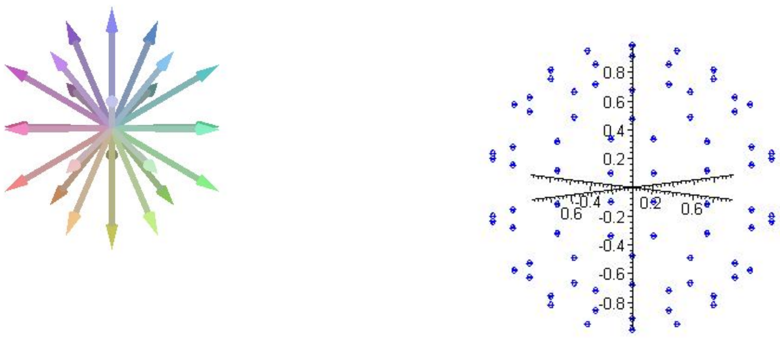

Figure 1: S_4 and S_8 Discrete Directions with Level Simetric

$$\mu_m \frac{d\Psi_{mx,g}}{dx}(x) + \sigma_{t,g}\Psi_{mx,g}(x) - \frac{1}{8}\left[\sigma_{s,1,g}\sum_{n=1}^{M} w_n \Psi_{nx,1}(x) + \sigma_{s,2,g}\sum_{n=1}^{M} w_n \Psi_{nx,2}(x)\right] = S_{mx,g}(x)\,, \tag{2}$$

where the mean angular flux in the in the discrete direction $\Omega_m = (\mu_m, \eta_m, \xi_m)$ is

$$\Psi_{mx,g}(x) = \frac{1}{bc}\int_0^c \int_0^b \Psi_{m,g}(x,y,z)dydz \tag{3}$$

The source term $S_{mx,g}(x)$ includes the external source and the transverse leakage terms.

$$S_{mx,g}(x) = \frac{1}{bc\mu_m}\left[Q_{x,g}(x) - \eta_m \int_0^c \left[\Psi_{m,g}(x,b,z) - \Psi_{m,g}(x,0,z)\right] dz\right] - -\frac{1}{bc\mu_m}\left[\xi_m \int_0^b \left[\Psi_{m,g}(x,y,c) - \Psi_{m,g}(x,y,0)\right] dy\right] \tag{4}$$

$$Q_{x,g}(x) = \int_0^c \int_0^b Q_g(x,y,z)dydz\,. \tag{5}$$

The transverse integrated S_N equations for the y and z spatial directions are obtained in a similar fashion.

Equation(2) forms a system of $2M$ linear ordinary differential equations in the $2M$ unknown functions $\Psi_{m,g}(x)$ in D. For $m = 1 : M$, we write Eq.(2) in the following explicit form

$$\begin{aligned} &\frac{d}{dx}\Psi_{mx,1}(x) + \frac{\sigma_{t,1}}{\mu_m}\Psi_{mx,1}(x) - \frac{1}{8\mu_m}\left[\sigma_{s,1,1}\sum_{n=1}^{M} w_n \Psi_{nx,1}(x) + \sigma_{s,2,1}\sum_{n=1}^{M} w_n \Psi_{nx,2}(x)\right] = \frac{S_{mx,1}(x)}{\mu_m} \\ &\frac{d}{dx}\Psi_{mx,2}(x) + \frac{\sigma_{t,2}}{\mu_m}\Psi_{mx,2}(x) - \frac{1}{8\mu_m}\left[\sigma_{s,1,2}\sum_{n=1}^{M} w_n \Psi_{nx,1}(x) + \sigma_{s,2,2}\sum_{n=1}^{M} w_n \Psi_{nx,2}(x)\right] = \frac{S_{mx,2}(x)}{\mu_m}\,. \end{aligned} \tag{6}$$

We apply the Laplace transform with respect x to Eq.(6). For $g = 1, 2$ we denote

$$\pounds\,\{\,S_{mx,g}(x)\} = \overline{S}_{mx,g}(s),\ \pounds\,\{\Psi_{mx,g}(x)\} = \overline{\Psi}_{mx,g}(s)$$

and

$$\pounds\left\{\frac{d\Psi_{mx,g}}{dx}(x)\right\} = s\overline{\Psi}_{mx,g}(s) - \Psi_{mx,g}(0)\,.$$

For $m = 1 : M$, we obtain two algebric systems of $2M$ equations

$$\begin{aligned}
&s\overline{\Psi}_{mx,1}(s) + \frac{\sigma_{t,1}}{\mu_m}\overline{\Psi}_{mx,1}(s) - \frac{\sigma_{s,1,1}}{8\mu_m}\sum_{n=1}^{M} w_n\overline{\Psi}_{nx,1}(s) - \frac{\sigma_{s,2,1}}{8\mu_m}\sum_{n=1}^{M} w_n\overline{\Psi}_{nx,2}(s)\\
&= \Psi_{mx,1}(0) + \frac{\overline{S}_{mx,1}(s)}{\mu_m}\\
&s\overline{\Psi}_{mx,2}(s) + \frac{\sigma_{t,2}}{\mu_m}\overline{\Psi}_{mx,2}(s) - \frac{\sigma_{s,2,2}}{4\mu_m}\sum_{n=1}^{M} w_n\overline{\Psi}_{nx,2}(s) - \frac{\sigma_{s,1,2}}{4\mu_m}\sum_{n=1}^{M} w_n\overline{\Psi}_{nx,1}(s)\\
&= \Psi_{m,2}(0) + \frac{\overline{S}_{mx,2}(s)}{\mu_m}\,.
\end{aligned} \tag{7}$$

We can be writte Eq.(7) in matrix form as

$$[\,s\,\mathbf{I} - \mathbf{A_x}\,]\begin{bmatrix}\overline{\Psi}_{mx,1}(s)\\ \\ \overline{\Psi}_{mx,2}(s)\end{bmatrix} = \begin{bmatrix}\Psi_{mx,1}(0)\\ \\ \Psi_{mx,2}(0)\end{bmatrix} + \frac{1}{\mu_m}\begin{bmatrix}\overline{S}_{mx,1}(s)\\ \\ \overline{S}_{mx,2}(s)\end{bmatrix} \tag{8}$$

where $\mathbf{I}$ is identity matrix and we have defined the $2M \times 2M$ matrix $\mathbf{A_x}$

$$\mathbf{A_x} = \begin{bmatrix}\mathbf{A_{x,11}} & \mathbf{A_{x,12}}\\ \\ \mathbf{A_{x,21}} & \mathbf{A_{x,22}}\end{bmatrix} \tag{9}$$

which is composed of the $M \times M$ submatrices $\mathbf{A_{x,g',g}}$, $g', g = 1, 2$.

In matrix (9) we have definedd submatrices

$$\mathbf{A_{x,11}} = \begin{bmatrix}
-\dfrac{8\sigma_{t,1} - \sigma_{s,1,1}\omega_1}{8\mu_1} & \dfrac{\sigma_{s,1,1}\omega_2}{8\mu_1} & \cdots & \dfrac{\sigma_{s,1,1}\omega_M}{8\mu_1}\\
\dfrac{\sigma_{s,1,1}\omega_1}{8\mu_2} & -\dfrac{8\sigma_{t,1} - \sigma_{s,1,1}\omega_2}{8\mu_2} & \cdots & \dfrac{\sigma_{s,1,1}\omega_M}{8\mu_2}\\
\vdots & \vdots & \ddots & \vdots\\
\dfrac{\sigma_{s,1,1}\omega_1}{8\mu_M} & \dfrac{\sigma_{s,1,1}\omega_2}{8\mu_M} & \cdots & -\dfrac{8\sigma_{t,1} - \sigma_{s,1,1}\omega_M}{8\mu_M}
\end{bmatrix}, \tag{10}$$

$$\mathbf{A_{x,22}} = \begin{bmatrix} -\frac{8\sigma_{t,2}-\sigma_{s,2,2}\omega_1}{8\mu_1} & \frac{\sigma_{s,2,2}\omega_2}{8\mu_1} & \cdots & \frac{\sigma_{s,2,2}\omega_M}{8\mu_1} \\ \frac{\sigma_{s,2,2}\omega_1}{8\mu_2} & -\frac{8\sigma_{t,2}-\sigma_{s,2,2}\omega_2}{8\mu_2} & \cdots & \frac{\sigma_{s,2,2}\omega_M}{8\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{s,2,2}\omega_1}{8\mu_M} & \frac{\sigma_{s,2,2}\omega_2}{8\mu_M} & \cdots & -\frac{8\sigma_{t,2}-\sigma_{s,2,2}\omega_M}{8\mu_M} \end{bmatrix} \quad (11)$$

and for $g', g = 1, 2$, which is composed of the $M \times M$ submatrices $\mathbf{A_{g',g}}$,

$$\mathbf{A_{x,21}} = \begin{bmatrix} \frac{\sigma_{s,1,2}\omega_1}{8\mu_1} & \frac{\sigma_{s,1,2}\omega_2}{8\mu_1} & \cdots & \frac{\sigma_{s,1,2}\omega_M}{8\mu_1} \\ \frac{\sigma_{s,1,2}\omega_1}{8\mu_2} & \frac{\sigma_{s,1,2}\omega_2}{8\mu_2} & \cdots & \frac{\sigma_{s,1,2}\omega_M}{8\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{s,1,2}\omega_1}{8\mu_M} & \frac{\sigma_{s,1,2}\omega_2}{8\mu_M} & \cdots & \frac{\sigma_{s,1,2}\omega_M}{8\mu_M} \end{bmatrix} \quad (12)$$

$$\mathbf{A_{x,12}} = \begin{bmatrix} \frac{\sigma_{s,2,1}\omega_1}{8\mu_1} & \frac{\sigma_{s,2,1}\omega_2}{8\mu_1} & \cdots & \frac{\sigma_{s,2,1}\omega_M}{8\mu_1} \\ \frac{\sigma_{s,2,1}\omega_1}{8\mu_2} & \frac{\sigma_{s,2,1}\omega_2}{8\mu_2} & \cdots & \frac{\sigma_{s,2,1}\omega_M}{8\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{s,2,1}\omega_1}{8\mu_M} & \frac{\sigma_{s,2,1}\omega_2}{48\mu_M} & \cdots & \frac{\sigma_{s,2,1}\omega_M}{8\mu_M} \end{bmatrix} \quad (13)$$

In addition, we have defined the M-dimensional vector functions

$$\overline{\Psi}_{mx,g}(s) = \left[\ \overline{\Psi}_{1x,g}(s) \ \ \overline{\Psi}_{2x,g}(s) \cdots \overline{\Psi}_{Mx,g}(s) \ \right]^T , \quad (14)$$

$$\Psi_{mx,g}(0) = [\ \Psi_{1x,g}(0) \ \ \Psi_{2x,g}(0) \cdots \Psi_{Mx,g}(0) \ ]^T , \quad (15)$$

and

$$\overline{S}_{mx,g}(s) = \left[\ \overline{S}_{1x,g}(s) \ \ \overline{S}_{2x,g}(s) \cdots \overline{S}_{Mx,g}(s) \ \right]^T . \quad (16)$$

The solution of the algebric sistem (8) is

$$\begin{bmatrix} \overline{\Psi}_{mx,1}(s) \\ \\ \overline{\Psi}_{mx,2}(s) \end{bmatrix} = [\, s\,\mathbf{I} - \mathbf{A_x}\,]^{-1} \left(\begin{bmatrix} \Psi_{mx,1}(0) \\ \\ \Psi_{mx,2}(0) \end{bmatrix} + \frac{1}{\mu_m} \begin{bmatrix} \overline{S}_{mx,1}(s) \\ \\ \overline{S}_{mx,2}(s) \end{bmatrix} \right) \tag{17}$$

In order to determine the angular flux we apply the inverse transform Laplace in (17).

$$\begin{bmatrix} \Psi_{mx,1}(x) \\ \\ \Psi_{mx,2}(x) \end{bmatrix} = \pounds^{-1} \left\{ [\, s\,\mathbf{I} - \mathbf{A_x}\,]^{-1} \left(\begin{bmatrix} \Psi_{mx,1}(0) \\ \\ \Psi_{mx,2}(0) \end{bmatrix} + \frac{1}{\mu_m} \begin{bmatrix} \overline{S}_{mx,1}(s) \\ \\ \overline{S}_{mx2}(s) \end{bmatrix} \right) \right\} \tag{18}$$

Then,

$$\begin{aligned} \begin{bmatrix} \Psi_{mx,1}(x) \\ \\ \Psi_{mx,2}(x) \end{bmatrix} &= \pounds^{-1} \left\{ [\, s\,\mathbf{I} - \mathbf{A_x}\,]^{-1} \right\} \begin{bmatrix} \Psi_{mx,1}(0) \\ \\ \Psi_{mx,2}(0) \end{bmatrix} \\ &+ \frac{1}{\mu_m} \, \pounds^{-1} \left\{ [\, s\,\mathbf{I} - \mathbf{A_x}\,]^{-1} \right\} * \begin{bmatrix} \overline{S}_{mx,1}(x) \\ \\ \overline{S}_{mx,2}(x) \end{bmatrix} \end{aligned} \tag{19}$$

where $*$ denote the convolution operation.

Furthermore, in order to determine $\pounds^{-1}\left\{ [\, s\,\mathbf{I} - \mathbf{A_x}\,]^{-1} \right\}$ we assume the diagonalizability of matrix $\mathbf{A_x}$, $\mathbf{A_x} = \mathbf{V_x D_x V_x}^{-1}$, to write

$$\begin{aligned} \pounds^{-1}\left\{ [\, s\,\mathbf{I} - \mathbf{A_x}\,]^{-1} \right\} &= \pounds^{-1}\left\{ \left[s\ \mathbf{V_x V_x}^{-1} - \mathbf{V_x D_x V_x}^{-1} \right]^{-1} \right\} \\ = \pounds^{-1} \left\{ \left[\mathbf{V_x} \,(\, s\,\mathbf{I} - \mathbf{D_x}\,)\, \mathbf{V_x}^{-1} \right]^{-1} \right\} &= \mathbf{V_x}\, \pounds^{-1} \{ [\, s\,\mathbf{I} - \mathbf{D_x}\,]^{-1} \} \,\mathbf{V_x}^{-1} \,. \end{aligned} \tag{20}$$

where $\mathbf{D_x}$ is an $M-$ order diagonal matrix of the eigenvalues of $\mathbf{A_x}$ and $\mathbf{V_x}$ is the matrix whose columns are M eigenvectors of $\mathbf{A_x}$.

We apply the inverse Laplace transform

$$\pounds^{-1}\left\{ (\, s\,\mathbf{I} - \mathbf{D_x})^{\mathbf{-1}} \right\} = e^{\mathbf{D_x} x} \,. \tag{21}$$

Substituing Eq.(21)in Eq.(20), we obtain

$$\pounds^{-1}\left\{ (sI - A_x)^{-1} \right\} = \mathbf{V}_x e^{\mathbf{D}_x x} \mathbf{V}_x^{-1} \,. \tag{22}$$

As a result, the analytical solution for the two-group S_N equations with linearly isotropic scattering (1).

$$\begin{bmatrix} \Psi_{mx,1}(x) \\ \\ \Psi_{mx,2}(x) \end{bmatrix} = \left[\mathbf{V_x}\, e^{\mathbf{D_x}\, x}\, \mathbf{V_x}^{-1}\right] \begin{bmatrix} \Psi_{mx,1}(0) \\ \\ \Psi_{mx,2}(0) \end{bmatrix}$$
$$+\frac{1}{\mu_m}\ \left[\mathbf{V}\, e^{\mathbf{D_x}\, x}\, \mathbf{V_x}^{-1}\right]\ *\ \begin{bmatrix} \overline{S}_{mx,1}(x) \\ \\ \overline{S}_{mx,2}(x) \end{bmatrix} \tag{23}$$

We proceed in a similar form with the S_N nodal equations transversally in the $x-y$ and $x-z$ planes and we obtain the following analytical solutions

$$\begin{bmatrix} \Psi_{my,1}(y) \\ \\ \Psi_{my,2}(y) \end{bmatrix} = \left[\mathbf{V_y}\, e^{\mathbf{D_y}\, y}\, \mathbf{V_y}^{-1}\right] \begin{bmatrix} \Psi_{my,1}(0) \\ \\ \Psi_{my,2}(0) \end{bmatrix}$$
$$+\frac{1}{\eta_m}\ \left[\mathbf{V}\, e^{\mathbf{D_y}\, y}\, \mathbf{V_x}^{-1}\right]\ *\ \begin{bmatrix} \overline{S}_{my,1}(y) \\ \\ \overline{S}_{my,2}(y) \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} \Psi_{mz,1}(z) \\ \\ \Psi_{mz,2}(z) \end{bmatrix} = \left[\mathbf{V_z}\, e^{\mathbf{D_z}\, z}\, \mathbf{V_z}^{-1}\right] \begin{bmatrix} \Psi_{mz,1}(0) \\ \\ \Psi_{mz,2}(0) \end{bmatrix}$$
$$+\frac{1}{\xi_m}\ \left[\mathbf{V}\, e^{\mathbf{D_z}\, z}\, \mathbf{V_z}^{-1}\right]\ *\ \begin{bmatrix} \overline{S}_{mz,1}(z) \\ \\ \overline{S}_{mz,2}(z) \end{bmatrix} \tag{25}$$

Now, we denote the mean angular flux as

$$\Psi_{x,g}(x) = \sum_{i=1}^{M} A_{l,g} V_{x_l} e^{r_l x} = \mathbf{V_x} e^{\mathbf{D_x} x} \mathbf{A_g}, \tag{26}$$

where $\mathbf{A_g} = [A_{1,g}, A_{2,g}, \cdots, A_{M,g}]^T$,

$$\Psi_{y,g}(y) = \sum_{l=1}^{M} B_{l,g} V_{y_l} e^{s_l y} = \mathbf{V_y} e^{\mathbf{D_y} y} \mathbf{B_g}, \tag{27}$$

where $\mathbf{B_g} = [B_{1,g}, B_{2,g}, \cdots, B_{g,M}]^T$, e

$$\Psi_{z,g}(z) = \sum_{l=1}^{M} C_{l,g} V_{y_l} e^{t_l y} = \mathbf{V_z} e^{\mathbf{D_z} z} \mathbf{C_g}, \tag{28}$$

where $\mathbf{C_g} = [C_{1,g}, C_{2,g}, \cdots, C_{M,g}]^T$.

Now, based on the physics of shielding problems, we assume that:
(1) the neutron flux attenuates exponentially with increasing distance from the source along the edges of each region inside the domain;
(2) the attenuation constant, λ, depends upon the nuclear data of the region the neutrons leave

behind as they stream across the system.
We can choose the attenuation constant as the macroscopic absorption cross section of the region the neutrons leave behind. With this heuristic approximation, we claim that for diffusive regions, where the macroscopic absorption cross sections are relatively small, the attenuation of the transverse leakage terms along the y and z directions is smoother than for highly absorbing regions, where the absorption event dominates.

Then, we define the transverse leakage terms as

$$\int_0^c \Psi_{m,g}(x,0,z)dz = \mathbf{D}_{m,g}\, e^{-sign(\mu_m)\lambda x}\,, \quad (29)$$

$$\int_0^c \Psi_{m,g}(x,b,z)dz = \mathbf{E}_{m,g}\, e^{-sign(\mu_m)\lambda x}\,, \quad (30)$$

$$\int_0^b \Psi_{m,g}(x,y,0)dy = \mathbf{F}_{m,g}\, e^{-sign(\mu_m)\lambda x}, \quad (31)$$

$$\int_0^b \Psi_{m,g}(x,y,c)dy = \mathbf{G}_{m,g}\, e^{-sign(\mu_m)\lambda x}, \quad (32)$$

$$\int_0^c \Psi_{m,g}(0,y,z)dz = \mathbf{H}_{m,g}\, e^{-sign(\eta_m)\lambda y}, \quad (33)$$

$$\int_0^c \Psi_{m,g}(a,y,z)dz = \mathbf{I}_{m,g}\, e^{-sign(\eta_m)\lambda y}, \quad (34)$$

$$\int_0^a \Psi_{m,g}(x,y,0)dx = \mathbf{J}_{m,g}\, e^{-sign(\eta_m)\lambda y}, \quad (35)$$

$$\int_0^a \Psi_{m,g}(x,y,c)dx = \mathbf{K}_{m,g}\, e^{-sign(\eta_m)\lambda y}, \quad (36)$$

$$\int_0^b \Psi_{m,g}(0,y,z)dy = \mathbf{L}_{m,g}\, e^{-sign(\xi_m)\lambda z}, \quad (37)$$

$$\int_0^b \Psi_{m,g}(a,y,z)dy = \mathbf{O}_{m,g}\, e^{-sign(\xi_m)\lambda z}, \quad (38)$$

$$\int_0^a \Psi_{m,g}(x,0,z)dx = \mathbf{P}_{m,g}\, e^{-sign(\xi_m)\lambda z}, \quad (39)$$

$$\int_0^a \Psi_{m,g}(x,b,z)dx = \mathbf{R}_{m,g}\, e^{-sign(\xi_m)\lambda z}. \quad (40)$$

Finally analytical solution is completely determined if to find the $30M$ unknowns present in the expressions Eq. (26)to Eq.(40). Thus, it is a system solves linear compatible of $30M$ equations, derived from the definitions of the mean angular flux in the $x = a$, $y = b$ e $z = c$, and the application of the boundary conditions.

3 SPECTRAL ANALYSIS OF THE TWO-GROUP S_N EQUATIONS WITH ISOTROPIC SCATTERING

The main purpose this section is to proof the diagonalizability of matrix $\mathbf{A_x}$ in order to determine $\mathcal{L}^{-1}\left\{[\,s\,\mathbf{I}-\mathbf{A_x}\,]^{-1}\right\} = \mathbf{V_x}\,e^{\mathbf{D_x}\,x}\,\mathbf{V_x}^{-1}$. For this reason , we perform a spectral analysis of the two-group slab-geometry S_N equations with isotropic scattering in a way that is very similar that was performed for the spectral Green's Function method by Barros and Larsen, (3).

We need to obtain a linearly independent set of any $2M$ vectors, the eigenvectors of the matrix $\mathbf{A_x}$.

To do this, we consider the homogeneous equations associated to the two-group slab-geometry S_N equations with isotropic Scattering Eq.(2)

$$\mu_m \frac{d\Psi_{mx,g}}{dx}(x) + \sigma_t \Psi_{mx,g}(x) - \frac{1}{8}\left[\sigma_{s,1,g}\sum_{n=1}^{M} w_n \Psi_{nx,1}(x) + \sigma_{s,2,g}\sum_{n=1}^{M} w_n \Psi_{nx,2}(x)\right] = 0\,, \quad (41)$$

We supose that , for $m = 1 : M$ and $g = 1, 2$, the solution of Eq.(41) is

$$\Psi_{mx,g}(x) = \alpha_{m,g}(\nu) e^{x\,\nu}\,. \quad (42)$$

Substituing Eq.(42) into Eq.(41) leads to

$$\nu\,\mu_m \alpha_{m,g}(\nu) e^{x\,\nu} + \sigma_{t,g}\alpha_{m,g}(\nu) e^{x\,\nu} = \frac{1}{8}\left[\sigma_{s,1,g}\sum_{n=1}^{M} w_n \alpha_{n,1}(\nu) e^{x\,\nu} + \sigma_{s,2,g}\sum_{n=1}^{M} w_n \alpha_{n,2}(\nu) e^{x\,\nu}\right]. \quad (43)$$

and we obtain the eigenvalue problem

$$(\nu\,\mu_m + \sigma_{t,g})\,\alpha_{m,g}(\nu) = \frac{1}{8}\left[\sigma_{s,1,g}\sum_{n=1}^{M} w_n \alpha_{n,1}(\nu) + \sigma_{s,2,g}\sum_{n=1}^{M} w_n \alpha_{n,2}(\nu)\right]. \quad (44)$$

In Eq.(44) we need to determine the eigenvalues ν and the m^{th} components the eigenvectors. We have

$$\alpha_{m,1}(\nu) = \frac{1}{8\,(\nu\,\mu_m + \sigma_{t,1})}\left[\sigma_{s,1,1}\sum_{n=1}^{M} w_n \alpha_{n,1}(\nu) + \sigma_{s,2,1}\sum_{n=1}^{M} w_n \alpha_{n,2}(\nu)\right] \quad (45)$$

and

$$\alpha_{m,2}(\nu) = \frac{1}{8\,(\nu\,\mu_m + \sigma_{t,2})}\left[\sigma_{s,1,2}\sum_{n=1}^{M} w_n \alpha_{n,1}(\nu) + \sigma_{s,2,2}\sum_{n=1}^{M} w_n \alpha_{n,2}(\nu)\right]. \quad (46)$$

Now, for $g = 1, 2$ we denote the normalization as

$$F_g(\nu) = \sum_{n=1}^{M} w_n \alpha_{n,g}(\nu)\,, \quad (47)$$

where, $F_g(\nu) = 0$ or $F_g(\nu) \neq 0$.

First, for $g = 1, 2$, we consider that

$$F_g(\nu) = \sum_{n=1}^{M} w_n \alpha_{n,g}(\nu) \neq 0\,. \quad (48)$$

and substituting into Eqs.(45) and (46), we obtain

$$\alpha_{m,1}(\nu) = \frac{1}{8\,(\nu\,\mu_m + \sigma_{t,1})}\left[\sigma_{s,1,1}\,F_1(\nu) + \sigma_{s,2,1}\,F_2(\nu)\,)\right], \tag{49}$$

and

$$\alpha_{m,2}(\nu) = \frac{1}{8\,(\nu\,\mu_m + \sigma_{t,2})}\left[\sigma_{s,1,2}\,F_1(\nu) + \sigma_{s,2,2}\,F_2(\nu)\right]. \tag{50}$$

Now, we multiply both Eqs.(49) and (50) by w_n and summing the resulting equations over all $m = 1 : M$, we have

$$\sum_{m=1}^{M} w_m\,\alpha_{m,1}(\nu) = \sum_{n=1}^{M} \frac{w_m}{8\,(\nu\,\mu_m + \sigma_{t,1})}\left[\sigma_{s,1,1}\,F_1(\nu) + \sigma_{s,2,1}\,F_2(\nu)\,)\right], \tag{51}$$

and

$$\sum_{m=1}^{M} w_m\,\alpha_{m,2}(\nu) = \sum_{n=1}^{M} \frac{w_m}{8\,(\nu\,\mu_m + \sigma_{t,2})}\left[\sigma_{s,1,2}\,F_1(\nu) + \sigma_{s,2,2}\,F_2(\nu)\right]. \tag{52}$$

Then appear the following homogeneous system of two equations in the two unknowns $F_1(\nu)$ and $F_2(\nu)$:

$$4\,F_1(\nu) = G_1(\nu)\left[\sigma_{s,1,1}\,F_1(\nu) + \sigma_{s,2,1}\,F_2(\nu)\,)\right], \tag{53}$$

$$4\,F_2(\nu) = G_2(\nu)\left[\sigma_{s,1,2}\,F_1(\nu) + \sigma_{s,2,2}\,F_2(\nu)\right], \tag{54}$$

represented in matricial form

$$\begin{bmatrix} G_1(\nu)\,\sigma_{s,1,1} - 8 & G_1(\nu)\,\sigma_{s,2,1} \\ G_2(\nu)\,\sigma_{s,1,2} & G_2(\nu)\,\sigma_{s,2,2} - 8 \end{bmatrix}\begin{bmatrix} F_1(\nu) \\ F_2(\nu) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{55}$$

In Eqs.(53), (54) and (55) we have defined the functions

$$G_g(\nu) = \sum_{n=1}^{M} \frac{w_m}{\nu\,\mu_m + \sigma_{t,g}}\,,\ \nu \neq \frac{-\sigma_{t,g}}{\mu_m}\,,\ g = 1,2. \tag{56}$$

There is non-trivial solution for system linear (55) if the determinant formed by the coefficients of $F_1(\nu)$ and $F_2(\nu)$ is different of zero . Then

$$(\,G_1(\nu)\,\sigma_{s,1,1} - 8\,)\,(\,G_2(\nu)\,\sigma_{s,2,2} - 8\,) - (\,G_1(\nu)\,\sigma_{s,2,1}\,)\,(\,G_2(\nu)\,\sigma_{s,1,2}\,) = 0\,, \tag{57}$$

is the *spectral characteristic equation* , a polynomial of degree $2N$ and the roots $\pm\,\nu_k\,, k = 1 : 2N$ are the eigenvalues for two-group equations Eq.(2). Due to the symmetry of the Gaussian quadrature set in Eq.(57) it has even powers of ν . As a result, all roots $\pm\,\nu_k\,, k = 1 : M$ appear in pairs and, they are all simple, but some of that are very near.

In. Fig.2 we represent the graffically the characteristic equation, Eq.(57), for S_4 quadrature, with media parameters: $\sigma_{t,1} = 1$,$\sigma_{t,2} = 1$, $\sigma_{s,1,1} = 0.99$, $\sigma_{s,2,2} = 0.98$, $\sigma_{s,1,2} = 0.008$ and $\sigma_{s,2,1} = 0.005$.

In order to determine the $2N$ the eigenvectors associated to eigenvalues obtained of the Eq.(57), we observe that the set the eigenvectors of a linear operator is not unique in the sense that their normalization is arbitrary. We need a set of $2M$ eigenvectors. Therefore we can chose

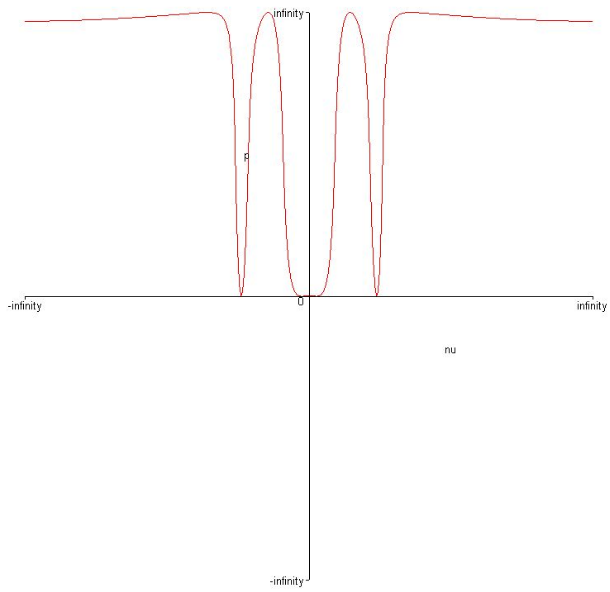

Figure 2: A typicaly distribuition of eingenvalues for S_4 quadrature

$$F_1(\nu) \;=\; \sum_{n=1}^{M} w_n \alpha_{n,1}(\nu) \;=\; 1\,, \tag{58}$$

and substituting Eq.(58)into the Eq.(53), we solve for $F_2(\nu)$, result

$$F_2(\nu) \;=\; \frac{4 \;-\; \sigma_{s,1,1}\,G_1(\nu)}{G_1(\nu)\,\sigma_{s,2,1}}\,. \tag{59}$$

Substituting this assumption and Eq.(58)into Eq.(49) and Eq.(50), for $k = 1 : 2N$ and $m = 1 : M$, we obtain the eigenvectors whose components $\alpha_{m,1}(\nu_k)$ and $\alpha_{m,2}(\nu_k)$ are

$$\alpha_{m,1}(\nu_k) \;=\; \frac{1}{8\;(\nu_k\,\mu_m \;+\sigma_{t,1}\,)}\left[\sigma_{s,1,1} \;+\; \sigma_{s,2,1}\;\;\frac{8 \;-\; \sigma_{s,1,1}\,G_1(\nu)}{G_1(\nu)\,\sigma_{s,2,1}}\right], \tag{60}$$

and

$$\alpha_{m,2}(\nu_k) \;=\; \frac{1}{8\;(\nu_k\,\mu_m \;+\; \sigma_{t,2}\,)}\left[\sigma_{s,1,2} \;+\; \sigma_{s,2,2}\;\;\frac{8 \;-\; \sigma_{s,1,1}\,G_1(\nu)}{G_1(\nu)\,\sigma_{s,2,1}}\right]. \tag{61}$$

Now, to obtain the others $2M - 2N$ eigenvalues we consider that, for $g = 1, 2$,

$$F_g(\nu) \;=\; \sum_{n=1}^{M} w_n \alpha_{n,g}(\nu) \;=\; 0\,, \tag{62}$$

and substituting in Eqs.(49) and (50), we obtain

$$\alpha_{m,1}(\nu)\;(\nu\,\mu_m \;+\sigma_{t,1}\,) \;=\; 0 \tag{63}$$

and

$$\alpha_{m,2}(\nu)\;(\nu\,\mu_m \;+\sigma_{t,2}\,) \;=\; 0\,. \tag{64}$$

Then, for $m = 1 : M$ and $g = 1, 2$, if we do

$$\nu = -\frac{\sigma_{t,g}}{\mu_m}\,, \tag{65}$$

then we can choose $\alpha_{m,g}(\nu) \neq 0$, to be valuated Eqs. (62), (63) e (64). The eigenvalues (65) can have multiplicity ≥ 1 and the components $\alpha_m(s)$, para $m = 1 : M$, are the corresponding linearly independent eigenvectors.

In Table 1 we summarize the multiplicities of positive eigenvalues. Due to the symmetry of the angular quadrature of the symmetry level that we use these multiplicities repeated for eigenvalue s negative.

Table 1: Number of the Positives Eigenvalues

N	$\frac{\sigma_{t,g}}{\mu_1}$	$\frac{\sigma_{t,g}}{\mu_2}$	$\frac{\sigma_{t,g}}{\mu_3}$	$\frac{\sigma_{t,g}}{\mu_4}$	$\frac{\sigma_{t,g}}{\mu_5}$	$\frac{\sigma_{t,g}}{\mu_6}$	$\frac{\sigma_{t,g}}{\mu_7}$	$\frac{\sigma_{t,g}}{\mu_8}$	Characteristic Equation Eq.56 (N)	Number of the Positives Eingenvalues $(M = N(N+2))$
2	6	-	-	-	-	-	-	-	2	8
4	6	14	-	-	-	-	-	-	4	24
6	6	14	22	-	-	-	-	-	6	48
8	6	14	22	30	-	-	-	-	8	80
10	6	14	22	30	38	-	-	-	10	120
12	6	14	22	30	38	46	-	-	12	168
14	6	14	22	30	38	46	52	-	14	224
16	6	14	22	30	38	46	52	60	16	288

4 CONCLUDING REMARKS

An analytical approach was used to derive a close form solution for the one dimensional integrated equations derived from a three dimensional neutron transport problem, that we refer to as the $LTS_N3D - Exp$ method for two-groups of the energy. In this method, the only approximation involved is in the transverse leakage terms of the transverse-integrated S_N equations. The scattering source terms are treated analytically. Based on the physics of radiation shielding problems, where the neutron flux attenuates exponentially with increasing distance from the source, we approximate the transverse leakage terms by exponential functions with prescribed attenuation constants.

Thus far, we have restricted ourselves to S_N with two energy problems. This is the reason why the resulting spectrum contain $2M$ eigenvalues and $2M$ eigenvectors. In general, though, the S_N problem is allowed to have an arbitrary number G of energy groups. In this case we shall perform a spectral analysis following the same steps explained in Section 3, except that instead of two equations for each direction m, we shall have G equations for each direction m. Therefore, the resulting spectrum will be contain $G \times 2M$ eigenvalues and $G \times 2M$ eigenvectors.A set of $G \times 2M$ eigenvectors linearly independent, corresponding to $G \times 2M$ eigenvalues, forms a basis for the $G \times 2M-$dimensional space solution.

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