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GEODETIC DATUM AND NO NET TRANSLATION (NNT) – NO NET ROTATION (NNR) CONDITIONS FROM TRANSFORMATION PARAMETERS, A REFERENCE FRAME AND A SELECTION MATRIX

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Abstract.

The definition of the geodetic datum is a fundamental issue in the solution of the inverse problems related with the adjustment of free geodetic networks. Taking into account the conventions given by the International Earth Rotation and Reference System Service (IERS) on the definition and realization of a Terrestrial Reference System (TRS), it is considered here that a geodetic datum is the set of all conventions, algorithms and constants necessaries to define and realize the origin, orientation, scale and their time evolution of a TRS in such a way that these attributes be accessible to the users through occupation, direct or indirect observation.

In this work, we deal with the adjustment of a two-dimensional trilateration network using coordinate based formulations within a Gauss-Markov Model (GMM), where the point positions are defined by means of coordinates (x,y) in a local Terrestrial Reference Cartesian Coordinate System TRS(x,y), which has not defined its position and orientation in a given epoch, causing a datum defect and a rank-deficient Singular Gauss-Markov Model (SGMM).

It is developed here, within this stochastic linear model for the adjustment of a two-dimensional free trilateration network of the type SGMM with datum defect, three linear condition equations namely minimum constraints: two "No Net Translation" (NNT) and one "No Net Rotation" (NNR) which define the datum in a given epoch respect to the position and orientation of the TRS(x,y) respectively, based in: a) three zero values conventionally adopted of three parameters of a plane coordinate Helmert transformation : two translation and one differential rotation, b) a Terrestrial Reference Frame TRF(xo,yo) known "a priori" considered "free of error" and c) a selection matrix $\bar{S}$, which allows to choose or exclude simultaneously the coordinate increments (dx,dy) of specifics network points.

1 INTRODUCTION

The definition of the geodetic datum is a fundamental issue in the solution of the inverse problems related with the adjustment of free geodetic networks. In this sense, taking into account the conventions given by the International Earth Rotation and Reference System Service (IERS) on the definition and realization of a Terrestrial Reference System (TRS), it is considered here that a geodetic datum is the set of all conventions, algorithms and constants necessaries to define and realize the origin, orientation, scale and their time evolution of a TRS in such a way that these attributes be accessible to the users through occupation, direct or indirect observation.

In this work, we deal with the adjustment of a two-dimensional trilateration network using coordinate based formulations within a Gauss-Markov Model (GMM), where the point positions are defined by means of coordinates (x,y) in a local Terrestrial Reference Cartesian Coordinate System TRS(x,y).

The TRS(x,y) is a trirectangular trihedron right-handed oriented, its vertex is a point **P**
not specified of the Earth's surface and is the origin **o** of the Cartesian coordinate system (x,y), the first and second rays are the **ox** and **oy** positive axis respectively with not specified orientations. The third ray is oriented "upward" aligned with the vertical in **P** and is orthogonal to the others two rays.

The lack of definition in the origin and orientation of the TRS(x,y) cause a datum defect and a rank-deficient Singular Gauss-Markov Model (SGMM).

Since the observed distances provide the scale, it is possible to find a Restricted LEast Square Solution (RLESS) for the adjustment of the network if the datum defect is eliminated by introducing in the SGMM three (minimum number equal to the number of datum defects) linear equations of condition to the unknown parameters (increments or coordinate differences) in the form of minimum constraints: two "No Net Translation" (NNT) and one "No Net Rotation" (NNR).

In the next section, it is developed within a stochastic linear model for the adjustment of a two-dimensional free trilateration network of the type SGMM with datum defect, three linear equations of condition namely minimum constraints: two "No Net Translation" (NNT) and one "No Net Rotation" (NNR) which define the datum in a given epoch respect to the position and orientation of the TRS(x,y) respectively, based in: a) three zero values conventionally adopted of three parameters of a plane coordinate Helmert transformation : two translation and one differential rotation, b) a Terrestrial Reference Frame TRF(xo,yo) known "a priori" considered "free of error" and c) a selection matrix $\bar{S}$, which allows to choose or exclude simultaneously the coordinate increments $dx_i = x_i - x_i^0$; $dy_i = y_i - y_i^0$, $i = 1 \dots k$ of specifics network points.

2 DATUM DEFINITION THROUGH THE NNT AND NNR CONDITIONS BASED IN TRANSFORMATION PARAMETERS, A PRIORI TRF (X_O,Y_O) AND A SELECTION MATRIX.

Let us consider a free geodetic network constituted by " k " physical points P_i with coordinates (x_i, y_i), $i = 1 \dots k$ in the $TRS\,(x, y)$, and related through " n " observed distances, and *not* being *defined* for any epoch, the position and orientation of the $TRS\,(x, y)$.

Moreover, let us consider known the coordinates (x_i^0, y_i^0), $i = 1 \dots k$ "a priori" or "approximated" from the reference frame $TRF\ (x_0, y_0)$.

The lack of definition in the origin and orientation of the TRS(x,y) cause a datum defect and a rank-deficient Singular Gauss-Markov Model (SGMM) for the adjustment of the network (Schaffrin,1985):

$$y - e = A\xi, r(A) =: q < m < n, d =: m - q, e \sim (0, \sigma_0^2 P^{-1} =: D\{y\}) \tag{1}$$

with,

$n =$ number of observations; $m =$ number of unknown parameters.

$r =$ rank; $d =$ number of datum defect; $D =$ Dispersion $P_{nxn} =$ symmetric positive-definite weight matrix; $o =$ order ; $\sigma_0^2 =$ unknown (observational) variance component ; E=expectation.

$y_{nx1} =$ vector of observations (increments)

$y_{nx1} = [y_{ij}] = [(s_{12}^{obs} - s_{12}^0), (s_{13}^{obs} - s_{13}^0), \dots, (s_{ij}^{obs} - s_{ij}^0), \dots, (s_{k-1,k}^{obs} - s_{k-1,k}^0)]^T$

$e_{nx1} = [e_{ij}]$ =vector of random errors (unknown)

$E\{e_{nx1}\} = 0$

$s_{ij}^0 = \sqrt{(\Delta x_{ij}^0)^2 + (\Delta y_{ij}^0)^2}$, $i = 1 \dots k$, $j = 1 \dots k$, $i < j$

$\Delta x_{ij}^0 = x_j^0 - x_i^0$; $\Delta y_{ij}^0 = y_j^0 - y_i^0$

$A_{nxm} =$ Design or coefficient matrix ("Jacobian")

$$A_{nxm} = \begin{bmatrix} \alpha_{12} \\ \dots \\ \alpha_{ij} \\ \dots \\ \alpha_{k-1,k} \end{bmatrix} ;\ \alpha_{ij_{1xm}} = [0, \dots, -\Delta x_{ij}^0, -\Delta y_{ij}^0, \dots, \Delta x_{ij}^0, \Delta y_{ij}^0, \dots, 0].(1/s_{ij}^0)$$

$\xi_{mx1} =$ Vector of unknown parameters (coordinate increments).

$\xi_{mx1} = X_{mx1} - X_{mx1}^0$

$X_{mx1} =$Vector of unknown coordinates of the points P_i of the TRF(x,y) expressed in the $TRS\,(x, y)$.

$X_{mx1} = [x_1, y_1 \dots x_k, y_k]^T$

X^0_{mx1} =Vector of known coordinates of P_i of the "a priori" or "approximated" $TRF\ (x_0, y_0)$.

$X^0_{mx1} = [x^0_1, y^0_1 \ldots x^0_k, y^0_k]^T$

$\xi_{mx1} = [dx_1 \quad dy_1 \quad \ldots \quad dx_k \quad dy_k]^T$; $dx_i = x_i - x^0_i$; $dy_i = y_i - y^0_i$, $i = 1 \ldots k$, $m = 2k$

The deficiency of rank or number of datum defects $d =: 3 = m - q$ shows that to define the origin and orientation in a given epoch of the $TRS\,(x, y)$ is necessary to introduce in (1) - as minimum - three additional parameters.

Therefore, to complete the datum definition of the network in (1) it is necessary to introduce as minimum three independent condition equations to *define* and *realize* in a given epoch the origin and orientation of the $TRS\,(x, y)$.

A Restricted LEast Square Solution (RLESS) can be found if the datum defect is eliminated by introducing in (1) three linear equations of condition to the unknown parameter (increments or coordinate differences) in the form of *minimum constraints* (Vacaflor, J.L., 2008):

$$K_{3xm}\xi_{mx1} = K_{3xm}E^T_{mx3}PT^*_{3x1} \ , \ o(K) = 3xm, \ r(K) = 3 \ , \tag{2}$$

$$R(A^T) \oplus R(K^T) = \Re^m$$

$$AE^T = 0 \ , \ o(E) = dxm \ , \ r(E) = d$$

$$\Rightarrow R(A^T) \overset{\perp}{\oplus} R(E^T) = \Re^m$$

with,

$$\xi = E^T PT^* \tag{3}$$

$$PT^* = [t^*_x, t^*_y, d\delta^*]^T \tag{4}$$

PT^* = Conventionally adopted numerical values of the transformation parameters : two translations t^*_x, t^*_y and one differential rotation $d\delta^*$ (of the $TRS\,(x, y)$) to *define* the position and orientation of the $TRS\,(x, y)$.

$$E_{3xm} = \begin{bmatrix} 1 & 0 & \ldots & 1 & 0 \\ 0 & 1 & \ldots & 0 & 1 \\ -y^0_1 & x^0_1 & \ldots & -y^0_k & x^0_k \end{bmatrix} \tag{5}$$

The Eq.(2) shows *explicitly* how the *minimum* datum constraints $K_{3xm}\xi_{mx1} = K_{3xm}E^T_{mx3}PT^*_{3x1}$ *define* and *realize* for a given epoch, the position and orientation of the $TRS\,(x, y)$ *using* PT^*_{3x1} and the coordinates of the frame $TRF\ (x_0, y_0)$ through E_{3xm}.(Vacaflor, J.L., 2008)

A particular class of minimum datum constraints arises from considering that K is by definition:

$$K := ES \ , \ o(S) = mxm \ , \ r = r(S) \geq m - q \ , \ m = 2k \tag{6}$$

$$S := Diag(s_{dx_1}, s_{dy_1}, ..., s_{dx_i}, s_{dy_i}, ... s_{dx_k}, s_{dy_k}),\ i = 1...k, \tag{7}$$

S: Selection matrix of coordinate differences
Introducing (6) in (2), leads to:

$$E_{3xm} S_{mxm} \xi_{mx1} = E_{3xm} S_{mxm} E^T_{mx3} PT^*_{3x1} \tag{8}$$

In (8) is possible to select or exclude the coordinate differences $dx_i = x_i - x_i^0$ or $dy_i = y_i - y_i^0$, $i = 1...k$ of the datum points assigning the values "0" or "1" respectively to the diagonal elements of S, namely, for example, if $s_{dx_1} = 1 \Rightarrow dx_1 = x_1 - x_1^0$ is selected, and if $s_{dx_1} = 0 \Rightarrow dx_1 = x_1 - x_1^0$ is exclude.

Once the structure of S is established, the minimum constraints (8) define the datum of the geodetic network of trilateration, namely:
a) *Define* for a given epoch, the position and orientation of the $TRS(x, y)$ *using* the conventionally adopted numerical values of the transformation parameters: PT^*_{3x1}.
b) *Realize* for a given epoch, the position and orientation of the $TRS(x, y)$ defined with PT^*_{3x1} *using* a numerical evaluation of a selected set by S of coordinate differences $dx_i = x_i - x_i^0$; $dy_i = y_i - y_i^0$, $i = 1...k$ of specific points of the network.

The resources used for this evaluation are: PT^*_{3x1} and the coordinates "a priori" of specific points - selected of the network - taken from a known frame considered "free of error" $TRF\ (x_0, y_0)$ given in E_{3xm}.
Now, if by definition:

$$s_{dx_i} =: s_{dy_i} =: \bar{s}_i \tag{9}$$

Hence, incorporating (9) in (7) the following selection matrix is obtained:

$$\bar{S} := Diag(\bar{s}_1, \bar{s}_1, ..., \bar{s}_i, \bar{s}_i, ... \bar{s}_k, \bar{s}_k)\ ,\ i = 1...k \tag{10}$$

Considering that $\bar{S}$ meets the rank condition:

$$r = r(\bar{S}) > m - q \tag{11}$$

Incorporating (10) in (6) with the condition (11) leads to:

$$\bar{K} := E\bar{S}\ ,\ o(\bar{S}) = mxm\ ,\ r = r(\bar{S}) > m - q\ ,\ m = 2k \tag{12}$$

With (12) the following minimum constraints can be formed:

$$E_{3xm} \bar{S}_{mxm} \xi_{mx1} = E_{3xm} \bar{S}_{mxm} E^T_{mx3} PT^*_{3x1} \tag{13}$$

In (13) is possible to select or exclude *simultaneously* the coordinate differences $dx_i = x_i - x_i^0$ or $dy_i = y_i - y_i^0$, $i = 1...k$ of specifics network points by assigning the values "0" or "1" respectively to the diagonal elements of $\bar{S}$, namely, for example, if $\bar{s}_1 = 1 \Rightarrow dx_1 = x_1 - x_1^0$ and $dy_1 = y_1 - y_1^0$ are selected, and if $\bar{s}_1 = 0$ both differences are excluded.
If by definition

$$PT^*_{3x1} := \begin{bmatrix} t^*_x \\ t^*_y \\ d\delta^* \end{bmatrix} := 0_{3x1} \tag{14}$$

and E_{3xm} of (5) is partitioned as follows:

$$E_{3xm} := \begin{bmatrix} E1_{2xm} \\ E2_{1xm} \end{bmatrix} \tag{15}$$

with,

$$E1_{2xm} := \begin{bmatrix} 1 & 0 & \dots & 1 & 0 \\ 0 & 1 & \dots & 0 & 1 \end{bmatrix} \tag{16}$$

$$E2_{1xm} := [-y^0_1 \quad x^0_1 \quad -y^0_2 \quad x^0_2 \quad \dots \quad -y^0_m \quad x^0_m] \tag{17}$$

Analogously, if PT^*_{3x1} of (14) is partitioned as follows:

$$PT^* := \begin{bmatrix} PT1^*_{2x1} \\ PT2^*_{1x1} \end{bmatrix} = 0_{3x1} \tag{18}$$

with,

$$PT1^*_{2x1} := \begin{bmatrix} t^*_x \\ t^*_y \end{bmatrix} \tag{19}$$

$$PT2^*_{1x1} := [d\delta] \tag{20}$$

Incorporating (15) and (18) in (13) the following minimum constraints are obtained:

$$\begin{bmatrix} E1_{2xm} \\ E2_{1xm} \end{bmatrix} \bar{S}.\xi = E\bar{S}E^T \begin{bmatrix} PT1^*_{2x1} \\ PT2^*_{1x1} \end{bmatrix} = \begin{bmatrix} 0_{2x1} \\ 0_{1x1} \end{bmatrix} \tag{21}$$

$\Rightarrow$

$$E1_{2xm}\bar{S}_{mxm}\xi_{mx1} = 0_{2x1} \qquad \text{NNT condition} \tag{22}$$

$$E2_{1xm}\bar{S}_{mxm}\xi_{mx1} = 0_{1x1} \qquad \text{NNR condition} \tag{23}$$

The condition equations (22) and (23) are called “No Net Translation” (NNT) and “No Net Rotation” (NNR), and:

a) *Define* for a given epoch, the position and orientation of the $TRS(x, y)$ respectively, *using* zero values conventionally adopted of the transformation parameters PT^*_{3x1}.

b) *Realize* for a given epoch, the position and orientation of the $TRS(x, y)$ respectively, defined with PT^*_{3x1} of (18), considering zero *the numerical evaluation of* $E1_{2xm}\bar{S}_{mxm}\xi_{mx1}$ and $E2_{1xm}\bar{S}_{mxm}\xi_{mx1}$ of a selected set by $\bar{S}$ of coordinate differences: $dx_i = x_i - x^0_i$; $dy_i = y_i - y^0_i$, $i = 1 \dots k$ of specifics network points.

3 CONCLUSIONS

In this work, it is developed within an stochastic linear model for the adjustment of a two-dimensional free trilateration network of the type SGMM with datum defect, three linear condition equations to the unknown parameters (increments or coordinate differences) namely *minimum constraints*: two "No Net Translation" (NNT) and one "No Net Rotation" (NNR) which define the datum of the network in a given epoch regarding to the position and orientation of the TRS(x,y) based in: a) three zero values conventionally adopted of three parameters of a plane coordinate Helmert transformation : two translation and one differential rotation, b) a Terrestrial Reference Frame TRF(xo,yo) known "a priori" considered "free of error" and c) a selection matrix $\bar{S}$, which allows to choose or exclude simultaneously the coordinate increments $dx_i = x_i - x_i^0$; $dy_i = y_i - y_i^0$, $i = 1 \ldots k$ of specifics network points.

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