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ABOUT THE APPLICATION OF BEM BASED ON THE SELF-REGULAR TRACTION BIE IN ELASTICITY

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Abstract: Self-regular BEM algorithm avoids Cauchy principal value or Hadamard finite part evaluations as regularization is applied prior to discretization. Smoothness requirement for traction BIE is more stringent ($C^{1,\alpha}$ for displacement). Boundary discretization into standard continuous elements leads to a loss of the smoothness required. Relaxation of the smoothness requirement has been proposed using C^0 continuous elements with collocation points at the intersection between elements. Some researchers claim that this procedure cannot be theoretically justified. Interpolation of displacement tangential derivative an 'relaxed continuity' hypothesis are pointed out as possible sources of error introduced on the discretization of self-regular traction-BIE. Discontinuous elements are implemented in order to verify the possible sources of error. Such elements allow the split of these sources of error. Numerical results show that the 'relaxed continuity' hypothesis seems to be the dominant source of error. Apparently the smoothness requirement for the self-regular traction-BIE must be preserved to guarantee converge.

1 INTRODUCTION

The boundary element method (BEM) has been shown to provide a convenient formulation to the analysis of many physical problems and to be a robust alternative to domain methods, such as the finite element method. The BEM is based on boundary integral equations (BIE) which contains singular integrals when the collocation point is on the boundary. Dealing with singular integrals is not straightforward. An effective technique to handle singular integrals is difficult to achieve. Several techniques have been presented, with varying degrees of success, to deal with the three singularity orders that appear on the BIEs: weakly singular, strongly singular and hypersingular. In spite of accurate results achieved with many of these techniques, they do not seem to be the most practical approach in the treatment of the singularities, as they require different approaches for the evaluation of the integrals present in the formulation according to the source point location and the order of singularity. The algorithms based on the self-regular form of the BIE are perhaps the most suitable approaches in the BEM (Cruse and Richardson, 1996; Jorge et al., 2001). The self-regular formulation allows the use of a single integration scheme in the evaluation of all integrals involved in the formulation, representing a great advantage over the classical formulation on BEM implementation. The self-regular formulation is achieved by re-writing the original BIE in a way that it is regularized before discretization. Thus, the evaluation of the singular integrals in the Cauchy principal value (CPV) or Hadamard finite part (HFP) sense, as usually employed in the classical BEM can be avoided. One of the first papers dealing with the development of the self-regular formulations has been presented by Rudolphi (1991), applying the so-called 'simple solutions'. The meaning of the original BIE is not changed due to regularization (Rudolphi, 1991).

The smoothness requirements are the same for self-regular formulation and its respective standard formulation. For elasticity problems, the smoothness requirement for the primary BIE (displacement-BIE) is met by the use of standard continuous elements based on C0 interpolation functions. Therefore, the numerical implementation of the displacement-BEM poses no significant problem. Nevertheless, when dealing with the gradient-based BIE (traction-BIE), the smoothness requirement is stronger, and standard continuous elements do not meet the requirement. In this case the displacement field should be $C^{1,\alpha}$ Hölder continuous, and the use of C^0 elements leads to a loss of this continuity at inter-element nodes. A priori only boundary elements that meet the smoothness requirement should be used.

Some approaches are presented in the literature to compute the gradient-based BIE in view of the smoothness requirement. Some authors use continuous elements based on C^1 interpolation functions, which include 'Overhauser' (Sladek and Sladek, 1995) and 'Hermite' (Gray and Soucie, 1993) elements. Other authors use continuous elements based on C^0 interpolation functions, but with all collocation points inside the element (Gallego and Domínguez, 1996). Some authors use a relaxation of the smoothness requirement for the density function in a way that standard continuous elements with collocation points at the intersection between elements can be used, and only piece-wise $C^{1,\alpha}$ continuity can be guaranteed. However, Martin and Rizzo (1996) claim that the use of the 'relaxed continuity' approach cannot be theoretically justified. Moreover, Krishnasamy et al. (1992) pointed out that a necessary condition, which in most cases is also sufficient to guarantee the existence of the BIE is to satisfy the smoothness requirement. The discussion was renewed by Martin et al. (1998), based on good numerical results achieved (Richardson et al., 1997), validating the 'relaxed continuity' hypothesis. However, those authors also point out that convergence and results accuracy cannot be assured since this approach is based on inconsistent reasoning.

Jorge et al. (2003), using a variational approach shows that it is indeed necessary to meet the smoothness requirement to achieve reliable solutions using the self-regular traction-BIE. On that paper, the continuity of the density function is enforced at inter-element nodes and the smoothness requirement is satisfied in a variational sense.

The current paper deals only with the self-regular gradient-based BIE. Previous works published on the literature (Richardson et al., 1997; Jorge et al., 2001; Jorge et al., 2003; Ribeiro et al., 2009) have already made comparison between the self-regular BEM formulation and the standard BEM formulation, and the reader should refer to these references for more information. The purpose of the current paper is to identify the main source of error that can be introduced when using the self-regular traction formulation. Standard discontinuous elements are used to investigate the sources of error introduced on the self-regular traction-BEM. When using continuous elements, the 'relaxed continuity' hypothesis is used on the self-regular traction-BEM implementation. This hypothesis implies that the displacement gradients, which are part of the regularizing term, are not single-valued at inter-element nodes as derived analytically. Different values for the displacement gradient are assumed depending on the collocation point location and element to be integrated. Therefore, the 'relaxed continuity' hypothesis can introduce errors on the self-regular traction-BEM. The use of discontinuous elements avoids the assumption of the 'relaxed continuity' hypothesis, as for these elements all collocation points are placed on the interior of the element where the continuity requirement is preserved. Another source of error that can be introduced on the self-regular traction-BEM is the evaluation of displacement tangential derivatives, which are also part of the regularizing term. The displacement tangential derivatives are locally obtained based on the exact derivative of the element interpolation functions, and as a result their approximations are one degree less than the approximation of the boundary variables. For discontinuous elements these source of error also exists. Thus, the use of discontinuous elements allows the split of the sources of error introduced on the self-regular traction-BEM to identify the most important one. Such errors do not occur on the standard traction-BEM.

The self-regular traction formulation is reviewed in this paper and the main features of the BEM implementation are presented. Numerical results are obtained comparing the accuracy of the self-regular traction-BEM with continuous and discontinuous elements. The main source of error introduced on the self-regular traction-BEM is highlighted.

2 SELF-REGULAR TRACTION BOUNDARY INTEGRAL EQUATION

The self-regular traction boundary integral equation, as presented by Cruse and Richardson (1999), is obtained through the regularization of the Somigliana stress identity (SSI)

$$\sigma_{ij}(p) = -\int_{\Gamma} u_k(Q) S_{kij}(p,Q)\, d\Gamma(Q) + \int_{\Gamma} t_k(Q) D_{kij}(p,Q)\, d\Gamma(Q) \tag{1}$$

where p is the source point (interior point), P is the regularizing point (boundary point), Q is the field point (boundary point), $u_i(Q)$ is the displacement field, $t_i(Q)$ the traction field and D_{jik} (p,Q) and S_{jik}(p,Q) are the fundamental solutions. A simple solution corresponding to a constant stress state in the body that is equal to the boundary stress at a surface point P is subtracted and added back to Eq. 1 to obtain the regularized form of the SSI

$$\sigma_{ij}(p) = \sigma_{ij}(P) - \int_{\Gamma} \left[u_k(Q) - u_k^L(Q)\right] S_{kij}(p,Q)\, d\Gamma(Q) + \int_{\Gamma} \left[t_k(Q) - t_k^L(Q)\right] D_{kij}(p,Q)\, d\Gamma(Q) \tag{2}$$

where P is the regularizing point (boundary point), $u_{kL}(Q)$ and $t_{kL}(Q)$ are the linear state of displacements and tractions associated with the boundary stress at P, and are given by

$$\begin{aligned} u_k^L(Q) &= u_k(P) + u_{k,m}(P)\left[x_m(Q) - x_m(P)\right] \\ t_k^L(Q) &= \sigma_{km}(P) n_m(Q) \end{aligned} \tag{3}$$

being $u_{k,m}(P)$ the displacement gradients at the source point, x_m the Cartesian co-ordinates of the source point P and field point Q, $\sigma_{km}(P)$ the stress components at P, and $n_m(Q)$ the components of the normal vector at the field point. Equation (2) leads to a smooth transition of the stress field at interior points to boundary points, since all the discontinuous and singular terms are cancelled out during the regularization process. This equation is regular for all interior points, including interior points close to the corners where the continuity requirement is satisfied. Equation (2) contains a term related to displacement tangential derivative, which is not part of the original SSI. Taking the limit as $p \rightarrow P$, and operating in Equation (2) with the surface normal at the source point, the self-regular traction-BIE is obtained.

$$\int_{\Gamma} \left[u_k(Q) - u_k^L(Q)\right] S_{kij}(P,Q) n_i(P)\, d\Gamma(Q) = \int_{\Gamma} \left[t_k(Q) - t_k^L(Q)\right] D_{kij}(P,Q) n_i(P)\, d\Gamma(Q) \tag{4}$$

Equation (4) is fully regular, and valid to all boundary points where the smoothness requirement is met. Unlike to the standard hypersingular BIE, Equation (4) does not require its integrands to be evaluated in the Hadamard finite part or Cauchy principal value sense.

3 BEM ALGORITHM

The main features of the self-regular traction-BEM algorithm implemented in the current work are described in this section. This algorithm requires an explicit representation of displacement gradients, as evaluated at the boundary. The displacement gradients are obtained for each boundary element in terms of the local displacement tangential derivatives and local tractions. Using Hooke’s law and the strain-displacement relations, the tractions are expressed in terms of displacement gradients. The displacement tangential derivatives are evaluated in terms of the intrinsic co-ordinate for each element. The displacement tangential derivatives are then represented in terms of the displacement gradients in the global Cartesian co-ordinate system making use of the components of the tangent vector and the Jacobian of the transformation of the system of co-ordinates. A system of equations relating local tractions and displacement tangential derivatives with the displacement gradients in the global co-ordinate system is written as

$$\begin{Bmatrix} t_1 \\ t_2 \\ u_{1,\xi} \\ u_{2,\xi} \end{Bmatrix} = \begin{bmatrix} \frac{2G}{1-2\upsilon}(1-\upsilon)n_1 & Gn_2 & Gn_2 & \frac{2G\upsilon}{1-2\upsilon}n_1 \\ \frac{2G\upsilon}{1-2\upsilon}n_2 & Gn_1 & Gn_1 & \frac{2G}{1-2\upsilon}(1-\upsilon)n_2 \\ -n_2 J(\xi) & 0 & n_1 J(\xi) & 0 \\ 0 & -n_2 J(\xi) & 0 & n_1 J(\xi) \end{bmatrix} \begin{Bmatrix} u_{1,1} \\ u_{2,1} \\ u_{1,2} \\ u_{2,2} \end{Bmatrix} \tag{5}$$

The algorithm implemented in this work allows the use of both, continuous and discontinuous elements. When discontinuous elements are used, the smoothness requirement for displacements is met, since for such elements all collocation points are placed at the interior of the element where $C^{1,\alpha}$ continuity is preserved. On the other hand, if continuous elements are used, $C^{1,\alpha}$ Hölder continuity is not preserved at inter-element nodes, and a ‘relaxed continuity’ hypothesis is assumed, allowing the displacement field to be only piece-wise $C^{1,\alpha}$ continuous.

When adopting the ‘relaxed continuity’ hypothesis, the regularizing terms, which contain displacement gradients, are no longer single-valued at inter-element nodes. Their values depend upon collocation point location and the element to be integrated in the following manner

$$u_k^{L'}(\xi) = \begin{cases} u_k(P) + u_{k,m}(P)\left[x_m(\xi) - x_m(P)\right] & P \in \Delta\Gamma_l \\ u_k(P) + \left(\frac{1}{M}\sum_{n=1}^{M} u_{k,m}(P_n)\right)\left[x_m(\xi) - x_m(P)\right] & P \notin \Delta\Gamma_l \end{cases} \tag{6}$$

where M is the number of elements sharing the collocation point. The state of linear traction is obtained in a similar fashion. The average nodal value of the gradient evaluated based on the elements sharing the collocation point is used on the evaluation of the integrals in all elements, unless the element to be integrated contains the collocation point. In this case, the element-based gradient values employed are locally evaluated based on the interpolation scheme of this element.

4 BEM ALGORITHM

A bidimensional elasticity problem is analysed using the self-regular traction-BEM presented in the previous section. Standard continuous and discontinuous isoparametric boundary elements based on quadratic, cubic and quartic interpolation functions are adopted in the discretization of the traction-BIE for bidimensional problems with finite domain. In case continuous elements are used, discontinuities in the surface normal or boundary traction are discretized using discontinuous elements at these discontinuities. The problem is taken to be plane stress. The adopted material constants were Poisson's ratio ν=0.3 and Young's modulus E=78850 units.

4.1 Numerical Example

In this example the analysis of a cantilever beam constrained on one edge (x=50) and subjected to a tangential load, $t=150 - 6y^2$, according to the figure 1, is presented.

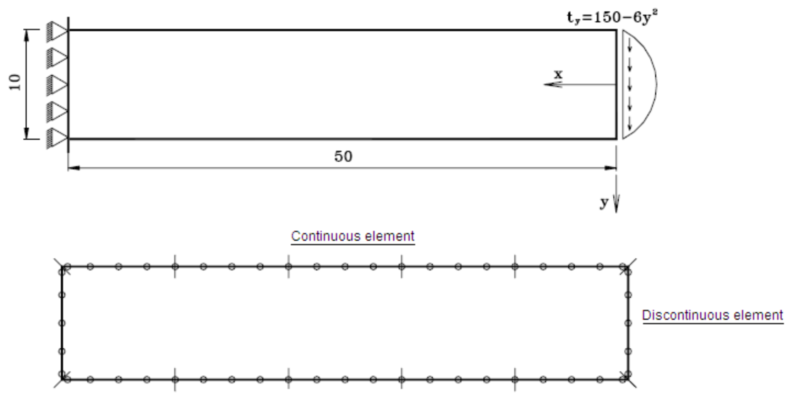

Figure 1: Cantilever beam subjected to a tangential load – geometry and coarsest mesh

The coarsest mesh for each of the three elemental interpolation functions is constructed with twelve elements of equal size as shown in Figure 1. In subsequent mesh refinements, each element of the previous mesh is divided into two elements of the same size. Numerical analyses are performed using meshes composed of continuous elements along the boundary and semi-continuous elements close to the points where traction is not uniquely defined (SRTBEM-SC) or meshes of discontinuous elements on the whole boundary (SRTBEM-D). Numerical integrations are performed using Gauss-Legendre schemes with 12, 16 and 32 points, denoted by 12GP, 16GP and 32GP, respectively, just to check the effect of integration order in the numerical results. BEM solution for vertical displacement at the free edge (x=0, y=0) (Figure 1), horizontal stress component (σ_x) at point (x=40, y =-2.0) and shear stress component (σ_{xy}) at point (x=40, y=0) for several meshes are obtained and compared with the analytical solution based on beam theory. Figures 2 to 4 show the errors on displacement and stresses for meshes of quadratic, cubic and quartic boundary elements, respectively.

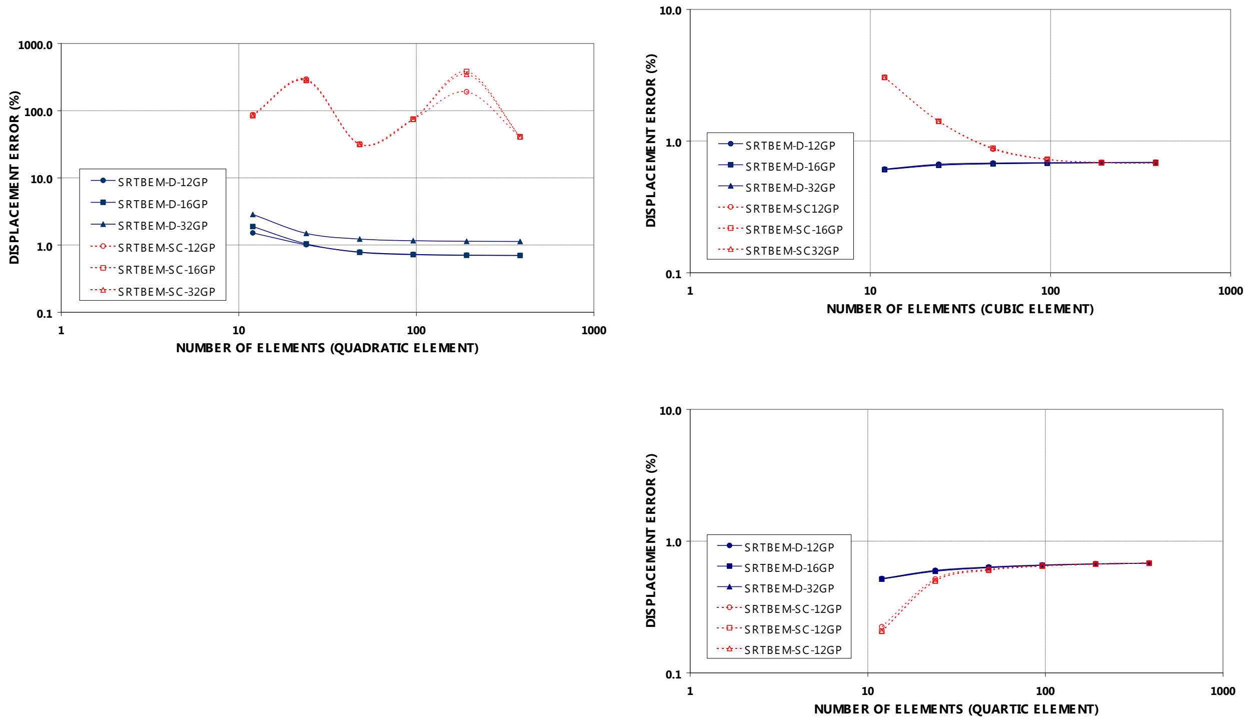

Figure 2: Displacement error analysis at boundary point (0; 0)

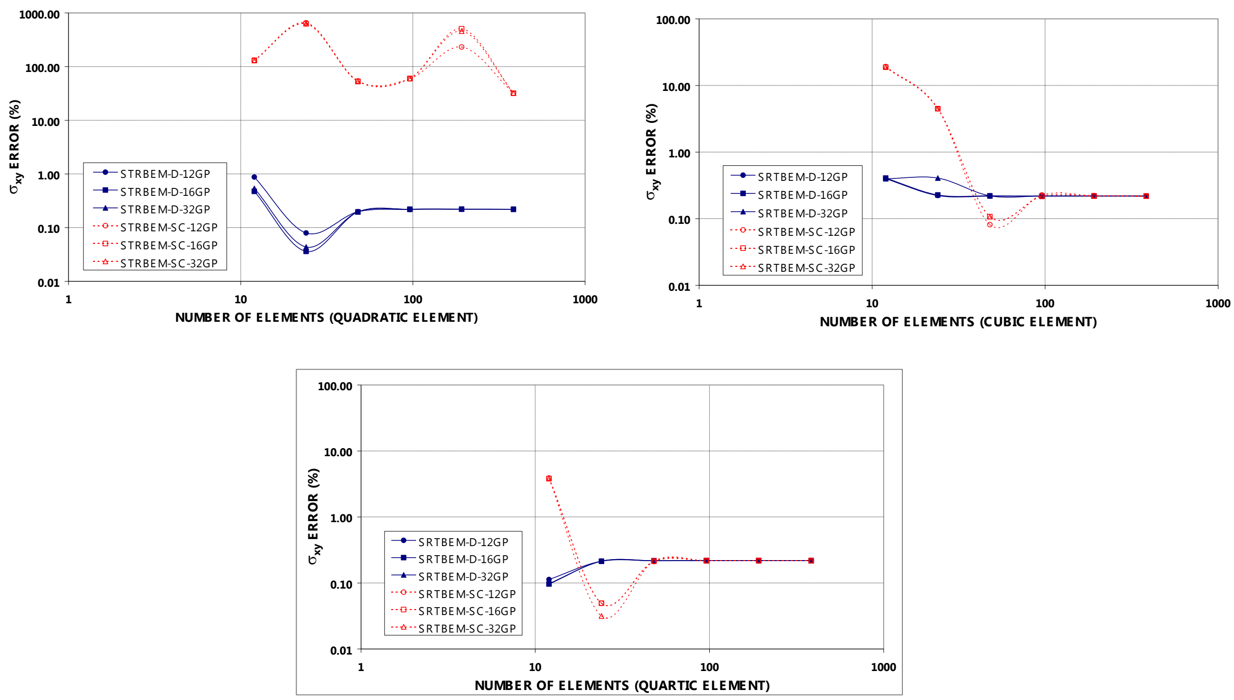

Figure 3: Shear stress error analysis (σ_{xy}) at internal point (40; 0)

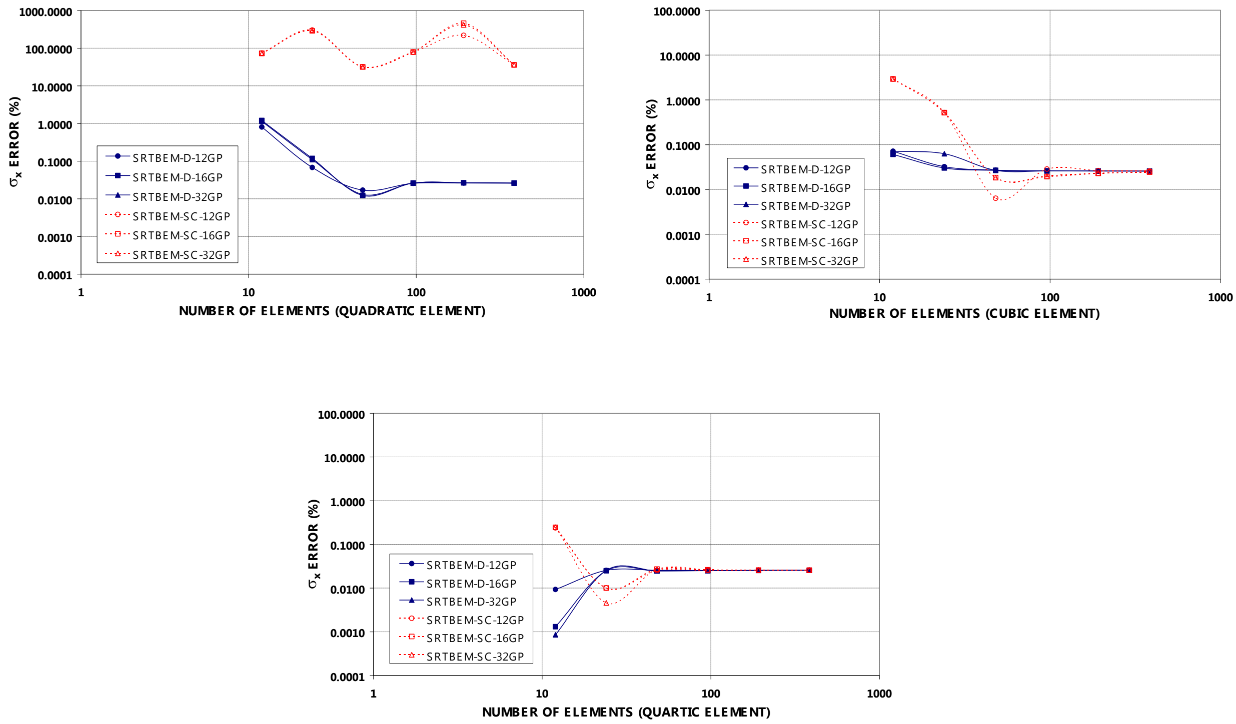

Figure 4: Horizontal stress component error analysis (σ_x) at internal point (40; 2)

From Figure 2 to 4 it can be seen that the use of discontinuous elements instead of continuous elements in most of cases improves BEM solution. For quadratic elements there is a substantial gain in results accuracy for all meshes and integration levels when discontinuous elements are used. For cubic elements some gain in accuracy is obtained when using discontinuous elements instead of continuous elements, especially for coarser meshes, mainly for meshes containing 48 elements or more; however this gain is smaller than for quadratic elements. In the case of quartic elements the solution accuracy using both discontinuous and continuous elements are almost the same, especially for integrations with 16 and 32 Gauss points. All these observations point out the 'relaxed continuity' hypothesis as the most important source of error on the self-regular traction-BEM. Discontinuous elements do not make use of such hypothesis as in this case all nodes are located at the interior of the element where the smoothness requirement is met. The different improvement rates observed for the three interpolation functions seem to be in some way related to the number of inter-element nodes to intra-element nodes. When quadratic continuous elements are used in a closed boundary, for each intra-element node there is one node at the junction between two elements. Thus, when adopting the 'relaxed continuity' hypothesis, for each collocation point where the displacement tangential derivative is single-valued, there is one collocation point where different values for the displacement gradient are assigned, according to the element to be integrated, which gives a proportion of 1:1. On the other hand, if cubic elements are used there are two intra-element nodes for each inter-element node, and for quartic elements the proportion is even higher being 3:1. Therefore, it seems that the influence of the assumption of the 'relaxed continuity' hypothesis is more noticeable for quadratic boundary elements than for higher order elements.

Jorge et al. (2003) using a variational formulation have also observed that the best global error reduction on the results for the self-regular traction BEM is obtained for quadratic elements, also eliminating some local error through some part of the boundary.

The increase on the number of Gauss points has presented no significant gain in results accuracy. This could be regarded to the fact that the Gauss points are never placed on the extremity of the boundary element, however for higher Gauss point order a quasi-singularity could appear as the Gauss point is each time closer to the collocation point, and this could be a source of error.

5 CONCLUSIONS

Sources of error for the 2D self-regular traction BEM algorithm are investigated in this paper, concentrated on: i) the interpolation of the displacement tangential derivative, ii) the number of Gauss points in an element, and iii) the lack of the required continuity at inter-element nodes, for continuous elements. The self-regular traction BIE is fully regular. Nevertheless, due to the hypersingular kernel present in the BIE, the displacement field is required to be $C^{1,\alpha}$ Hölder continuous, which cannot be met using standard continuous elements. To overcome this problem two approaches are implemented. In the first approach a 'relaxed continuity' hypothesis was assumed, allowing the displacement field to be only piece-wise $C^{1,\alpha}$ Hölder continuous. In the second approach discontinuous elements are used, preserving the smoothness requirement at all collocation points.

The use of continuous elements with the 'relaxed continuity' approach in the self-regular traction-BEM has not presented results accurate enough, especially with quadratic elements. The use of discontinuous elements instead of continuous elements led to a gain in result accuracy. This gain is substantial for quadratic interpolation, but is lower for cubic interpolation. For quartic interpolation accuracy in results obtained for continuous and discontinuous BEM meshes are almost the same.

The use of discontinuous elements allowed that the sources of error on the SRTBEM were split to identify the most important one. The interpolation of the displacement tangential derivative does not seem to be the most important error source on the implementation of the self-regular traction-BEM. The discontinuity of the displacement gradients at the collocation point shared by two elements, which is the basis of the 'relaxed continuity' hypothesis, seems to be the dominant source of error in this formulation. The influence of the 'relaxed continuity' hypothesis on results accuracy seems to be in some way related to the proportion of intra-element nodes to inter-element nodes, as quadratic elements present a lower proportion of intra-element to inter-element nodes than cubic and quartic elements.

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