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ON A RESIDUAL LOCAL PROJECTION METHOD FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract.
This work proposes a new residual local projection stabilized finite element method for the incompressible Navier-Stokes equations. The method adds to the Galerkin formulation new fluctuation terms which are symmetric and easily computable at the element level. The method is proved to be well-posed for the linearized model using the pair of spaces P_1/P_l, $l = 0, 1$ *with continuously and discontinuously pressure interpolations. Next, we establish a new hierarchical a posteriori error estimator, and introduce a cheap strategy to recover a locally mass conservative velocity field in the discontinuous pressure case, a property usually neglected in the stabilized finite element context. Several numerical tests illustrate theoretical results.*

Keywords: *finite element, Navier-Stokes equations, stabilized method, boundary layer, a posteriori error estimator*

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1. INTRODUCTION

The steady incompressible Navier-Stokes problem consists of finding the velocity and pressure $(\boldsymbol{u}, p)$ as the solution of

$$(\nabla \boldsymbol{u})\,\boldsymbol{u} - \nu\,\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \nabla\cdot \boldsymbol{u} = 0 \qquad \text{in}\,\Omega\,, \qquad (1)$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on}\,\partial\Omega\,,$$

where $\nu \in \mathbb{R}^+$ is the fluid viscosity and $\boldsymbol{f}$ is a given regular data in Ω. Adopting standard notations for Sobolev spaces, the weak form associated to (1) reads: Find $(\boldsymbol{u}, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ such that

$$\mathbf{A}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = (\boldsymbol{f}, \boldsymbol{v})_\Omega \quad \text{for all } (\boldsymbol{v}, q) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)\,, \qquad (2)$$

where

$$\mathbf{A}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) := ((\nabla \boldsymbol{u})\,\boldsymbol{u}, \boldsymbol{v})_\Omega + \nu\,(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_\Omega - (p, \nabla\cdot \boldsymbol{v})_\Omega + (q, \nabla\cdot \boldsymbol{u})_\Omega\,. \qquad (3)$$

Stabilized finite element methods for the incompressible Navier-Stokes equations handle two numerical difficulties: the first is the well known inf-sup condition (see Girault and Raviart (1986)), which prevents some of the most interesting and easy to use low order pairs of finite elements from being used by the Galerkin method. Also, boundary layers ought to be accuratly captured if we want to avoid non-physical spurious oscillations on solutions. In general, stabilized finite element methods add extra terms to the Galerkin formulation to circumvent both cited shortcomings.

Recently, a new family of residual-based stabilized methods, called RELP (Residual Local Projection), has been casted and analyzed in Barrenechea and Valentin (2010a,b). As a result of a Petrov-Galerkin enrichment (see Barrenechea et al. (2007, 2009); Franca et al. (2009) for the idea applied to the Darcy problem), a new kind of fluctuation term arises as part of a static condensation procedure, which prevents additional degrees of freedom. Consequently, the RELP method allows us to adopt low order polynomials for velocity and pressure while keep fluctuations contributions element-wise computable.

This work aims at extending the RELP method proposed in Barrenechea and Valentin (2010b) to the incompressible Navier-Stokes equations adopting the pair of spaces $\mathbb{P}_1^2/\mathbb{P}_l$, $l = 0,\,1$. Afterwards, we introduce a new *a posteriori* error estimator based on a hierarchical strategy which, when combined with the RELP method, produces oscillatory-free numerical solutions.

The plan of the paper is as follows. In §2. we introduce the RELP method, and we revisit main theoretical results for the linearized Navier-Stokes equations in §3.. Next in §4., we introduce the hierarchical *a posteriori* estimator. In §5. several numerical results attest the good performance of our method and conclusions lay in §6..

1.1 Notations

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of regular triangulations of Ω, built up using triangles K with boundary ∂K and characteristic length $h_K := diam(K)$, and $h := \max\{h_K : K \in \mathcal{T}_h\}$. For simplicity, we suppose the boundary of Ω to be polygonal. Associated to this triangulation, the discrete space for the velocity $\mathbf{V}_h$ is the usual space of vector-valued piecewise linear continuous functions with zero trace on $\partial\Omega$. To approximate the pressure we use Q_h^l, $l = 0, 1$, the space of piecewise polynomial functions of degree l with zero mean value on Ω. If $l = 1$, the space of pressures may contain continuous or discontinuous functions. The set of internal edges F

of the triangulation is denoted $\mathcal{E}_h$ with $h_F = |F|$. We denote by $\boldsymbol{n}$ the normal outward vector on ∂K; also, $[v]$ stands for the jump of v across F, and Π_S, where $S \subset \mathbb{R}^2$, is the orthogonal projection onto the constant space, i.e., $\Pi_S(q) := \frac{(q,1)_S}{|S|}$.

2. THE METHOD

We extend the RELP method introduced in Barrenechea and Valentin (2010b) to the non-linear incompressible Navier-Stokes equations as follows: *Find* $(\boldsymbol{u}_1, p_l) \in \mathbf{V}_h \times Q_h^l$ *such that*

$$\mathbf{B}((\boldsymbol{u}_1, p_l), (\boldsymbol{v}_1, q_l)) \;=\; \mathbf{F}(\boldsymbol{v}_1, q_l) \quad \forall (\boldsymbol{v}_1, q_l) \in \mathbf{V}_h \times Q_h^l \, . \tag{4}$$

The weak forms $\mathbf{B}(.,.)$ and $\mathbf{F}(.)$ are split into the Galerkin contributions and the additional terms over elements and internal edges, i.e., $\mathbf{B}(.,.) := \mathbf{A}(.,.) + \mathbf{B}_{\mathcal{T}}(.,.) + \mathbf{B}_{\mathcal{E}}(.,.)$, where

$$\begin{aligned}
\mathbf{B}_{\mathcal{T}}((\boldsymbol{u}_1, p_l), (\boldsymbol{v}_1, q_l)) &:= \sum_{K \in \mathcal{T}_h} \frac{\alpha_K}{\nu} \left(\chi_h(p_l + \mathbf{x} \cdot (\nabla \boldsymbol{u}_1)\, \Pi_K \boldsymbol{u}_1), \chi_h(q_l + \mathbf{x} \cdot (\nabla \boldsymbol{v}_1)\, \Pi_K \boldsymbol{u}_1)\right)_K \\
&\quad + \frac{\gamma_K}{\nu} (\chi_h(\boldsymbol{u}_1 \cdot \mathbf{x}\, \nabla \cdot \boldsymbol{u}_1), \chi_h(\boldsymbol{u}_1 \cdot \mathbf{x}\, \nabla \cdot \boldsymbol{v}_1))_K \\
\mathbf{B}_{\mathcal{E}}((\boldsymbol{u}_1, p_l), (\boldsymbol{v}_1, q_l)) &:= \sum_{F \in \mathcal{E}_h} \tau_F \left(\Pi_F([\nu\, \partial_{\boldsymbol{n}} \boldsymbol{u}_1 + p_l \mathbf{I} \cdot \boldsymbol{n}]), \Pi_F([\nu\, \partial_{\boldsymbol{n}} \boldsymbol{v}_1 + q_l \mathbf{I} \cdot \boldsymbol{n}])\right)_F ,
\end{aligned}$$

and $\mathbf{F}(\boldsymbol{v}_1, q_l) := (\boldsymbol{f}, \boldsymbol{v}_1)_\Omega + \mathbf{F}_{\mathcal{T}}(\boldsymbol{v}_1, q_l)$, where

$$\mathbf{F}_{\mathcal{T}}(\boldsymbol{v}_1, q_l) := \sum_{K \in \mathcal{T}_h} \frac{\alpha_K}{\nu} (\chi_h(\mathbf{x} \cdot \boldsymbol{f}), \chi_h(q_l + \mathbf{x} \cdot (\nabla \boldsymbol{v}_1)\, \Pi_K \boldsymbol{u}_1))_K \, ,$$

and $\chi_h := \mathbf{I} - \Pi_K$ is the fluctuation operator. The positive piecewise constants α_K and γ_K are given by

$$\alpha_K := \max\{1, Pe_K\}^{-1} \quad \text{and} \quad \gamma_K := \max\left\{1, \frac{Pe_K}{24}\right\}^{-1} , \tag{5}$$

where $Pe_K := \frac{|\boldsymbol{u}_1|_K h_K}{18\,\mu}$ and $|\boldsymbol{u}_1|_K := \frac{\|\boldsymbol{u}_1\|_{0,K}}{|K|^{\frac{1}{2}}}$.

We adapt the stabilization parameter from Barrenechea and Valentin (2010b), given once and for all by $\tau_F = \frac{h_F}{12\,\nu}$ if $\|\boldsymbol{u}_1\|_{0,F} = 0$, else,

$$\tau_F = \frac{1}{2\|\boldsymbol{u}_1\|_{0,F}} - \frac{1}{\|\boldsymbol{u}_1\|_{0,F} Pe_F} + \frac{1}{\|\boldsymbol{u}_1\|_{0,F}(e^{Pe_F} - 1)} , \tag{6}$$

where, for $F = K^+ \cap K^- \in \mathcal{E}_h$,

$$Pe_F = \frac{\|\boldsymbol{u}_1\|_{0,F}\, h_F}{\nu} \, . \tag{7}$$

Remark: For large Pe_F we can approximate τ_F as follows

$$\tau_F \approx \frac{1}{2\|\boldsymbol{u}_1\|_{0,F}} - \frac{1}{\|\boldsymbol{u}_1\|_{0,F} Pe_F} \, . \tag{8}$$

This simplified form helps to avoid overflows in the dominated convection regime (i.e., when Pe_F is large). $\square$

Remarks:

1. The stabilizing terms in (4) include those used in Dohrmann and Bochev (2004) to stabilize the Stokes problem plus terms meant to stabilize the convection. These convective terms are different from those used in the LPS method (see, for example, Braack and Burman (2006)), and this fact appears as the main difference between the present method and previously existing alternatives, such as He and Li (2008) and Ge et al. (2009), for the Navier-Stokes equations.

2. Furthermore, the shape of the fluctuation terms allows us to compute them in an easy way at the element level, without the need of an enrichment of the finite element space as in Ganesan et al. (2008), or any patch-wise computation as in Becker and Braack (2001).

3. When the pressure is approximated by piecewise constant functions, the stabilization of the inf-sup deficiency of the $\mathbb{P}_1^2 \times \mathbb{P}_0$ pair relies on the jump term $\sum_{F\in\mathcal{E}_h} \tau_F\,([p_0],[q_0])_F$, since $(\chi_h(p_0),\chi_h(q_0))_K$ vanishes. In the case of linear discontinuous pressures, the jump terms present a minimal stabilization needed to control a norm of the pressure.

4. The discrete velocity $\boldsymbol{u}_1$ itself is not locally mass conservative, but there is an easy way to post-process it to build a locally mass conservative velocity field. To this end, let $\boldsymbol{\varphi}_F$ be the local basis function for the lowest order Raviart-Thomas finite element space given by $\boldsymbol{\varphi}_F(\mathbf{x}) = \pm\frac{h_F}{2|K|}\,(\mathbf{x}-\mathbf{x}_F)$, and $\mathbf{x}_F$ is the node opposite to the edge F. Let also $\boldsymbol{u}_{nc}$ be the Raviart-Thomas field given by

$$\boldsymbol{u}_{nc} := \sum_{F\subseteq\partial K\cap\Omega} \tau_F\,\Pi_F([\nu\,\partial_{\boldsymbol{n}}\boldsymbol{u}_1 + p_l\mathbf{I}\cdot\boldsymbol{n}]\cdot\boldsymbol{n})\boldsymbol{\varphi}_F\,. \tag{9}$$

Then,

$$\nabla\cdot(\boldsymbol{u}_1+\boldsymbol{u}_{nc})\Big|_K = 0\,,$$

in each $K\in\mathcal{T}_h$. We remark also that, once the discrete solution $(\boldsymbol{u}_1,p_l)$ is computed, the computation of $\boldsymbol{u}_{nc}$ does not involve any further computational cost.

5. The method, as well as the analysis presented below, may be naturally extended to cover affine quadrilateral meshes and three-dimensional problems.

6. The method (4) is based on the one presented in Araya et al. (2009) now modified to handle high Reynolds number flows. □

3. WELL-POSEDNESS

Theoretical aspects of the RELP method for the linearized Navier-Stokes equations using the pairs of interpolation $\mathbb{P}_1/\mathbb{P}_0$ is revisited in this section (see Barrenechea and Valentin (2010b) for further details). It is seen as the model to be solved in a Newton iterative algorithm. It is worth mentioning that similar results may be proved to the others pair of spaces.

First, we recall that the linearized Navier-Stokes or Oseen problem reads: Find $(\boldsymbol{u},p)$ such that

$$\begin{aligned}&(\nabla\boldsymbol{u})\,\mathbf{a}-\nu\,\Delta\boldsymbol{u}\,+\,\nabla p=\,\boldsymbol{f},\quad \nabla\cdot\boldsymbol{u}=\,0 \qquad \text{in}\,\Omega\,,\\ &\boldsymbol{u}=\,\mathbf{0}\quad \text{on}\,\partial\Omega\,,\end{aligned} \tag{10}$$

where, to avoid unnecessary technicalities, we suppose that $\mathbf{a}|_K\in\mathbb{R}^2$ for all $K\in\mathcal{T}_h$, and that $[\mathbf{a}\cdot\boldsymbol{n}]=0$ for each $F\in\mathcal{E}_h$.

Let $\|\cdot\|_h$ be the mesh-dependent norm given by

$$\|(\boldsymbol{v},q)\|_h := \Big[\nu\,|\boldsymbol{v}|_{1,\Omega}^2 + \sum_{K\in\mathcal{T}_h}\frac{\alpha_K h_K^2}{\nu}\left[\|(\nabla\boldsymbol{v})\,\mathbf{a}+\nabla q\|_{0,K}^2 + \frac{\gamma_K\,h_K^2\|\mathbf{a}\|_{0,K}}{\nu}\,\|\nabla\cdot\boldsymbol{v}\|_{0,K}^2\right] + \sum_{F\in\mathcal{E}_h}\tau_F\,\|\Pi_F([\nu\partial_{\boldsymbol{n}}\boldsymbol{v}+q\mathbf{I}\cdot\boldsymbol{n}])\|_{0,F}^2\Big]^{1/2}, \qquad (11)$$

for all $(\boldsymbol{v},q)\in\mathbf{V}_h\times Q_h^0$. The method (4) applied to the weak form of the Oseen problem (10) is well-posed thanks to the following result.

Lemma 1 *There exits a positive constant C, independent of h and ν and $\mathbf{a}$, such that, for all $(\boldsymbol{v}_1,q_0)\in\mathbf{V}_h\times Q_h^0$, it holds*

$$\mathbf{B}((\boldsymbol{v}_1,q_0),(\boldsymbol{v}_1,q_0)) \geq C\,\|(\boldsymbol{v}_1,q_0)\|_h^2\,, \qquad (12)$$

and thus (4) *has an unique solution.*

Proof: The demonstration follows closely Barrenechea and Valentin (2009, 2010b) □.

4. AN A POSTERIORI ERROR ESTIMATOR

In what follows, we sketch some of the main results of Araya et al. (2011) which is a work in progress. First, let us denote

$$\|(\boldsymbol{v},q)\|_\Omega := \left\{\nu\,|\boldsymbol{v}|_{1,\Omega}^2 + \frac{1}{\nu}\,\|q\|_{0,\Omega}^2\right\}^{1/2},$$

and, for each $K\in\mathcal{T}_h$ and $F\in\mathcal{E}_h$, we define the following residuals

$$\mathcal{R}_K := \Big(\boldsymbol{f}+\nu\Delta\boldsymbol{u}_1-(\nabla\boldsymbol{u}_1)\boldsymbol{u}_1-\nabla p_0\Big)\Big|_K, \qquad (13)$$

$$\mathcal{R}_F := [\nu\partial_{\boldsymbol{n}}\boldsymbol{u}_1+p_0\mathbf{I}\cdot\boldsymbol{n}]\Big|_F. \qquad (14)$$

Next, the *a posteriori* estimator is given by

$$\eta := \left\{\sum_{K\in\mathcal{T}_h}\eta_K^2 + \sum_{K\in\mathcal{T}_h}\frac{h_K^2}{\nu}\left[\|(\nabla\boldsymbol{u}_1)\chi_h(\boldsymbol{u}_1)\|_{0,K}^2 + \|\nabla\cdot\boldsymbol{u}_1\,\Pi_K\boldsymbol{u}_1\|_{0,K}^2\right.\right\}^{\frac{1}{2}}, \qquad (15)$$

where,

$$\eta_K^2 := \frac{h_K^2}{\nu}\|\mathcal{R}_K\|_{0,K}^2 + \frac{1}{2}\sum_{F\in\mathcal{E}_h}\frac{h_F}{\nu}\,\|\mathcal{R}_F\|_{0,F}^2 + \nu\,\|\nabla\cdot\boldsymbol{u}_h\|_{0,K}^2\,. \qquad (16)$$

The following result establishes the correspondence between the error and the estimator.

Theorem 2 *Let $(\boldsymbol{u},p)$ be the solution of* (1) *and $(\boldsymbol{u}_1,p_0)$ the solution of* (4) *. Then, the following a posteriori error estimates hold*

$$\|(\boldsymbol{u}-\boldsymbol{u}_1,p-p_0)\|_\Omega \leq C_1\,\max\left\{1,\frac{\|\boldsymbol{u}_1\|_{1,\Omega}}{\nu}\right\}\,\eta \qquad (17)$$

$$\eta_K \leq C_2\,\|(\boldsymbol{u}-\boldsymbol{u}_1,p-p_0)\|_{\omega_K}\,, \qquad (18)$$

where η and η_K are defined, respectively, in (15) *and* (16)*, and the positives constants C_1 and C_2 are independent on h and ν, but can depend on $\boldsymbol{u}$ and p.*

5. NUMERICAL VALIDATIONS

5.1 An analytical solution

Here we consider $\Omega = (0,1) \times (0,1)$, $\nu = 10^{-2}$ and we set $\boldsymbol{f}$, and the boundary conditions such that the exact solution reads

$$\boldsymbol{u}(x,y) = \begin{pmatrix} \frac{1-e^{y/\nu}}{1-e^{1/\nu}} \\ \frac{1-e^{x/\nu}}{1-e^{1/\nu}} \end{pmatrix}, \quad p(x,y) = x - y\,. \tag{19}$$

In Figure 1 we observe optimal convergence histories for the $\mathbb{P}_1/\mathbb{P}_0$ element and for all the variables, a result which anticipates theoretical a priori estimates to be presented in a forthcoming work. Figure 2 points out that the *a posteriori* estimator stays close to the true error as the total of degree of freedom (d.o.f) increases. Moreover, Figure 3 shows that the estimator drives the mesh refinement toward the sharp gradient regions.

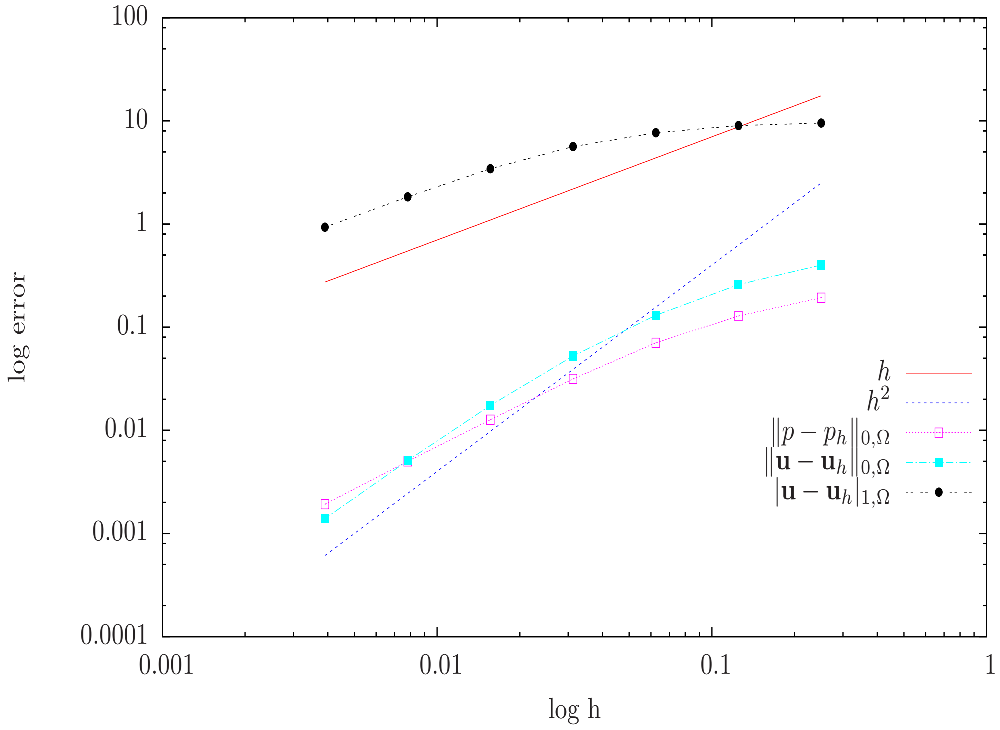

Figure 1: Convergence history for the RELP method.

5.2 The lid-driven cavity problem

This test uses the same domain of the previous section, we set $\boldsymbol{f} = \mathbf{0}$, and the boundary conditions are $\boldsymbol{u} = \mathbf{0}$ on $[\{0\} \times (0,1)] \cup [(0,1) \times \{0\}] \cup [\{1\} \times (0,1)]$ and $\boldsymbol{u} = (1,0)^t$ on $(0,1) \times \{1\}$. Here the viscosity is set as $\nu = \frac{1}{5000}$. In Figures 4 and 5 we depict the adapted mesh and the streamlines of the solution, respectively, using the $\mathbb{P}_1/\mathbb{P}_0$ element. The use of the new method along with the *a posteriori* estimator predicts the regions where the mesh must be refined in order to achieve oscillatory-free solutions.

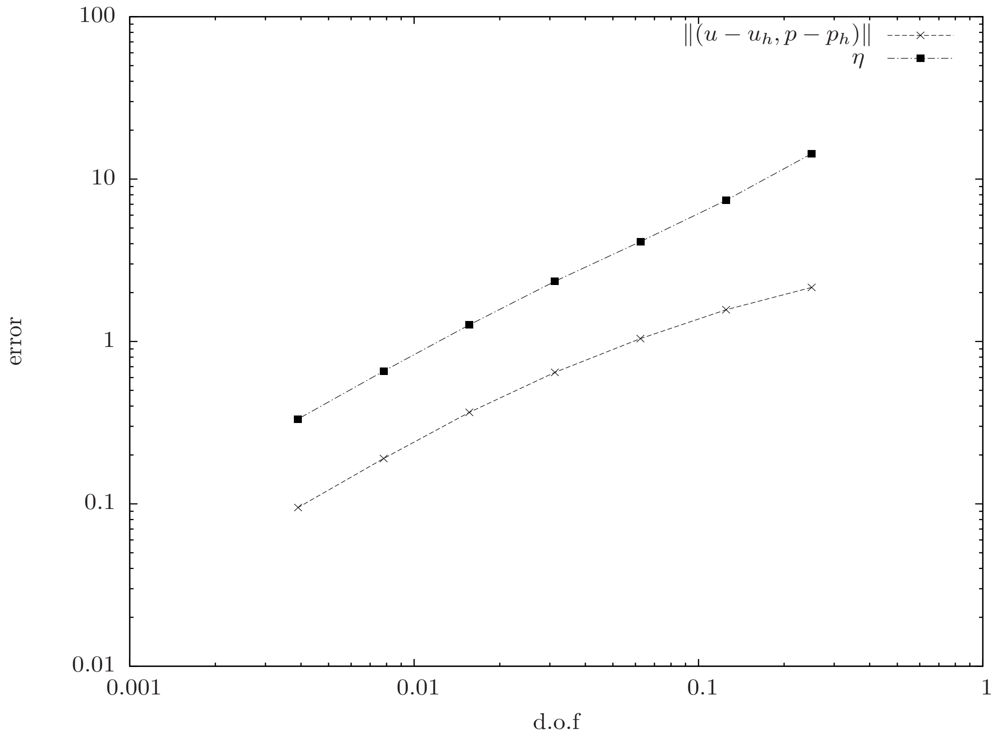

Figure 2: Comparison between the index of effectivity and the error, with respect to the degree of freedom.

6. CONCLUSION

A new parameter-free stabilized method has been developed for the incompressible Navier-Stokes equations. The method recovered stability for the equal order linear interpolation pairs as well as for the simplest element, while induced the right dose of numerical diffusion to capture boundary layers. In addition, we introduce a sharp *a posteriori* error estimate for the fully non-linear Navier-Stokes model. Thereby, the combination of the RELP method and the new *a posteriori* error estimator made the approach a simple low cost and locally conservative alternative to solve the Navier-Stokes equations. Numerical tests validated theoretical results.

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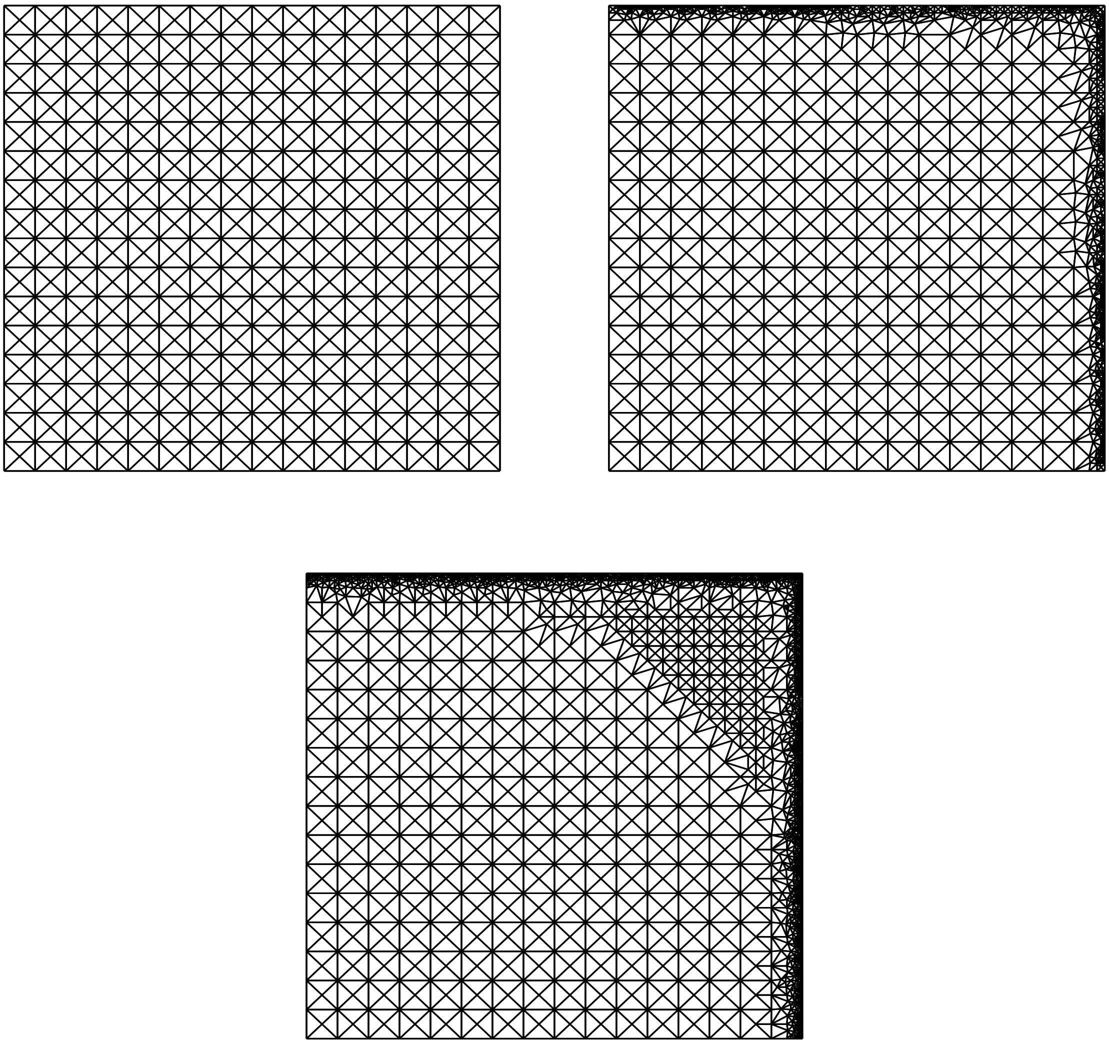

Figure 3: Sequence of adapted meshes.

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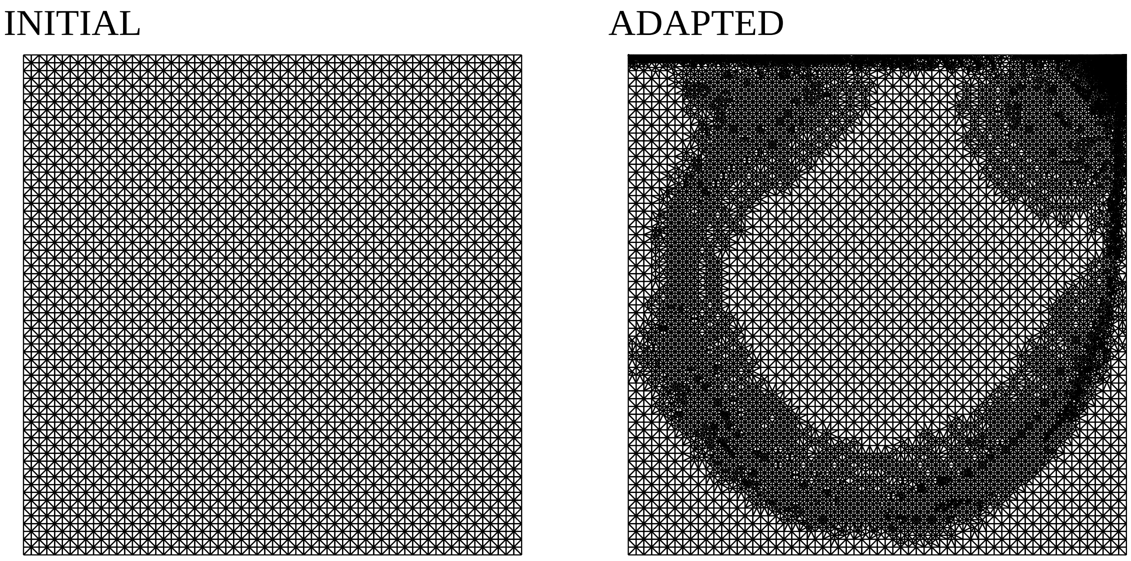

Figure 4: The initial (left) and the adapted (right) meshes.

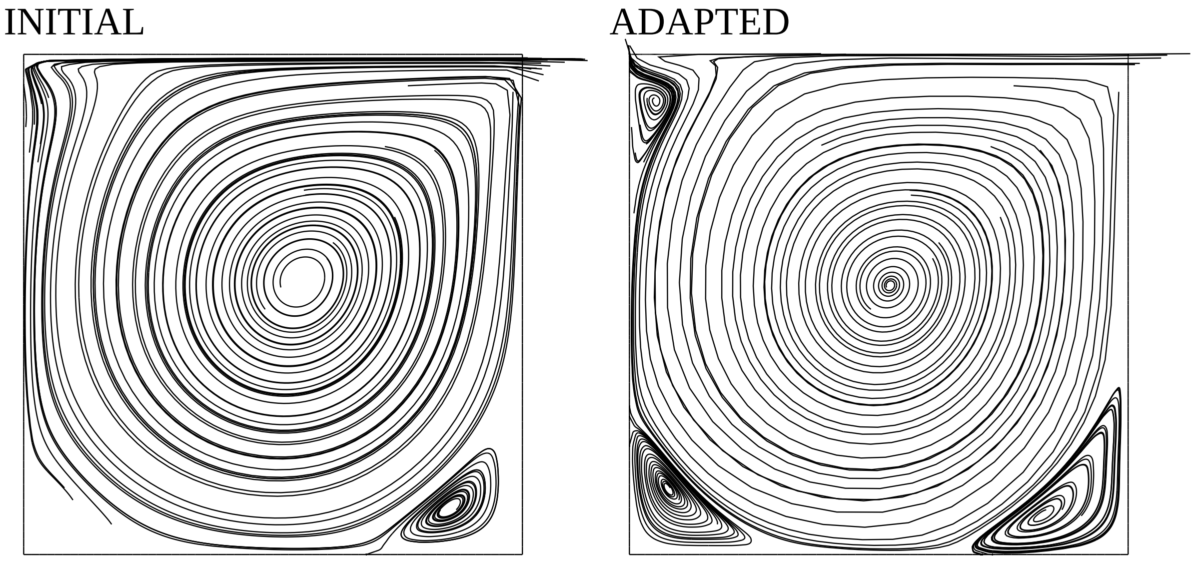

Figure 5: Streamlines using the initial (left) and the adapted (right) meshes