Mecánica Computacional Vol XXIX, págs. 4661-4685 (artículo completo)
Eduardo Dvorkin, Marcela Goldschmit, Mario Storti (Eds.)
Buenos Aires, Argentina, 15-18 Noviembre 2010

A NON-LINEAR STABILIZED FINITE ELEMENT METHOD WITH SHOCK CAPTURING AND REDUCED SMEARING FOR ADVECTION PROBLEMS: THE CONSISTENT SHOCK CAPTURING WITH REDUCED SMEARING METHOD

Eduardo G. Dutra do Carmo[a], **Gustavo B. Alvarez**[b]

[a]*Programa de Engenharia Nuclear, COPPE - Universidade Federal do Rio de Janeiro Ilha do Fundão, 21945-970, P.B. 68509, Rio de Janeiro, RJ, Brazil, carmo@lmn.con.ufrj.br, http://www.con.ufrj.br/*

[b]*Departamento de Ciências Exatas, EEIMVR - Universidade Federal Fluminense Av. dos Trabalhadores, 420, Vila Sta. Cecília, 27255-125, Volta Redonda, RJ, Brazil, benitez.gustavo@gmail.com, http://www.professores.uff.br/gbenitez*

Keywords: Stabilization, GLS, Shock Capturing, Convection Dominated Problems, Convection Diffusion Problem.

Abstract. In previous works the GPR method (Galerkin Projected Residual Method) was introduced. The GPR formulation has been applied with success to Helmholtz problem and to diffusion-reaction singularly perturbed problem. Based on the initial ideas of the GPR method, we developed the Galerkin Symmetrical Projected Residual Method (GSPR) for convection dominated diffusion-convection problems. The GSPR method is a linear finite element method with stabilization properties similar to SUPG method. However, for practical problems this method is not sufficiently stable and accurate.

In this work, we developed the new non-linear stabilized finite element method, based on the ideas of the shock-capturing stabilization. The method introduces new ideas about the upwind function and the stabilizing parameter τ. We observed that the stabilizing parameter is dependent on the degree of the interpolation polynomial, on the geometry of the element, on the advective field β, on the boundary conditions prescribed for the problem on Γ^- and on the value of $meas(\Gamma_e \cap (\Gamma - \Gamma^-))$. A variety of upwind functions can be chosen to improve the spurious oscillations. The strategy to choose the stabilizing parameter is based on numerical experiment and on the requirements that overshooting and undershooting localized in narrow regions along sharp layers are not observed without leading to excessive smearing of the layers (CSCRS - Consistent Shock Capturing with Reduced Smearing Method). Some numerical tests for 2D problems are presented.

1 INTRODUCTION

The advection-diffusion equation model several physical phenomena. The Galerkin finite element method (Galerkin FEM) is often used to obtain numerical solutions for this boundary value problem. In general, only for purely diffusive problem the Galerkin approximate solution is the optimal solution. It is well known that the Galerkin FEM is unstable and inaccurate for this equation with dominate advection (Brooks and Hughes, 1982; Johnson et al. 1984), unless the mesh is very fined. Its numerical solution presents spurious oscillations that not corresponding with the physical solution of the problem. The instability of the Galerkin FEM is due to the lack of control on the gradient of the solution.

Stable and accurate numerical solution for this problem has been one of the largest challenge. A great variety of FEM have been developed to guarantee stability and accuracy, but it is impossible to list all works in this direction (Hughes et al. 1986; Hughes and Mallet, 1986; Galeão and Carmo, 1988; Carmo and Galeão, 1991; Brezzi et al. 1992; Baiocchi et al. 1993; Codina, 1993; Franca and Farhat, 1995; Agarwal and Pinsky, 1996; John et al. 1998; Shih and Elman, 1999; Ramage, 1999; John, 2000; Papastavrou and Verfürth, 2000; Farhat et al. 2001; Hauke and Olivares, 2001; Hauke, 2002; Carmo and Alvarez, 2003; Carmo and Alvarez, 2004; Knobloch, 2006; John and Schmeyer, 2008; Ramakkagari and Flaherty, 2008; Hsu et al. 2010; Hsieh and Yang, 2009; Chiu and Shue, 2009; Tobiska, 2009). Comparisons between different methods and a recent bibliographical review can be found in (Codina, 1998; John and Knobloch, 2007; John and Knobloch, 2008). Many of these attempts have used continuous finite element spaces (Hughes et al. 1989; Franca and carmo, 1989; Hughes, 1995; Oñate, 1998; Ilinca et al. 2000; Franca and Valentin, 2000; Knopp et al. 2002; Nesliturk and harari, 2003; Burman and Hansbo, 2004; Franca et al. 2005; Lube and Rapin, 2006) and discontinuous finite element spaces (Zienkiewicz et al. 2003; Hughes et al 2006; Hughes et al 2006; Gómez et al. 2007) to development of new FEM. The main challenge is to find a consistent formulation in continuous or discontinuous finite element spaces, such that, its approximate solution is stable and accurate. In this paper we will deal only with continuous finite dimensional spaces.

Methods as SUPG (Brooks and Hughes, 1982) and GLS (Hughes et al. 1989) possess control of the derivative in the streamline direction. However, these methods present spurious oscillations for problems with sharp layers. It is well known that other stabilization terms should be added to the SUPG or GLS methods to guarantee larger stability. One common strategy has been to add stabilization terms of the type capture operators. Unfortunately, the stabilized methods based on "capture operators" are non-linear methods even for linear diffusion-advection problem (Hughes and Mallet, 1986; Galeão and Carmo, 1988). In general, these non-linear methods are stable for problems with boundary layers, but they are inaccurate for smooth problem (undesirable crosswind diffusion effect). In (Carmo and Alvarez, 2003; Carmo and Alvarez, 2004) the authors developed the SAUPG and GLSAU methods solving in part the difficulty mentioned above.

The stabilized finite element methods based on capture operators possess besides the control of the derivative in the streamline direction an extra control of the derivative in another direction (generally, the direction of the approximate gradient). This extra control in another direction allows that methods based on capture operator possess larger stability than the linear methods, such as GLS or SUPG methods. On the other hand, these methods based on capture operators lead to excessive smearing of the layers. The reduction of the smearing at internal layers is extremely important and it is the main objective of this paper. Therefore, in this work we should consider advection-reaction problems only. The diffusion-advection problems with boundary layers will be analyzed in another work.

For advection-reaction problems, we developed in this paper the Consistent Shock Capturing with Reduced Smearing method (CSCRS), where the fundamental ideas of the shock capturing methods in (Galeão and Carmo, 1988; Codina, 1993; Carmo and Alvarez, 2003; Carmo and Alvarez, 2004) are preserved. The goal of this method consists in suppressing the spurious oscillations without an excessive smearing of the layers (accuracy at layers). There are different ways of defining criteria to guarantee the goal mentioned above. We introduce new ideas to design the upwind function based on two criteria. The first of them consists of several numerical experiments to determine the relationship between the stabilization parameter and the data of the continuous and discrete problems. The second criterion is to reduce or to eliminate overshooting and undershooting localized in narrow regions along sharp layers, without producing excessive smearing of the layers (CSCRS - Consistent Shock Capturing with Reduced Smearing Method). As outcome we obtain a non-linear FEM with excellent stability and accuracy properties. For problems with sharp layers, the CSCRS method is more stable than the methods given in (Galeão and Carmo, 1988; Codina, 1993; Carmo and Alvarez, 2003; Carmo and Alvarez, 2004) and the sharp gradients at layers are preserved (no excessive crosswind smearing appears). Also, the CSCRS method is accurate like the SUPG or GLS methods for smooth problem.

The paper is organized as follows. The model boundary-value problem and its variational formulation are briefly presented in Section 2. In Section 3, we introduce the stabilized FEM with shock capturing. We design the stabilizing parameter and the upwind function in Section 4. Some numerical experiments to evaluate the performance of the new formulation are presented in Section 5. Finally, Section 6 contains some conclusions and final remarks.

2 STATEMENT OF THE PROBLEM

Let $\Omega \subset \mathrm{IR}^d$ $(d \geq 1)$ be a bounded domain with a Lipschitz continuous boundary Γ and consider the following advection-reaction problem

$$L(\boldsymbol{\beta}, c, u) \equiv \boldsymbol{\beta} \cdot \nabla u + cu = f \quad \text{in } \Omega \tag{1}$$

$$u = g \quad \text{on } \Gamma^- \text{ with } \Gamma^- = \{\mathbf{x} \in \Gamma;\ \boldsymbol{\beta} \cdot \mathbf{n} < 0\} \text{ and } meas(\Gamma^-) > 0\,, \tag{2}$$

where $\mathbf{n}$ denotes the outward normal unit vector defined almost everywhere on Γ and $meas(\circ)$ denotes the Lebesgue measure. We assume the hypotheses that follow.

(H.1a) $\boldsymbol{\beta} \in L^\infty(\Omega)^d$, $\nabla \cdot \boldsymbol{\beta} \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $g \in L^2(\Gamma^-)$,

(H.1b) $-\nabla \cdot \boldsymbol{\beta} + 2c \geq c_0 \geq 0$ a.e. in Ω,

where c_0 denotes a real constant and the spaces $L^2(\Omega)$ and $L^\infty(\Omega)$ are as defined in (Adams, 1975). The weak formulation of the problem defined by (1) and (2) consists of finding $u \in S_g$ satisfying

$$A_G(w,v) \equiv \int_\Omega [\boldsymbol{\beta} \cdot \nabla w + cw] v \, d\Omega = b_G(v) \equiv \int_\Omega f v d\Omega \ \ \forall\, v \in V_g\,, \tag{3}$$

where $S_g = \left\{ w \in H_{\boldsymbol{\beta}}{}^1(\Omega);\, w = g \text{ on } \Gamma^- \right\}$, $V_g = \left\{ w \in H_{\boldsymbol{\beta}}{}^1(\Omega);\, w = 0 \text{ on } \Gamma^- \right\}$ and $H_{\boldsymbol{\beta}}{}^1(\Omega) = \left\{ w \in L^2(\Omega);\, \boldsymbol{\beta} \cdot \nabla w \in L^2(\Omega) \right\}$.

3 STABILIZED FEM WITH SHOCK CAPTURING

Here we briefly present the theoretical background for the stabilized FEM with shock capturing. With this goal, consider $\Omega^h = \{\Omega_1, \ldots, \Omega_{ne}\}$ as being a partition of Ω into non degenerated finite elements Ω_e, such that Ω_e can be mapped in standard elements by isoparametric mappings satisfying $\Omega_e \cap \Omega_{e'} = \emptyset$ if $e \neq e'$ and $\Omega \cup \Gamma = \bigcup_{e=1}^{ne} (\Omega_e \cup \Gamma_e)$, where Γ_e denotes the boundary of Ω_e. Let $k \geq 1$ be an integer and consider $P^k(\Omega_e)$ as being the space of polynomials of degree less than or equal to k in the local coordinates. Let $H^{h,k} = \{\eta \in C^0(\Omega \cup \Gamma); \eta_e \in P^k(\Omega_e)\}$, $S^{h,k} = \{\eta \in H^{h,k}; \eta = g^h \text{ on } \Gamma_g\}$, $V^{h,k} = \{\eta \in H^{h,k}; \eta = 0 \text{ on } \Gamma_g\}$ are the finite dimension spaces and g^h the usual interpolate of g, where $C^0(\Omega \cup \Gamma)$ is as given in (Adams, 1975). We assume that $\boldsymbol{\beta}$, c and f are continuous in Ω_e and we defined the mesh parameters

$$\bar{h}_e = \begin{cases} \left[\int_{\Omega_e} d\Omega \right]^{\frac{1}{d}} & \text{if } \Omega_e \text{ is a quadrilateral or a hexahedron,} \\ 2\left[\int_{\Omega_e} d\Omega \right]^{\frac{1}{d}} & \text{if } \Omega_e \text{ is a triangle,} \\ 6\left[\int_{\Omega_e} d\Omega \right]^{\frac{1}{d}} & \text{if } \Omega_e \text{ is a tetrahedron,} \end{cases} \tag{4}$$

$$h_{e,*} = 2\frac{|\boldsymbol{\beta}|}{\mathbf{J}_e^{-1}\boldsymbol{\beta}} \text{ and } h_e = \sup\{\bar{h}_e, h_{e,*}\}, \tag{5}$$

where $\mathbf{J}_e^{-1}$ is the inverse Jacobian matrix and "*sup*" denotes supremum.

Let $\tau^{e,GLS}$ be the stabilization parameter of the GLS method and P_{reac} another parameter defined as

$$\tau^{e,GLS} = \frac{h_e}{2|\boldsymbol{\beta}|}, \quad P_{reac} = \frac{h_e c}{2|\boldsymbol{\beta}|}. \tag{6}$$

In this paper, we assume $P_{reac} << 1$, i.e., advection-reaction problems with dominated advection. To define the stabilization parameter we consider $\tau^e(w, \boldsymbol{\beta})$ as being a dimensionless real function that depend continuously on $w \in \mathrm{IR}$ for each $\boldsymbol{\beta}$ fixed, such that $\forall \boldsymbol{\beta}$ we have $0 < \tau^e(w, \boldsymbol{\beta}) \leq 1$ if $0 \leq w \leq 1$. We need the spaces $H^{\infty, L(\boldsymbol{\beta}, c, \circ), e} = \left\{ w \in H^1_{\boldsymbol{\beta}}(\Omega_e); L(\boldsymbol{\beta}, c, \circ) \in L^\infty(\Omega_e) \right\}$ and $H^{\infty, L(\boldsymbol{\beta}, c, \circ)} = \left\{ w \in H^1_{\boldsymbol{\beta}}(\Omega); w_e \in H^{\infty, L(\boldsymbol{\beta}, c, \circ), e} \right\}$. We consider $F^{e,*}$ and F^e as being two functions from $H^{\infty, L(\boldsymbol{\beta}, c, \circ), e}$ into IR and given as follow

$$F^{e,*}(w) = \begin{cases} \dfrac{|L(\boldsymbol{\beta}, c, w) - f|}{|\boldsymbol{\beta}||\nabla w|} & \text{if } |\boldsymbol{\beta}||\nabla w| \neq 0, \\ 0 \quad \text{if } |\boldsymbol{\beta}||\nabla w| = 0, \end{cases} \tag{7}$$

$$F^e(w) = \begin{cases} F^{e,*}(w) & \text{if } F^{e,*}(w) \le 1, \\ 1 \ \text{ if } F^{e,*}(w) > 1. \end{cases} \tag{8}$$

Let $\gamma^{e,k,\boldsymbol{\beta}}$ be a positive real constant independent of h_e and possibly dependent of the data of the continuous problem (such as the advection field $\boldsymbol{\beta}$ and the boundary conditions on Γ^-) and of the discrete problem (such as the geometry of the element, the degree of the interpolating polynomial and the usual aspect ratio). It can be verified that the function "$|\boldsymbol{\beta}|^2 \gamma^{e,k,\boldsymbol{\beta}} \tau^e(F^e(w,\boldsymbol{\beta}))\tau^{e,GLS}$" satisfies the condition (H.4)* given in (Knopp et al. 2002) $\forall w \in H^{\infty,L(\boldsymbol{\beta},c,\circ),e}$ and $\forall \Omega_e$. This condition demands that the capture term is non-negative and has an upper bound.

By following the reference (Codina, 1993) we defines the following $d\times d$ matrix

$$\mathbf{D}^e = \mathbf{I} - \frac{\boldsymbol{\beta}\times\boldsymbol{\beta}}{|\boldsymbol{\beta}|}, \tag{9}$$

where "**I**" denotes the identity matrix and "×" denotes the tensorial product between vectors of IR^d.

Finally, a general class of stabilized FEM with shock capture can be obtained through the variational formulation that follows. Find $u^h \in S^{h,k}$ satisfying the variational equation

$$A_{GLS}(u^h, v^h) + A_{SC}(u^h; u^h, v^h) = b_{GLS}(v^h) \ \ \forall \ v^h \in V^{h,k}, \tag{10}$$

$$A_{GLS}(w,v) \equiv A_G(w,v) + \sum_{e=1}^{ne} \int_{\Omega_e} \tau^{e,GLS} L(\boldsymbol{\beta}, c, w_e) L(\boldsymbol{\beta}, c, v_e)\, d\Omega, \tag{11}$$

$$A_{SC}(w;\eta,v) \equiv \sum_{e=1}^{ne} \int_{\Omega_e} |\boldsymbol{\beta}|^2 \gamma^{e,k,\boldsymbol{\beta}} \tau^e(F^e(w_e,\boldsymbol{\beta}))\tau^{e,GLS} [D^e \nabla \eta_e \cdot \nabla v_e \cdot]\, d\Omega, \tag{12}$$

$$b_{GLS}(v) \equiv b_G(v) + \sum_{e=1}^{ne} \int_{\Omega_e} \tau^{e,GLS} f L(\boldsymbol{\beta}, c, v_e)\, d\Omega, \tag{13}$$

where $(w,\eta,v) \in (H^{\infty,L(\boldsymbol{\beta},c,\circ)})^3$ and w_e, v_e and η_e are the respective restrictions of w, v and η to Ω_e. This method satisfies all the conditions of the theorem 3.5 given in (Knopp et al. 2002).

An important point of this stabilized method is the optimal choice of the stabilization parameter. It is well known that the stability and accuracy of this method depends on this choice. Until the present moment, no theoretical support exists to determine the stabilization parameter that can be optimum in some sense. A great amount of papers has been devoted to the choice of the stabilization parameter. Some works in this direction can be found in (Brezzi et al. 1992; John and Knobloch, 2007; John and Knobloch, 2008; Hughes, 1995; Christie and Mitchell, 1978; Mizukami, 1985; Stynes and Tobiska, 1995; Roos et al. 1996; Ramage, 1999; Fischer et al. 1999; Harari et al. 2001; Elman and Ramage, 2002; Principe and Codina, 2010; Carmo et al. 2008; Carmo et al. 2008) for finite elements of first order and in (Carmo and Alvarez, 2004; Heinrich, 1980; Codina et al. 1992; Almeida and Silva, 1997; Tezduyar and Osawa, 2000; Akin et al. 2003; Galeão et al. 2004; Akin and Tezduyar, 2004) for finite elements of higher order. Many of these works are based on analysis done for the one-dimensional case. Other attempts are based on heuristic, intuitive and experimental criteria. A recent and good bibliographical review about the stabilization parameter for several finite element methods can be found in (John and Knobloch, 2007).

4 A NEW SCHEME FOR THE CONSTRUCTION OF THE PARAMETRIC FUNCTION " $\gamma^{e,k,\boldsymbol{\beta}}\tau^e(F^e(w,\boldsymbol{\beta}))$ "

The main objective of this section consists of determining the relationship between the stabilization parameter and the data of the problem. Several numerical experiments suggest that the stabilization parameter depends on the data of the continuous problem (such as the advection field $\boldsymbol{\beta}$ and the boundary conditions on Γ^-) and on the discrete problem (such as the degree of the interpolating polynomial, the geometry of the element and the aspect ratio of the element). The construction of a suitable parameter function will be based on the reduction of spurious oscillations and on the reduction of the smearing at internal layers as described in (John and Knobloch, 2007; John and Knobloch, 2008). Since spurious oscillations are far more undesirable than moderately smeared layers, spurious oscillations will be weighted higher. Another requirement is to maintain the order of approach of the GLS method for smooth problems. We note that for each $w \in H^{\infty,L(\boldsymbol{\beta},c,\circ)}$ a variety of functions can be chosen to represent $\tau^e(F^e(w_e,\boldsymbol{\beta}))$ satisfying the requirements mentioned above. The class of functions of the type $\left[Q^{e,\tau}(F^e(w)\right]^{P^{e,\tau}(w,\boldsymbol{\beta})}$ where $P^{e,\tau}(w,\boldsymbol{\beta})>0$ and $Q^{e,\tau}(F^e(w))$ is a polynomial in $F^e(w)$, is one of the possible functions that can satisfy the demands above, since the necessary information for this, can be inserted in the definition of the parameters through an appropriate numerical experimentation. Inspired in the VCAU method given in (Carmo and Galeão, 1991), SAUPG method given in (Carmo and Alvarez, 2003) and in the method given in (Codina, 1993) we propose the following function

$$\tau^e(F^e(w,\boldsymbol{\beta})) = \left[Q^{e,\tau}(F^e(w))\right]^{P^{e,\tau}(w,\boldsymbol{\beta})}, \tag{14}$$

$$Q^{e,\tau}(F^e(w)) = [1-\lambda]Q^{e,\tau,0}(F^e(w)) + \lambda F^e(w), \tag{15}$$

$$Q^{e,\tau,0}(F^e(w)) = F^e(w) + \sum_{i=2}^{N_\tau}\left[-1\right]^{[i-1]} C^{e,i}[F^e(w)]^i, \tag{16}$$

where $w \in H^{\infty,L(\boldsymbol{\beta},c,\circ)}$, $0 \leq \lambda \leq 1$ is a parameter to be determined, $N_\tau \geq 2$, $C^{e,i}=0$ or $C^{e,i}=1$ and $P^{e,\tau}(w,\boldsymbol{\beta})>0 \quad \forall w \in H^{\infty,L(\boldsymbol{\beta},c,\circ)}$ and $\forall\boldsymbol{\beta}$. In general, these parameters and $\gamma^{e,k,\boldsymbol{\beta}}$ depend on the data of the continuous problem and on the discrete problem as was already commented previously. From conditions $0 \leq \lambda \leq 1$, $N_\tau \geq 2$, $C^{e,i}=0$ or $C^{e,i}=1$ and from the definition of $Q^{e,\tau}(F^e(w))$ follows that $Q^{e,\tau}(F^e(w)) \leq 1 \quad \forall w \in H^{\infty,L(\boldsymbol{\beta},c,\circ)}$.

In order to incorporate in $\tau^e(F^e(\circ),\circ)$ the information of the regularity of the solution we defined $\forall w \in L^1(\Omega_e)$ the function

$$\overline{W}^{e}(w) = \frac{\int_{\Omega_e} |w| \, d\Omega}{\int_{\Omega_e} d\Omega}, \tag{17}$$

and $\forall w \in H^{\infty, L(\boldsymbol{\beta}, c, \circ)}$ with $\mu \geq 0$ we define the set

$$\Pi^{h}(w, \mu) = \left\{ \Omega_e \in \Omega^{h}; \overline{W}^{e}(F^{e}(w)) > \mu \right\}. \tag{18}$$

For each $w \in H^{\infty, L(\boldsymbol{\beta}, c, \circ)}$ and for $i=0$ we still needed to define the following parametric functions

$$N^{0}(w) = \text{ number of elements of } \Pi^{h}(w, 0), \tag{19}$$

$$M_0^0(w) = \begin{cases} \frac{1}{N^0(w)} \sum_{\Omega_e \in \Pi^h(w,0)} \overline{W}^{e}(F^{e}(w)) \text{ if } N^0(w) \geq 1, \\ \varepsilon \text{ if } N^0(w) = 0, \end{cases} \tag{20}$$

$$M_1^0(w) = \begin{cases} \frac{1}{N^0(w)} \sum_{\Omega_e \in \Pi^h(w,0)} \left[\overline{W}^{e}(F^{e}(w)) \right]^{-1} \text{ if } N^0(w) \geq 1, \\ \frac{1}{M_0^0(w)} \text{ if } N^0(w) = 0, \end{cases} \tag{21}$$

$$\left[\sigma^0(w) \right]^2 (w) = \begin{cases} \frac{1}{N^0(w)} \sum_{\Omega_e \in \Pi^h(w,0)} \left[\overline{W}^{e}(F^{e}(w)) - M_0^0(w) \right]^2 \text{ if } N^0(w) \geq 1, \\ 0 \text{ if } N^0(w) = 0, \end{cases} \tag{22}$$

$$\mu^0(w) = \begin{cases} M_0^0(w) \frac{\left| M_0^0(w) - \sigma^0(w) \right|}{\left| M_0^0(w) + \sigma^0(w) \right|} \text{ if } \left| M_0^0(w) + \sigma^0(w) \right| > 0, \\ \varepsilon \text{ if } \left| M_0^0(w) + \sigma^0(w) \right| = 0, \end{cases}, \tag{23}$$

$$M_2^0(w) = \inf \left\{ M_1^0(w), \frac{4}{M_0^0(w)} \right\}, \tag{24}$$

$$\aleph^0(w) = \inf \left\{ M_0^0(w), \sigma^0(w) \right\}, \tag{25}$$

$$E_{\aleph,0}^0(w) = M_2^0(w) \frac{\mu^0(w)}{M_0^0(w)}, \tag{26}$$

$$E_{\aleph,1}^0(w) = E_{\aleph,0}^0(w) \left[\frac{\mu^0(w)}{M_0^0(w)} \right]^{\aleph^0(w)}, \tag{27}$$

$$E_{\aleph,2}^0(w) = \sup \left\{ 0, E_{\aleph,1}^0(w) - E_{smooth} \right\}, \tag{28}$$

$$E^0_{\aleph,3}(w) = E^0_{\aleph,2}(w) M^0_2(w)\,, \tag{29}$$

$$P^{e,\tau,0}(w) = \begin{cases} \varepsilon^{-1} \text{ if } E^0_{\aleph,3} > 200 \text{ and } M^0_2(w) > 10, \\ \left[M^0_2(w)\right]^{200} \text{ if } E^0_{\aleph,3} > 200 \text{ and } M^0_2(w) \le 10, \\ \left[M^0_2(w)\right]^{E^0_{\aleph,3}} \text{ if } E^0_{\aleph,3} \le 200 \text{ and } M^0_2(w) \le 10, \\ \left[10\right]^{E^0_{\aleph,3}} \text{ if } E^0_{\aleph,3} \le 200 \text{ and } M^0_2(w) > 10, \end{cases} \tag{30}$$

and for other integers $i \ge 1$ we define the next parametric functions

$$N^i(w) = \text{ number of elements of } \Pi^h(w, \mu^{i-1}(w))\,, \tag{31}$$

$$M^i_0(w) = \begin{cases} \frac{1}{N^i(w)} \sum_{\Omega_e \in \Pi^h(w,\mu^{i-1})} \overline{W}^e(F^e(w)) \text{ if } N^i(w) \ge 1, \\ \varepsilon \text{ if } N^i(w) = 0, \end{cases} \tag{32}$$

$$M^i_1(w) = \begin{cases} \frac{1}{N^i(w)} \sum_{\Omega_e \in \Pi^h(w,\mu^{i-1})} \left[\overline{W}^e(F^e(w))\right]^{-1} \text{ if } N^i(w) \ge 1, \\ \frac{1}{M^i_0(w)} \text{ if } N^i(w) = 0, \end{cases} \tag{33}$$

$$\left[\sigma^i(w)\right]^2(w) = \begin{cases} \frac{1}{N^i(w)} \sum_{\Omega_e \in \Pi^h(w,\mu^{i-1})} \left[\overline{W}^e(F^e(w)) - M^i_0(w)\right]^2 \text{ if } N^i(w) \ge 1, \\ 0 \text{ if } N^i(w) = 0, \end{cases} \tag{34}$$

$$\mu^i(w) = \begin{cases} M^i_0(w) \frac{\left|M^i_0(w) - \sigma^i(w)\right|}{\left|M^i_0(w) + \sigma^i(w)\right|} \text{ if } \left|M^i_0(w) + \sigma^i(w)\right| > 0, \\ \varepsilon \text{ if } \left|M^i_0(w) + \sigma^i(w)\right| = 0, \end{cases} \tag{35}$$

$$M^i_2(w) = \inf\left\{M^i_1(w), \frac{4}{M^i_0(w)}\right\}, \tag{36}$$

$$\aleph^i(w) = \inf\left\{M^i_0(w), \sigma^i(w)\right\}, \tag{37}$$

$$E^i_{\aleph,0}(w) = M^i_2(w) \frac{\mu^i(w)}{M^i_0(w)}\,, \tag{38}$$

$$E^i_{\aleph,1}(w) = E^i_{\aleph,0}(w) \left[\frac{\mu^i(w)}{M^i_0(w)}\right]^{\aleph^i(w)}\,, \tag{39}$$

$$E^i_{\aleph,2}(w) = \sup\left\{0, E^i_{\aleph,1}(w) - E_{smooth}\right\}, \tag{40}$$

$$E^i_{\aleph,3}(w) = E^i_{\aleph,2}(w) M^i_2(w)\,, \tag{41}$$

$$P^{e,\tau,i}(w) = \begin{cases} \varepsilon^{-1} \text{ if } E^i_{\aleph,3} > 200 \text{ and } M^i_2(w) > 10, \\ \left[M^i_2(w)\right]^{200} \text{ if } E^i_{\aleph,3} > 200 \text{ and } M^i_2(w) \le 10, \\ \left[M^i_2(w)\right]^{E^i_{\aleph,3}} \text{ if } E^i_{\aleph,3} \le 200 \text{ and } M^i_2(w) \le 10, \\ [10]^{E^i_{\aleph,3}} \text{ if } E^i_{\aleph,3} \le 200 \text{ and } M^i_2(w) > 10, \end{cases} \tag{42}$$

where ε represents a prescribed infinitesimal ($\varepsilon = 10^{-14}$ for a PC). The symbol E_{smooth} represents a positive real constant such that if $E^i_{\aleph,1}(w) > E_{smooth}$, then the solution is considered smooth. Numerical experiments suggest i=1, $E_{smooth} = 1$ for mesh with quadrilateral or hexahedron elements and $E_{smooth} = 0.5$ for mesh with triangle or tetrahedron elements. Therefore, for each $w \in H^{\infty, L(\boldsymbol{\beta}, c, \circ)}$ we propose

$$P^{e,\tau}(w, \boldsymbol{\beta}) = F^e_0(\boldsymbol{\beta}) P^{e,\tau,1}(w)\,, \tag{43}$$

where $F^e_0(\boldsymbol{\beta}) \ge 1$ is a factor that is independent of w and dependent of the data of the continuous problem and of the discrete problem.

We note that in every Ω_e should have $0 < Q^{e,\tau}(F^e(u^{h,GLS}_e)) < 1$ and $P^{e,\tau}(u^{h,GLS}_e, \boldsymbol{\beta}) << 1$ for smooth problems, where $u^{h,GLS}_e$ denotes the solution of the GLS method. Therefore, in general, $\tau^e(F^e(u^{h,GLS}_e), \boldsymbol{\beta}) << 1 \;\; \forall \Omega_e$ in smooth problems and the solution u^h is very similar to $u^{h,GLS}$ since the capture operator is very small. For problems with internal layers (no smooth solution) we should have $E^i_{\aleph,1}(w) \le E_{smooth}$, and therefore, $P^{e,\tau}(u^h, \boldsymbol{\beta}) = F^e_0(\boldsymbol{\beta}) \;\; \forall \Omega_e$. In this case, u^h is different of $u^{h,GLS}$ since the capture operator is not disabled. Therefore, the function $P^{e,\tau}(\circ, \boldsymbol{\beta})$ defined above, allows that the function $\tau^e(F^e(\circ), \boldsymbol{\beta})$ possesses information of the regularity of the problem through the function $P^{e,\tau,1}(\circ)$.

In order to incorporate in $P^{e,\tau}(w, \boldsymbol{\beta})$ information from the advection field $\boldsymbol{\beta}$ and from the geometry of the element, $\forall \Omega_e$ we defined

$$\Upsilon^{e,-} = \int_{\Gamma^-_e} d\Gamma\,, \qquad \Upsilon^{e,+} = \int_{\Gamma^+_e} d\Gamma\,, \tag{44}$$

$$\Upsilon^{e,\max} = \sup\left\{ meas(\Gamma_{e,i});\quad i = 1, \ldots, N^e_{face} \right\}, \tag{45}$$

$$\Gamma^-_e = \{\mathbf{x} \in \Gamma_e;\; \boldsymbol{\beta} \cdot \mathbf{x} < 0\}\;\;,\quad \Gamma^+_e = \Gamma_e - \Gamma^-_e\;\;, \tag{46}$$

$$F^e_1(\boldsymbol{\beta}) = \begin{cases} F^{e,T}_1(\boldsymbol{\beta}) \text{ if } \Omega_e \text{ is a triangle or tetrahedron,} \\ F^{e,Q}_1(\boldsymbol{\beta}) \text{ if } \Omega_e \text{ is a quadrilateral or hexahedron,} \end{cases} \tag{47}$$

$$F_1^{e,Q}(\boldsymbol{\beta}) = \begin{cases} 1 \text{ if } \Upsilon^{e,-} \geq \Upsilon^{e,+}, \\ \left[\dfrac{\Upsilon^{e,-}}{\Upsilon^{e,+}} \right]^{\left[1+\frac{\Upsilon^{e,-}}{\Upsilon^{e,+}}\right]} \text{ if } \Upsilon^{e,-} < \Upsilon^{e,+}, \end{cases} \tag{48}$$

$$F_1^{e,T}(\boldsymbol{\beta}) = F_{D,1}^{e,T}(\boldsymbol{\beta}, D^{e,T}) \begin{cases} 1 \text{ if } (\Upsilon^{e,\max} \neq \Upsilon^{e,-} \text{ and } \Upsilon^{e,\max} \neq \Upsilon^{e,+}) \text{ or } \Upsilon^{e,-} \geq \Upsilon^{e,+}, \\ \left[\dfrac{\Upsilon^{e,-}}{\Upsilon^{e,+}} \right]^{\left[1+\frac{\Upsilon^{e,-}}{\Upsilon^{e,+}}\right]} \text{ otherwise}, \end{cases} \tag{49}$$

$$D^{e,T} = \begin{cases} \dfrac{\left[2+\sqrt{2}\right]\bar{h}_e}{\int_{\Gamma_e^-} d\Gamma} \text{ if } \Omega_e \text{ is a triangle}, \\ \dfrac{\left[3+\sqrt{3}\right]\left[\bar{h}_e\right]^2}{2\int_{\Gamma_e^-} d\Gamma} \text{ if } \Omega_e \text{ is a tetrahedron}, \end{cases} \tag{50}$$

$$F_0^e(\boldsymbol{\beta}) = F_1^e(\boldsymbol{\beta}) F_{std,1}^{e,k}(\alpha^{e,\inf}), \tag{51}$$

$$\alpha^{e,\inf} = \inf\left\{ \alpha^{e,i}; \quad \Gamma_{e,i} \subset \Gamma_e^{-,0}, \ i = 1,\ldots, N_{face}^e \right\}, \tag{52}$$

$$\Gamma_e^{-,0} = \{\mathbf{x} \in \Gamma_e; \ \boldsymbol{\beta} \cdot \mathbf{x} \leq 0\} , \tag{53}$$

$$\alpha^{e,i} = \frac{\int_{\Gamma_{e,i}} \chi^{e,i}(\boldsymbol{\beta})\, d\Omega}{\int_{\Gamma_{e,i}} d\Omega} \text{ with } i = 1,\ldots, N_{face}^e, \tag{54}$$

$$\chi^{e,i}(\boldsymbol{\beta}) = \begin{cases} \dfrac{|\boldsymbol{\beta} \cdot \mathbf{n}_e|}{|\boldsymbol{\beta}|} \text{ if } |\boldsymbol{\beta}| > 0, \\ 0 \text{ if } |\boldsymbol{\beta}| = 0, \end{cases} , \tag{55}$$

where N_{face}^e denotes the number of faces of Ω_e, $\Gamma_{e,i}$ denotes the face of number "i", $\mathbf{n}_e$ denotes the outward normal unit vector defined almost everywhere on Γ_e. The function $F_{std,1}^{e,k}(\circ)$ is based on a standard isoparametric element, where k denotes the degree of the interpolating polynomial. This function should be obtained through appropriate numerical experiments. $F_{D,1}^{e,T}(\circ,\circ)$ is a function that carry out the information of the distortion of the element in relation to the corresponding standard element. It is known that higher accuracy can be obtained with triangular mesh when the advection field is aligned with one face of the triangle (Iliescu, 1999; Skalický and Roos, 1999). Therefore, it is desirable to have some indicator of alignment of the advection field $\boldsymbol{\beta}$ with the face of larger measure for triangle or tetrahedron. We remark that " $\Upsilon^{e,\max} \neq \Upsilon^{e,-}$ and $\Upsilon^{e,\max} \neq \Upsilon^{e,+}$ " can be used as an indicator of this alignment.

There are different ways of defining numerical experiments to determine the function $F^{e,k}_{std,1}(\circ)$. We considered that good choices are the numerical experiments based on the following advection problem

$$L(\boldsymbol{\beta},0,u) = \boldsymbol{\beta}_{test} \cdot \nabla u = 0 \quad \text{in } \Omega_{test} = (0,1)\times(0,1)\,, \tag{56}$$

$$u = g_{test} \quad \text{on } \Gamma^-_{test} \text{ with } \Gamma^-_{test} = \{\mathbf{x} \in \Gamma_{test};\ \boldsymbol{\beta}_{test} \cdot \mathbf{n}_{test} < 0\}\ , \tag{57}$$

$$g_{test} = \begin{cases} 1 \text{ if } 0 \le x \le 1 \text{ and } y = 1, \\ 1 \text{ if } 0.65 \le y < 1 \text{ and } x = 0, \\ 0 \text{ if } 0 \le y < 0.65 \text{ and } x = 0, \end{cases} \tag{58}$$

$$\boldsymbol{\beta}_{test} = (\cos(\theta_{test}),\text{-}\sin(\theta_{test})) \text{ with } \theta_{test} \in [0, \tfrac{\pi}{2}]\,, \tag{59}$$

where Γ_{test} denotes the boundary of Ω_{test} and $\mathbf{n}_{test}$ denotes the outward normal unit vector defined almost everywhere on Γ_{test}.

The numerical tests are accomplished for a uniform partition $N_{test} \times N_{test}$ of Ω_{test} in square elements, where $N_{test} = 40$ was the adopted value. Therefore, the standard isoparametric element used to determine the function $F^{e,k}_{std,1}(\circ)$ is the square element.

Chosen the criterion to determine $F^{e,k}_{std,1}(\circ)$, the next step consists of specifying the functions $Q^{e,\tau}(F^e(\circ))$ and $F^{e,T}_{D,1}(\circ,\circ)$ as also the parameter $\gamma^{e,k,\boldsymbol{\beta}}$. Inspired in the VCAU method given in (Carmo and Galeão, 1991) and in the SAUPG method given in (Carmo and Alvarez, 2003), we specified $N_\tau = 2$ and $C^{e,2} = 1$ and have the function $Q^{e,\tau,0}(F^e(\circ))$ defined $w \in H^{\infty,L(\boldsymbol{\beta},c,\circ)}$ as follows

$$Q^{e,\tau,0}(F^e(w_e)) = F^e(w_e) - [F^e(w_e)]^2\,. \tag{60}$$

With the objective that the parameter $\gamma^{e,k,\boldsymbol{\beta}}$ possesses in himself the information of the geometry of the element and of the smoothness of the boundary conditions on Γ^-_{test}, we propose the parameter $\gamma^{e,k,\boldsymbol{\beta}}$ given below

$$\gamma^{e,k,\boldsymbol{\beta}} = \gamma^{geom,e}\gamma_1^{e,k,\boldsymbol{\beta}}(\alpha^{e,\inf}, \overline{\omega}^e)\,, \tag{61}$$

$$\gamma_1^{e,k,\boldsymbol{\beta}}(\alpha^{e,\inf}, \overline{\omega}^e) = \begin{cases} 1 \text{ if } \overline{\omega}^e = 0, \\ \gamma^{-,e,k,\boldsymbol{\beta}}(\alpha^{e,\inf}) \text{ if } \overline{\omega}^e = 1, \end{cases} \tag{62}$$

$$\overline{\omega}^e = \begin{cases} 0 \text{ if } \Gamma_e \cap \Gamma^- = \emptyset \text{ or data on } \Gamma_e \cap \Gamma^- \text{ are specified as smooth,} \\ 1 \text{ if } \Gamma_e \cap \Gamma^- \neq \emptyset \text{ or data on } \Gamma_e \cap \Gamma^- \text{ are specified as non-smooth,} \end{cases} \tag{63}$$

where $\alpha^{e,\inf}$ is another parameters to be determined.

The function $\gamma^{-,e,k,\boldsymbol{\beta}}(\alpha^{e,\inf})$ is determined through numerical experiments for the advection problem above, where $\overline{\omega}^e = 1$ is specified in every element such that the point $(0, 0.65) \in \Gamma_e$ and $\overline{\omega}^e = 0$ in the other elements. The parameter $\gamma^{geom,e}$ possesses in himself the information of the geometry of the element.

The dependence of $F_{D,1}^{e,T}(\circ,\circ)$ on the field $\boldsymbol{\beta}$ was observed as significant for some meshes where a strong distortion of the elements and strong degree of nonalignment with the field $\boldsymbol{\beta}$ were present. Already the dependence of the distortion $D^{e,T}$ was observed for all meshes with distorted elements. Several functions were experimented and the better performance was observed for

$$F_{D,1}^{e,T}(\boldsymbol{\beta}, D^{e,T}) = \begin{cases} 1.1 \text{ if } D^{e,T} < 0.85, \\ -\frac{2}{3} D^{e,T} + \frac{5}{3} \text{ if } 0.85 \le D^{e,T} \le 1, \\ 1.0 \text{ if } D^{e,T} > 1. \end{cases} \tag{64}$$

From definitions above follows that

$$F_{std,1}^{e,k}(\alpha^{e,\inf}) = \begin{cases} F_{std,1}^{e,k,0}(\alpha^{e,\inf}) \text{ if } \alpha_0^{e,\inf} \le \alpha^{e,\inf} \le 1, \\ F_{std,1}^{e,k,0}(\alpha_0^{e,\inf}) \text{ if } 0 \le \alpha^{e,\inf} < \alpha_0^{e,\inf}, \end{cases} \tag{65}$$

$$P^{e,\tau}(w, \boldsymbol{\beta}) = P^{e,\tau,1}(w) F_1^e(\boldsymbol{\beta}) F_{std,1}^{e,k}(\alpha^{e,\inf}) \tag{66}$$

and consequently $w \in H^{\infty, L(\boldsymbol{\beta}, c, \circ)}$ and $\forall \boldsymbol{\beta}$ we have

$$\gamma^{e,k,\boldsymbol{\beta}} \tau^e(F^e(w_e, \boldsymbol{\beta})) = \gamma^{geom,e} \gamma_1^{e,k,\boldsymbol{\beta}}(\alpha^{e,\inf}, \overline{\omega}^e) \left[Q^{e,\tau}(F^e(w_e)) \right]^{P^{e,\tau}(w_e, \boldsymbol{\beta})}. \tag{67}$$

All numerical experiments were accomplished for k=1 and hence the results that proceed are valid for linear, bilinear and trilinear elements only. For each $\lambda \in \{\frac{i}{N_\lambda}; i \in [0, N_\lambda]; N_\lambda = 8\}$ and $\gamma^{geom,e} \in \{0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2.0\}$ fixed, the scheme to determine $\gamma^{e,k,\boldsymbol{\beta}} \tau^e(F^e(w_e, \boldsymbol{\beta}))$ consists of the following steps:

0) We consider a partition of $\Omega_{test} = (0,1) \times (0,1)$ in 40 square elements.

1) We make $\overline{\omega}^e = 0 \ \ \forall \Omega_e$ and hence $\gamma_1^{e,k,\boldsymbol{\beta}}(\alpha^{e,\inf}, \overline{\omega}^e) = 1 \ \ \forall \Omega_e$.

2) For an integer $N_\theta > 4$ we define $\Delta\theta_{test} = \dfrac{\pi}{4N_\theta}$ and we solved the advection problem above by using the stabilized FEM with shock capturing and by considering the advection field $\boldsymbol{\beta} \in \{(\cos(\theta_{test}^i), -\sin(\theta_{test}^i)), (\sin(\theta_{test}^i), -\cos(\theta_{test}^i)); \ \theta_{test}^i = i\Delta\theta_{test} \text{ with } i = 1, \ldots, N_\theta\}$. We determine $F_{std,1}^{e,k,0}(\alpha^{e,\inf})$ in way to have a best reduction of the spurious oscillations and smearing at internal layers. The experiments suggest $F_{std,1}^{e,k,0}(\alpha^{e,\inf}) = F_{std,1}^{e,k,0}(\sin(0.165149 \text{ rd}))$ as appropriate for $\alpha^{e,\inf} \le \sin(0.165149 \text{ rd})$, where "rd" denotes radians. The comment above suggests $\alpha^{e,\inf} = \sin(0.165149 \text{ rd})$ as a good choice for this parameter.

3) The function $F_{std,1}^{e,k,0}(\alpha^{e,\inf})$ is approximate by the usual interpolate of the FEM in one-dimension. The table 1 presents the values obtained for $F_{std,1}^{e,k,0}(\alpha^{e,\inf})$ and $\alpha^{e,\inf}$ with

numerical simulation described from steps (0) to (2).

Nodal Point	$\alpha^{e,\inf}$	$F_{std,1}^{e,k,0}(\alpha^{e,\inf})$
1	0.00000000000000	1.75000
2	0.164398987305357	1.75000
3	0.242535625036333	1.80000
4	0.316227766016838	1.85000
5	0.447213595499958	1.85000
6	0.707106781186548	1.90000
7	1.00000000000000	1.90000

Table 1- Coordinates of the nodal points and the value of $F_{std,1}^{e,k,0}(\alpha^{e,\inf})$

4) With the parameters and the functions determined through the steps (1) to (3), we make new numerical experiments to determine $\gamma_1^{e,k,\boldsymbol{\beta}}(\alpha^{e,\inf},\overline{\omega}^e)=1$ for $\overline{\omega}^e=1$ in every element such that $(0,0.65)\in\Gamma_e$. The most appropriate value for all direction of the advection field and for all geometry of the element is $\gamma^{-,e,k,\boldsymbol{\beta}}(\alpha^{e,\inf})=1.5\ \ \forall\alpha^{e,\inf}$. Finally, the numerical experiments suggest that the best value for λ is λ=0.75 and for $\gamma^{geom,e}$ is

$$\gamma^{geom,e}=\begin{cases}\frac{1}{2} & \text{for quadrilateral element or hexahedron element,}\\ \frac{3}{2} & \text{for triangle element or tetrahedron element.}\end{cases} \tag{68}$$

We observed that the scheme presented above, is a first attempt to determine a parametric function that brings in himself the information of the continuous problem and of the discrete problem.

5 NUMERICAL RESULTS

We selected some numerical examples to compare the solution of the CSCRS method with other shock capturing methods presented in (Galeão and Carmo, 1988; Codina, 1993; Carmo and Alvarez, 2003) and the GLS method. Results in representative sections are presented to show a better visualization of the spurious oscillations and the smearing effect at internal layers. The same problem is solved for several meshes, in order to confirm that the stabilizing parameter depends on the geometry of the element and on the distortion of the element for general meshes, and that for triangular meshes depends also on the alignment of the advection field $\boldsymbol{\beta}$ with the face of the larger measure. The method presented in (Codina, 1993) will be denoted by DCCD (Discontinuity Capturing Crosswind Dissipation). In all numerical examples with internal layers the SAUPG' solution (Carmo and Alvarez, 2003) is very similar to the CAU' solution (Galeão and Carmo, 1988). By the comment above we present the results for the CAU method only. The GLS, DCCD and CAU methods were implemented as in original papers.

5.1 Problem with internal layer in a square domain

In this experiment $\Omega=(0,1)\times(0,1)$, $\boldsymbol{\beta}\in\{(2,-1),(1,-1),(1,-2)\}$, $f=0$, $c=0$ and g is given as

$$g = \begin{cases} 0.5 \text{ in } (1,1), \\ 1 \text{ if } 0 \le x < 1 \text{ and } y = 1, \\ 1 \text{ if } x = 0 \text{ and } 0.75 < y < 1, \\ 0 \text{ if } x = 0 \text{ and } 0 < y < 0.75. \end{cases} \tag{69}$$

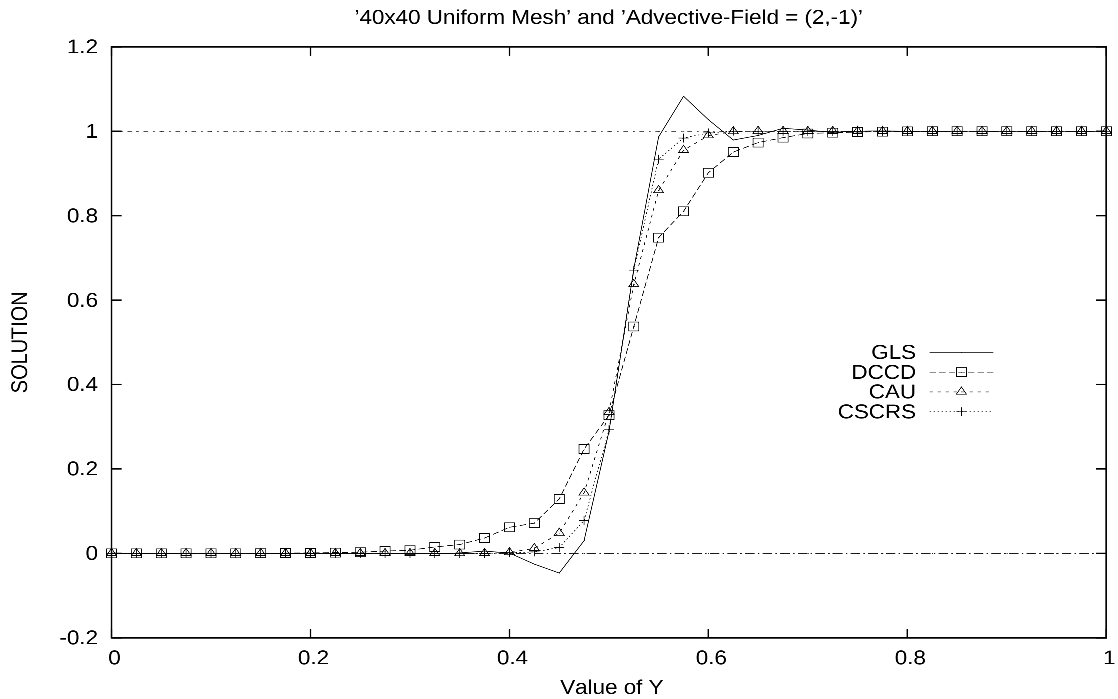
This problem was solved for three meshes and the three advection fields above in each mesh. The first mesh is a *40×40* uniform mesh with square elements. For this mesh can be observed from figure 1 to 2 that the CSCRS method reduces the spurious oscillations and the smearing effect at internal layer in relation to other methods. The second mesh is show in figure 3 (top) and has 3200 triangular elements aligned with the advection field $\boldsymbol{\beta}$. Numerical results for this mesh are presented in figure 4. In this case we did not present results for $\boldsymbol{\beta} = (1,-1)$ because all methods have the same solution, which is nodally exact. Again, in figure 4 we can see that the CSCRS method reduce the spurious oscillations and the smearing effect at internal layer in relation to other methods. The third mesh is shown in figure 3 (bottom) and has 1600 triangular elements no aligned with the advection field $\boldsymbol{\beta}$. Numerical results for this mesh are presented in figure 5. We observed spurious oscillations for the CAU and DCCD methods. However, the spurious oscillations are more pronounced for the CAU method. Practically, the CSCRS method eliminates these spurious oscillations. On the other hand, a little larger smearing for $\boldsymbol{\beta} = (1,-1)$ and $\boldsymbol{\beta} = (1,-2)$ is observed in the CSCRS method. This suggests that the stabilization parameter also depends on other variables not considered here when the mesh is constituted of triangular elements no aligned with the field $\boldsymbol{\beta}$.

5.2 Problem with smooth solution in a square domain

In this experiment $\Omega = (0,1)\times(0,1)$, $\boldsymbol{\beta} = (-y+0.5, x-0.5)$, $f = 0$, $c = 0$ and g is given as follows

$$g = \begin{cases} 0 \text{ if } x = 0 \text{ or } x = 1 \text{ or } y = 0 \text{ or } y = 1, \\ \sin(2\pi y) \text{ if } x = 0.5 \text{ and } 0 < y < 0.5. \end{cases} \tag{70}$$

This problem was solved for two meshes. The first mesh is a uniform mesh with *40×40* square elements. The second mesh is shown in figure 2 and has 3200 triangular elements no aligned with the advection field $\boldsymbol{\beta}$. Figure 7 present 2D plots comparing the solutions of the GLS, CAU, DCCD and CSCRS methods for both meshes in section $y = 0.5$. In this case the exact solution is smooth. Therefore, the solutions of the GLS and CSCRS methods are very similar because for problems with smooth solution the parameter $\tau^e(F^e(u_e^{h,GLS}),\boldsymbol{\beta}) << 1 \ \forall\Omega_e$ and the capture operator is approximately disabled. The solutions of the CAU and DCCD methods show evident loss accuracy.



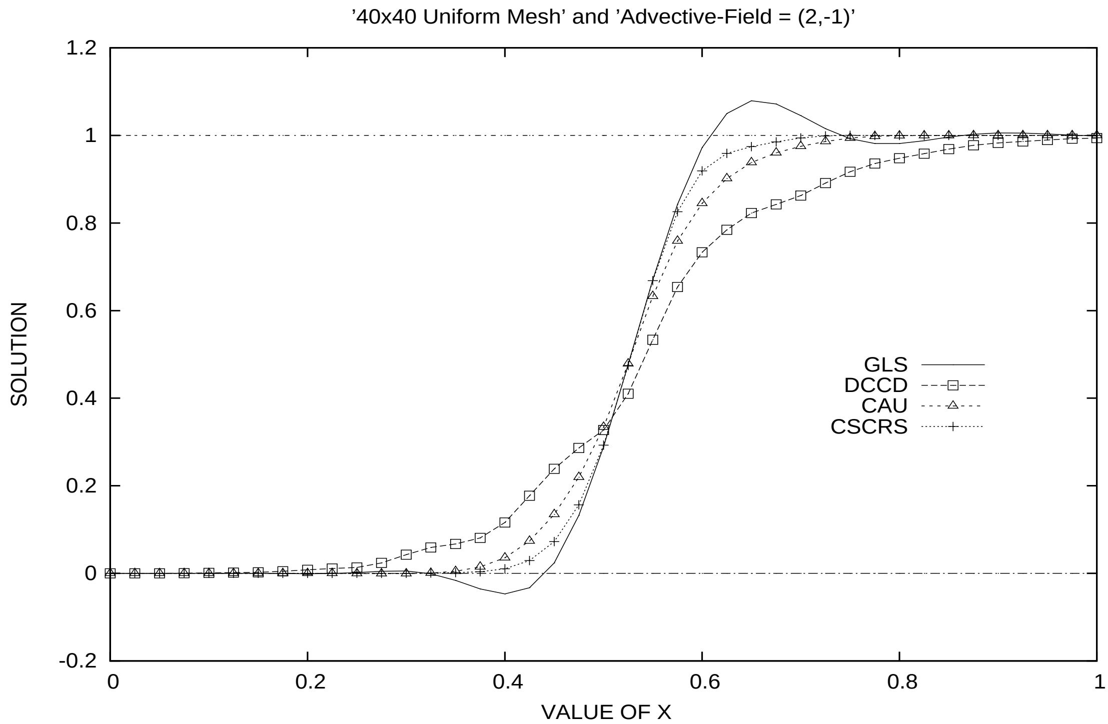
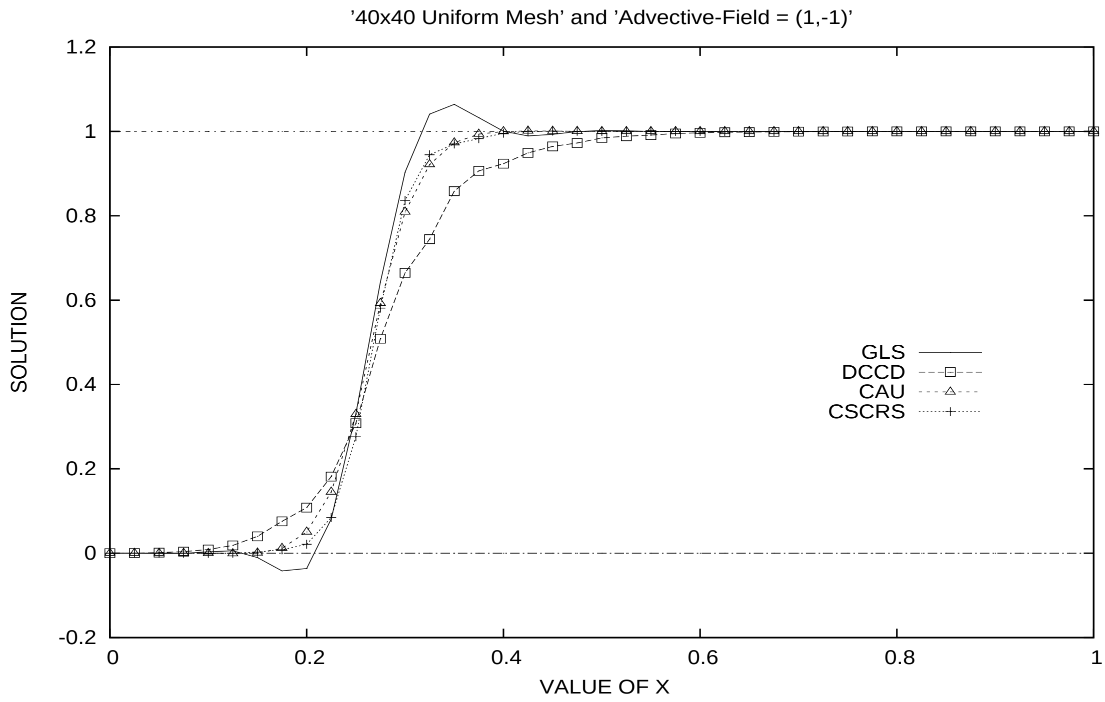


Figure 1: Solutions for $\boldsymbol{\beta} = (1, -2)$ in section $x = 0.5$ along the y direction (top) and in section $y = 0.5$ along the x direction (bottom).



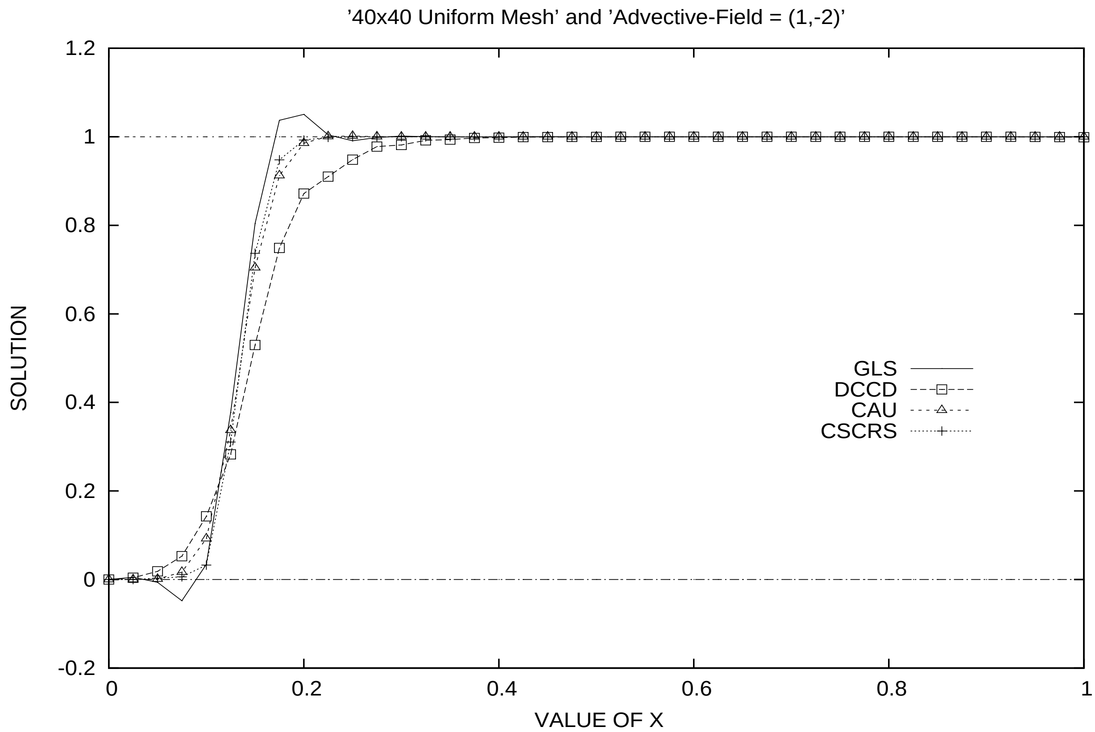
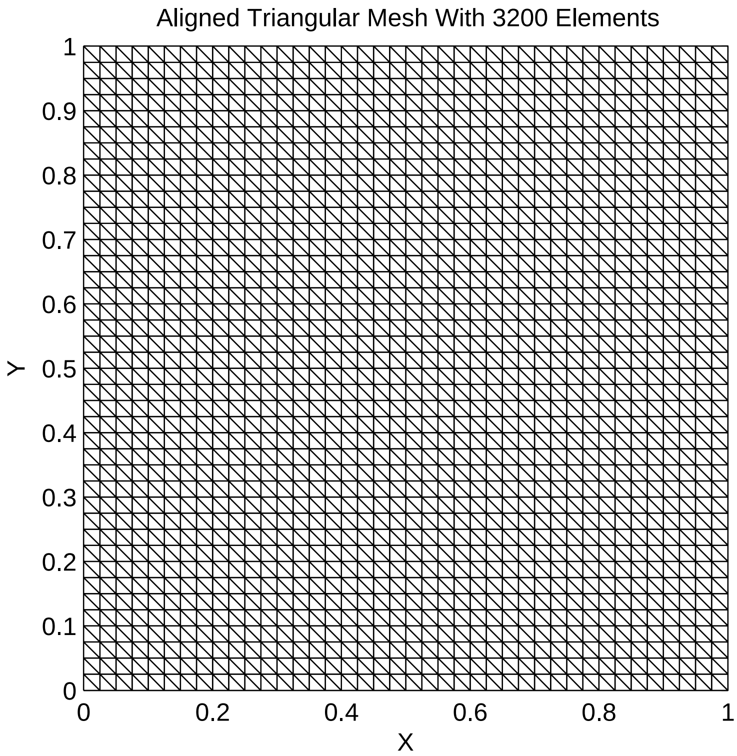


Figure 2: Solutions for $\boldsymbol{\beta} = (1,-1)$ (top) and $\boldsymbol{\beta} = (1,-2)$ (bottom) in section $y = 0.5$ along the x direction.



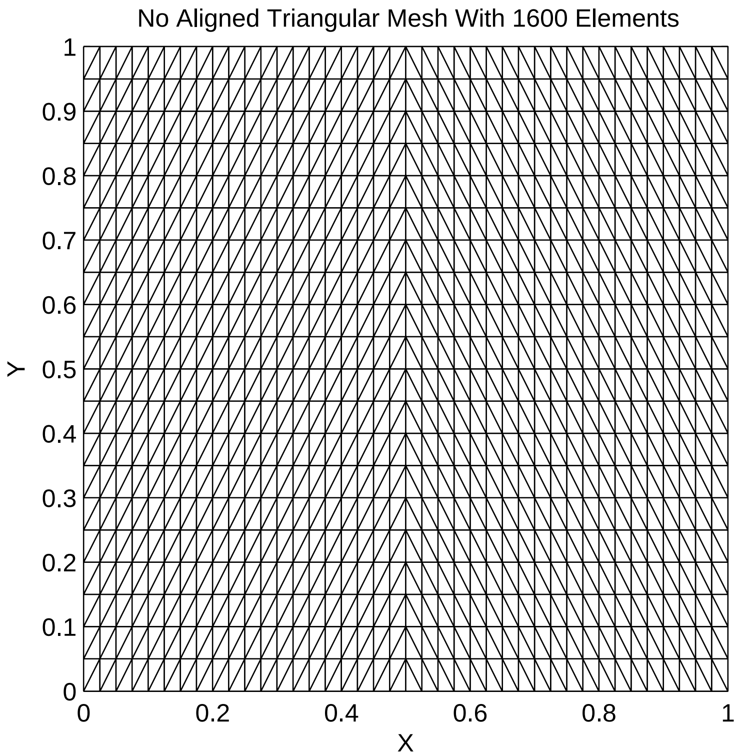


Figure 3: Two triangular meshes of linear element: $\boldsymbol{\beta}$ is aligned (top) and nonaligned (bottom) to the mesh.

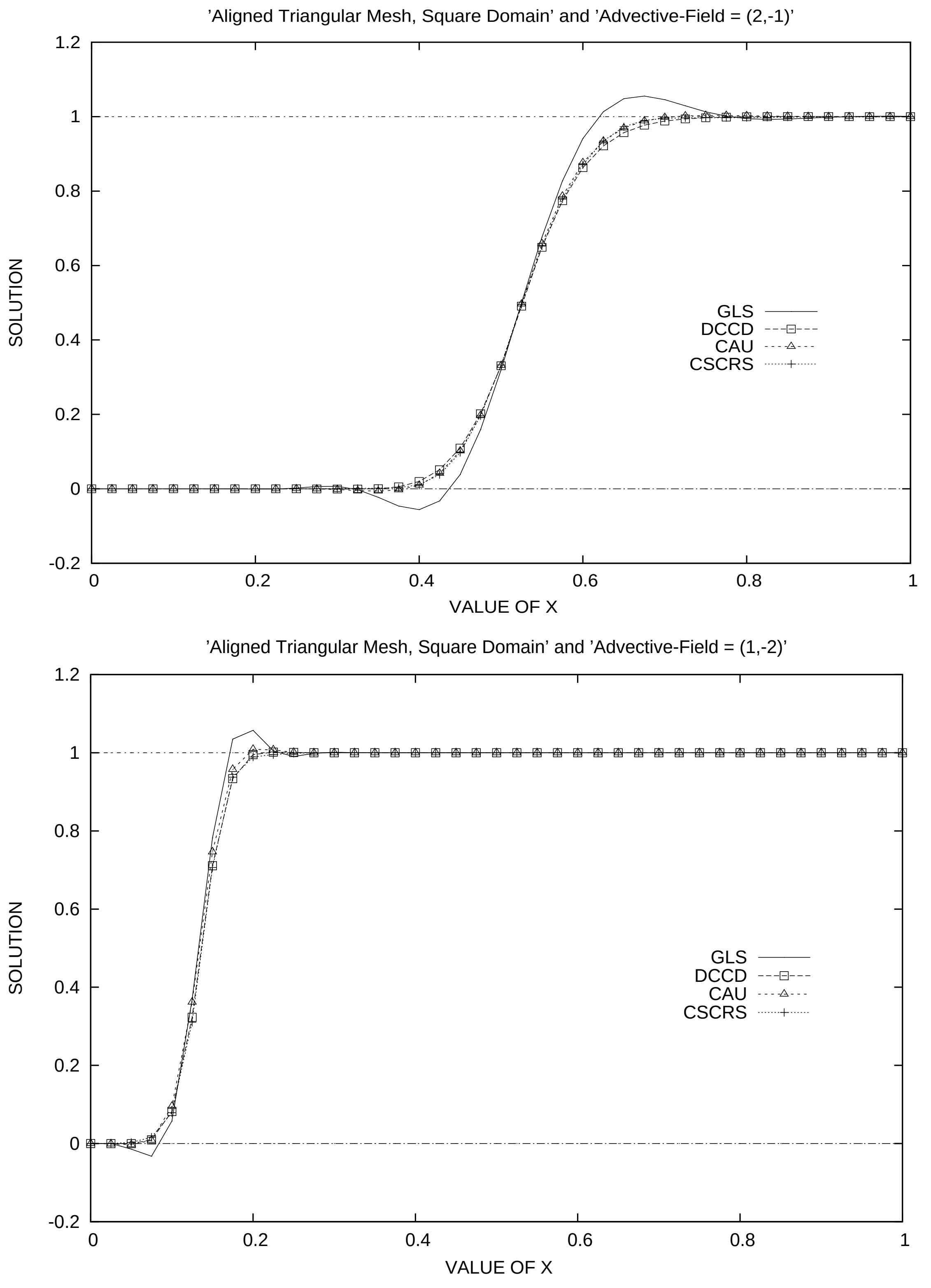


Figure 4: Solutions for $\boldsymbol{\beta} = (2,-1)$ (top) and $\boldsymbol{\beta} = (1,-2)$ (bottom) in section $y = 0.5$ along the x direction.

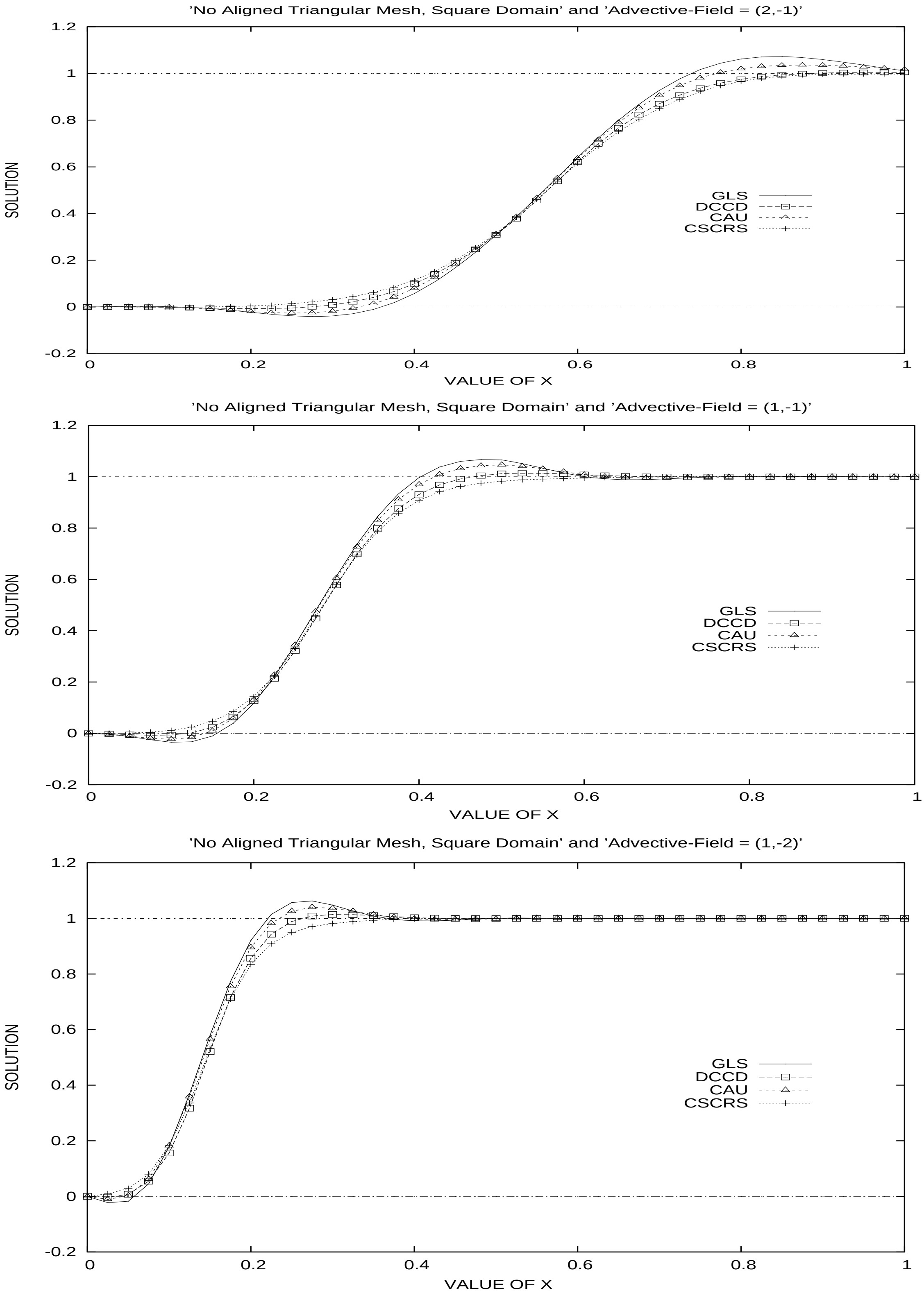


Figure 5: Solutions for $\boldsymbol{\beta} = (2,-1)$ (top), $\boldsymbol{\beta} = (1,-1)$ and $\boldsymbol{\beta} = (1,-2)$ (bottom) in section $y = 0.5$ along the x direction.

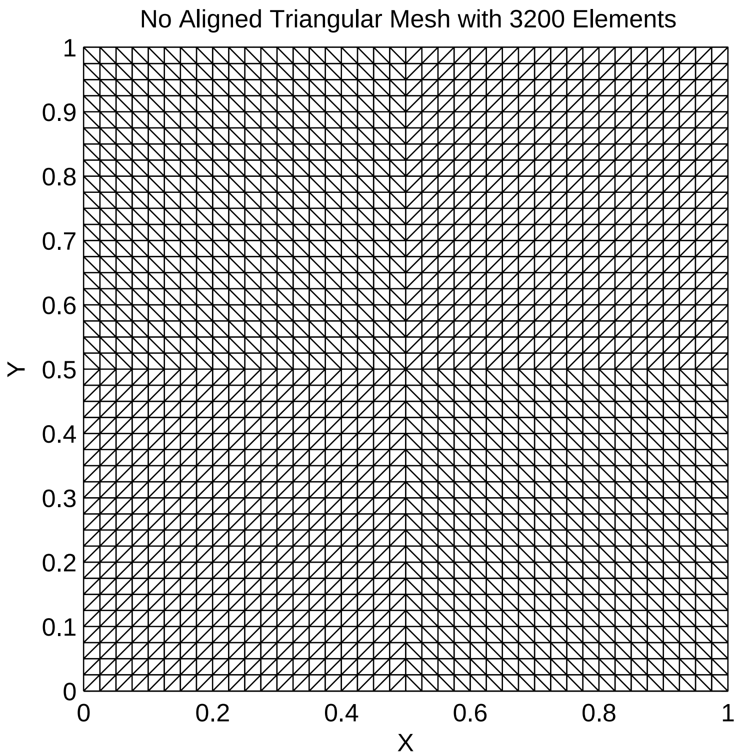


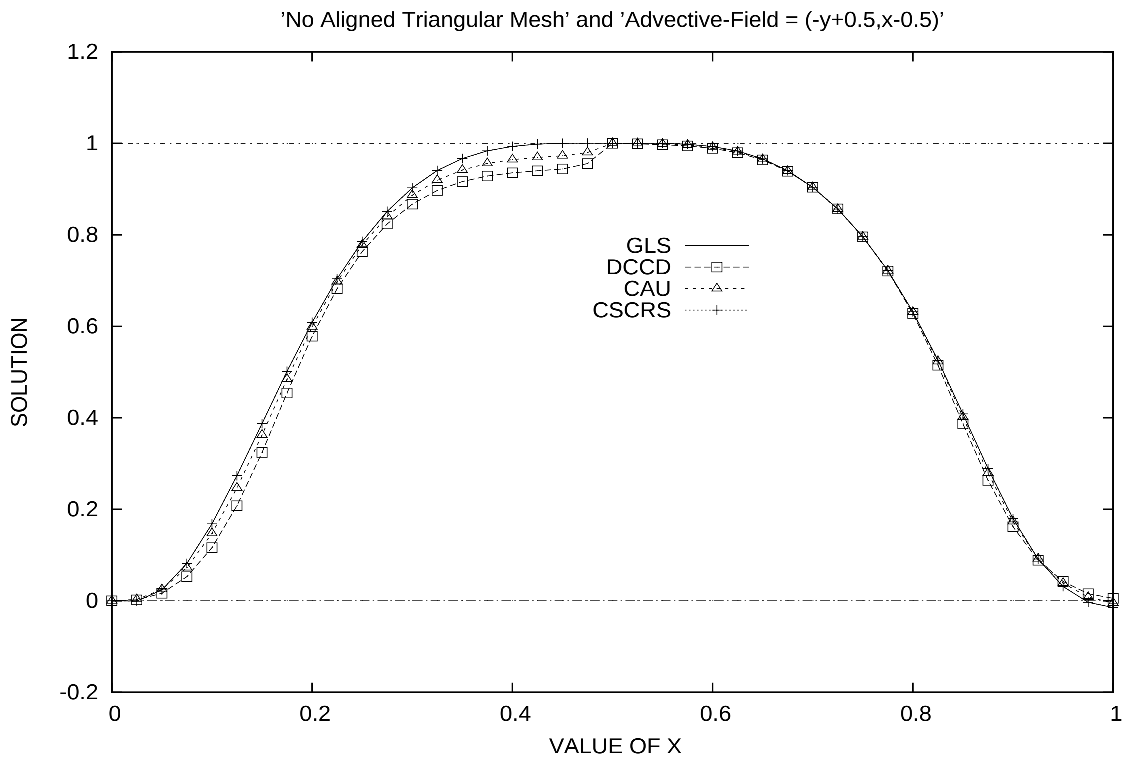
Figure 6: Nonaligned triangular mesh with the $\boldsymbol{\beta} = (-y + 0.5, x - 0.5)$.

6 CONCLUSIONS

We developed a new stabilized finite element method based on the ideas of shock-capturing stabilization. The CSCRS method introduces new ideas about the upwind function and the stabilization parameter. The strategy to choose the stabilization parameter is based on numerical experiments. The numerical experiments suggest that the stabilization parameter depends on the degree of the interpolating polynomial, the geometry of the element, the advection field $\boldsymbol{\beta}$ and on the data prescribed for the problem on Γ^{-}. Although we lack a theoretical error estimate, the simplicity of the new scheme, as well as the stabilization parameter must be considered for practical purposes.

In short, the CSCRS method presents the following properties:

- it is a nonlinear stabilized accurate method,
- its computational algorithm can be easily implemented,
- its stability at internal layers is superior to the CAU, SAUPG, GLSAU and DCCD methods,
- its solution preserve the sharp gradient at internal layers, i.e., no excessive crosswind smearing appears,
- its accuracy is very similar to the SUPG or GLS methods for smooth problem.



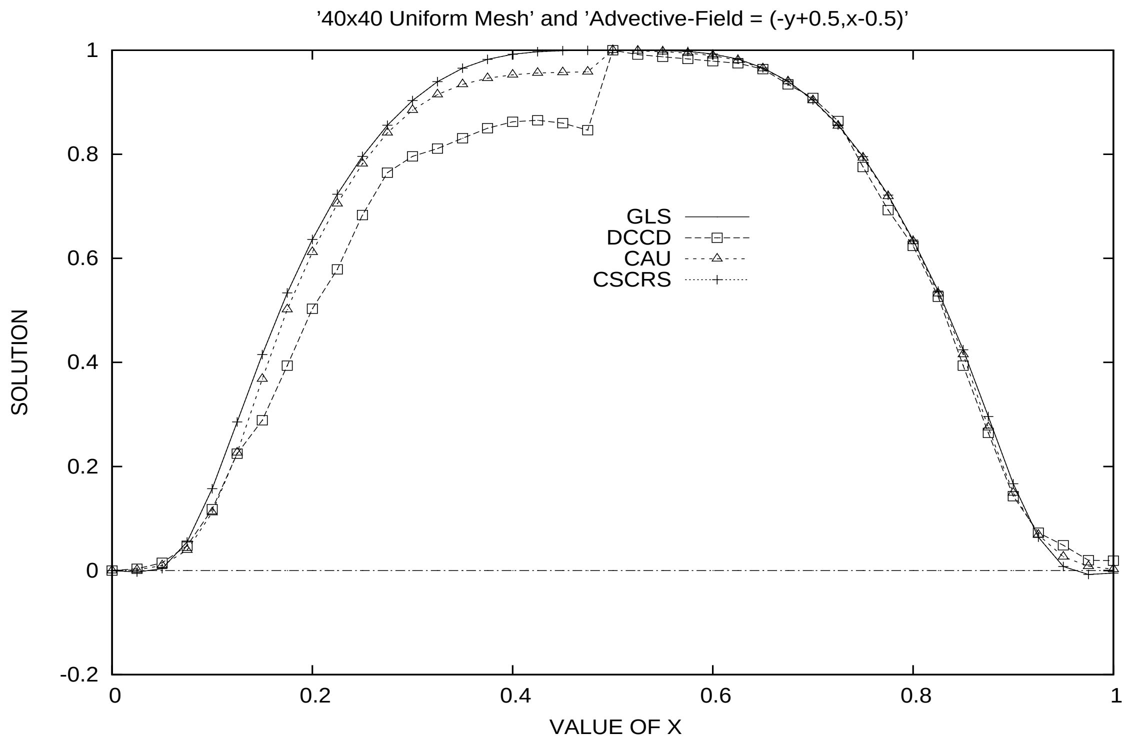


Figure 7: Solutions of GLS, CAU, DCCD and CSCRS methods in section $y = 0.5$ along the x direction for triangular mesh (top) and quadrilateral mesh (bottom).

The strategy presented here are our first insight to obtain better stabilization near of the internal layers. This strategy deserves further numerical studies and serves as starting point to identify other construction of a suitable parametric function. We highlighted that the relationship between the stabilization parameter and the advection field not is entirely established.

Acknowledgement: The authors wish to thank the Brazilian research funding agencies CNPq and FAPERJ for their support to this work.

REFERENCES

Brooks, A. N. and Hughes, T. J. R., Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. *Comput. Methods Appl. Mech. Engrg.*, 32:199-259, 1982.

Johnson, C., Nävert, U. and Pitkaranta, J., Finite element methods for linear hyperbolic problems. *Comput. Methods Appl. Mech. Engrg.*, 45:285-312, 1984.

Hughes, T. J. R., Mallet, M. and Mizukami, A., A new finite element formulation for computational fluid dynamics: II. Beyond SUPG. *Comput. Methods Appl. Mech. Engrg.*, 54:341-355, 1986.

Hughes, T. J. R. and Mallet, M., A new finite element formulation for computational fluid dynamics: IV. A discontinuity capturing operator for multidimensional advective-diffusive systems. *Comput. Methods Appl. Mech. Engrg.*, 58:329-336, 1986.

Galeão, A. C. and Do Carmo, E. G. D., A consistent approximate upwind Petrov-Galerkin method for convection-dominated problems. *Comput. Methods Appl. Mech. Engrg.*, 68:83-95, 1988.

Do Carmo, E. G. D. and Galeão, A. C., Feedback Petrov-Galerkin methods for convection-diminated problems. *Comput. Methods Appl. Mech. Engrg.*, 88:1-16, 1991.

Brezzi, F., Bristeau, M. O., Franca, L. P., Mallet, M. and Rogé, G., A relationship between stabilized finite element methods and the Galerkin with bubble functions. *Comput. Methods Appl. Mech. Engrg.*, 96:117-129, 1992.

Baiocchi, C., Brezzi, F. and Franca, L. P., Virtual bubbles and Galerkin-least-squares type methods Ga.L.S. *Comput. Methods Appl. Mech. Engrg.*, 105:125-141, 1993.

Codina, R., A discontinuity-capturing crosswind-dissipation for the finite element solution of the convection-diffusion equation, *Comput. Methods Appl. Mech. Engrg.*, 110:325-342, 1993.

Franca, L. P. and Farhat, C., Bubble functions prompt unusual stabilized finite element methods. *Comput. Methods Appl. Mech. Engrg.*, 123:299-308, 1995.

Agarwal, A. N. and Pinsky, P. M., Stabilized element residual methods (SERM): A posteriori error estimation for advection-diffusion equation. *J. Comput. Appl. Math.*, 74:3-17, 1996.

John, V., Matthies, G., Schieweck, F. and Tobiska, L., A streamline-diffusion method for nonconforming finite element approximations applied to convection-diffusion problems. *Comput. Methods Appl. Mech. Engrg.*, 166:85-97, 1998.

Shih, Y. T. and Elman, H. C., Modified streamline diffusion schemes for convection-diffusion problems. *Comput. Methods Appl. Mech. Engrg.*, 174:137-151, 1999.

Ramage, A., A multigrid preconditioner for stabilised discretisations of advection-diffusion problems. *J. Comput. Appl. Math.*, 110:187-203, 1999.

John, V., A numerical study of a posteriori error estimators for convection-diffusion equations. *Comput. Methods Appl. Mech. Engrg.*, 190:757-781, 2000.

Papastavrou, A. and Verfürth, R., A posteriori error estimator for stationary convection-diffusion problems: a computational comparison. *Comput. Methods Appl. Mech. Engrg.*, 189:449-462, 2000.

Farhat, C., Harari, I. and Franca, L., The discontinuous enrichment method. *Comput. Methods Appl. Mech. Engrg.*, 190:6455-6479, 2001.

Hauke, G. and Olivares, A. G., Variational subgrid scale formulations for the advection-diffusion-reaction equation. *Comput. Methods Appl. Mech. Engrg.*, 190:6847-6865, 2001.

Hauke, G., A simple subgrid scale stabilized method for the advection-diffusion-reaction equation. *Comput. Methods Appl. Mech. Engrg.*, 191:2925-2947, 2002.

Dutra do Carmo, E. G. and Alvarez, G. B., A new stabilized finite element formulation for scalar convection-diffusion problems: the streamline and approximate upwind/Petrov-Galerkin method. *Comput. Methods Appl. Mech. Engrg.*, 192:3379-3396, 2003.

Dutra do Carmo, E. G. and Alvarez, G. B., A new upwind function in stabilized finite element formulations, using linear and quadratic elements for scalar convection-diffusion problems. *Comput. Methods Appl. Mech. Engrg.*, 193:2383-2402, 2004.

Knobloch, P., Improvements of the Mizukami-Hughes method for convection-diffusion equations. *Comput. Methods Appl. Mech. Engrg.*, 196:579-594, 2006.

John, V. and Schmeyer, E., Finite element methods for time-dependent convection-diffusion-reaction equations with small diffusion. *Comput. Methods Appl. Mech. Engrg.*, 198:475-494, 2008.

Ramakkagari, V., Flaherty, J. E., A new stable method for singularly perturbed convection-diffusion equations. *Comput. Methods Appl. Mech. Engrg.*, 197:1507-1524, 2008.

Hsu, M.-C., Bazilevs, Y., Calo, V.M., Tezduyar, T.E. and Hughes, T.J.R., Improving stability of stabilized and multiscale formulations in flow simulations at small time steps. *Comput. Methods Appl. Mech. Engrg.*, 199:828-840, 2010.

Hsieh, P-W. and Yang, S-Y., On efficient least-squares finite element methods for convection dominated problems. *Comput. Methods Appl. Mech. Engrg.*, 199:183-196, 2009.

Chiu, P.H. and Sheu, T. W. H., On the development of a dispersion-relation-preserving dual-compact upwind scheme for convection-diffusion equation. *Journal of Computational Physics*, 228:3640-3655, 2009.

Tobiska, L., On the relationship of local projection stabilization to other stabilized methods for one-dimensional advection-diffusion equations. *Comput. Methods Appl. Mech. Engrg.*, 198:831-837, 2009.

Codina, R., Comparison of some finite element methods for solving the diffusion-convection-reaction equation. *Comput. Methods Appl. Mech. Engrg.*, 156:185-210, 1998.

John, V. and Knobloch, P., On spurious oscillations at layers diminishing (SOLD) methods for convection--diffusion equations: Part I - A review. *Comput. Methods Appl. Mech. Engrg.*, 196:2197-2215, 2007.

John, V. and Knobloch, P., On spurious oscillations at layers diminishing (SOLD) methods for convection-diffusion equations: Part II - Analysis for P1 and Q1 finite elements. *Comput. Methods Appl. Mech. Engrg.*, 197:1997-2014, 2008.

Hughes, T. J. R., Franca, L. P. and Hulbert, G. M., A new finite element formulation for computational fluid dynamics: VII. The Galerkin-least-squares method for advective-diffusive equations. *Comput. Methods Appl. Mech. Engrg.*, 73:173-189, 1989.

Franca, L. P. and Do Carmo, E. G. D., The Galerkin gradient least squares method. *Comput. Methods Appl. Mech. Engrg.*, 74:41-54, 1989.

Hughes, T. J. R., Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid scale models, bubbles and the origins of stabilized methods. *Comput. Methods Appl. Mech. Engrg.*, 127:387-401, 1995.

Oñate, E., Derivation of stabilized equation for numerical solution of advective-diffusive transport and fluid flow problems. *Comput. Methods Appl. Mech. Engrg.*, 151:233-265, 1998.

Ilinca, F., Hétu, J. F. and Pelletier, D., On stabilized finite element formulations for incompressible advective-diffusive transport and fluid flow problems. *Comput. Methods Appl. Mech. Engrg.*, 188:235-255, 2000.

Franca, L. P. and Valentin, F., On an improved unusual stabilized finite element method for the advective-reative-diffusive equation. *Comput. Methods Appl. Mech. Engrg.*, 190:1785-1800, 2000.

Knopp, T., Lube, G. and Rapin, G., Stabilized finite element methods with shock capturing for advection--diffusion problems. *Comput. Methods Appl. Mech. Engrg.*, 191:2997-3013, 2002.

Nesliturk, A. and Harari, I., The nearly-optimal Petrov-Galerkin method for convection-diffusion problems. *Comput. Methods Appl. Mech. Engrg.*, 192:2501-2519, 2003.

Burman, E. and Hansbo, P., Edge stabilization for Galerkin approximations of convection--diffusion--reaction problems. *Comput. Methods Appl. Mech. Engrg.*, 193:1437-1453, 2004.

Franca, L. P., Madureira, A. L. and Valentin, F., Towards multiscale functions: enriching finite element spaces with local but not bubble-like functions. *Comput. Methods Appl. Mech. Engrg.*, 194:3006-3021, 2005.

Lube, G. and Rapin, G., Residual-based stabilized higher-order FEM for advection-dominated problems. *Comput. Methods Appl. Mech. Engrg.*, 195:4124-4138, 2006.

Zienkiewicz, O. C., Taylor, R. L., Sherwin, S. J. and Peiró, J., On discontinuous Galerkin methods. *Int. J. Numer. Meth. Engng.*, 58:1119-1148, 2003.

Hughes, T.J.R., Scovazzi, G., Bochev, P.B. and Buffa, A., A multiscale discontinuous Galerkin method with the computational structure of a continuous Galerkin method. *Comput. Methods Appl. Mech. Engrg.*, 195:2761-2787, 2006.

Hughes, T.J.R., Masud, A., Wan, J., A stabilized mixed discontinuous Galerkin method for Darcy flow. *Comput. Methods Appl. Mech. Engrg.*, 195:3347-3381, 2006.

Gómez, H., Colominas, I., Navarrina, F. and Casteleiro, M., A discontinuous Galerkin method for a hyperbolic model for convection-diffusion problems in CFD. *Int. J. Numer. Meth. Engng.*, 71:1342-1364, 2007.

Adams, R. A., *Sobolev spaces*. New York, Academic Press, 1975.

Christie, I., Mitchell, A.R., Upwinding of high order Galerkin methods in conduction--convection problems. *Int. J. Numer. Methods Engrg.*, 14:1764-1771, 1978.

Mizukami, A., An implementation of the streamline-upwind/Petrov--Galerkin method for linear triangular elements. *Comput. Methods Appl. Mech. Engrg.*, 49:357-364, 1985.

Stynes, M., Tobiska, L., Necessary L2-uniform convergence conditions for difference schemes for two-dimensional convection-diffusion problems, *Comput. Math. Appl.*, 29:45-53, 1995.

Roos, H.-G., Stynes, M. and Tobiska, L., *Numerical Methods for Singularly Perturbed Differential Equations. Convection-Diffusion and Flow problems*. Springer, Berlin, 1996.

Ramage, A., A multigrid preconditioner for stabilised discretisations of advection-diffusion problems. *J. Comput. Appl. Math.*, 110:187-203, 1999.

Fischer, B., Ramage, A., Silvester, D.J. and Wathen, A.J., On parameter choice and iterative convergence for stabilised discretisations of advection--diffusion problems. *Comput. Methods Appl. Mech. Engrg.*, 179:179-195, 1999.

Harari, I., Franca, L.P. and Oliveira, S.P., Streamline design of stability parameters for advection-diffusion problems. *J. Comput. Phys.*, 171:115-131, 2001.

Elman, H.C. and Ramage, A., An analysis of smoothing effects of upwinding strategies for the convection-diffusion equation. *SIAM J. Numer. Anal.*, 40:254-281, 2002.

Principe, J., Codina, R., On the stabilization parameter in the subgrid scale approximation of scalar convection diffusion reaction equations on distorted meshes. *Comput. Methods Appl. Mech. Engrg.*, 199:1386-1402, 2010.

Dutra do Carmo, E. G., Alvarez, G. B., Loula, A. F. D. and Rochinha, F. A., A nearly optimal Galerkin Projected Residual finite element method for Helmholtz problem. *Comput. Methods Appl. Mech. Engrg.*, 197:1362-1375, 2008.

Dutra do Carmo, E. G., Alvarez, G. B., Rochinha, F. A. and Loula, A. F. D., Galerkin projected residual method applied to diffusion-reaction problems. *Comput. Methods Appl. Mech. Engrg.*, 197:4559-4570, 2008.

Heinrich, J.C., On quadratic elements in finite element solution of steady-state convection-diffusion equation. *Int. J. Numer. Methods Engrg.*, 15:1041-1052, 1980.

Codina, R., Oñate, E. and Cervera, M., The intrinsic time for the streamline upwind Petrov-Galerkin formulation using quadratic elements. *Comput. Methods Appl. Mech. Engrg.*, 94:239-262, 1992.

Almeida, R.C., Silva, R.S., A stable Petrov--Galerkin method for convection-dominated problems. *Comput. Methods Appl. Mech. Engrg.*, 140:291-304, 1997.

Tezduyar, T. E. and Osawa, Y., Finite element stabilization parameters computed from element matrices and vectors. *Comput. Methods Appl. Mech. Engrg.*, 190:411-430, 2000.

Akin, J.E., Tezduyar, T.E., Ungor, M. and Mittal, S., Stabilization parameters and Smagorinski turbulence model. *J. Appl. Mech.*, 70:2-9, 2003.

Galeão, A.C., Almeida, R.C., Malta, S.M.C. and Loula, A.F.D., Finite element analysis of convection dominated reaction-diffusion problems. *Appl. Numer. Math.*, 48:205-222, 2004.

Akin, J.E. and Tezduyar, T.E., Calculation of the advective limit of the SUPG stabilization parameter for linear and higher-order elements. *Comput. Methods Appl. Mech. Engrg.*, 193:1909-1922, 2004.

Iliescu, T., A flow-aligning algorithm for convection-dominated problems. *Int. J. Numer. Meth. Engng.*, 46:993-1000, 1999.

Skalický, T. and Roos, H.-G., Anisotropic mesh refinement for problems with internal and boundary layers. *Int. J. Numer. Meth. Engng.*, 46:1933-1953, 1999.