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RELIABILITY ANALYSIS OF NONLINEAR REINFORCED CONCRETE BEAMS

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Abstract. This paper presents the application of reliability analysis to nonlinear reinforced concrete beams. Failure is assumed to occur when the structure presents displacements bigger than a prescribed limit. A First Order Reliability Method (FORM) is used, and the results are compared to the ones given by Monte Carlo simulation. Since the structural model is nonlinear, special techniques are used for sensitivity analysis. These techniques allow one to consider almost any variable as probabilistic in this problem. It is also proven that the reliability index given by this problem is a lower bound for the reliability index when collapse of the structure is considered. Finally, two examples are presented in order to validate the proposed approach.

1 INTRODUCTION

Reinforced concrete structures present significant nonlinear behavior and consequently nonlinear analysis of this kind of structure has been subject of research for many years (Lin, 1955; Ferry Borges and Arantes e Oliveira, 1963; Ass-Jacobsen and Grenancher, 1974; Kang and Scordelis, 1980; Chen, 1982). In the last decades very accurate structural models were proposed, that can take into account most aspects of the nonlinear behavior of concrete structures (Abdollahi, 1994; Gomes and Awruch, 2001; Oliver et al., 2008; Di Luzio, 2009; Hinchberger, 2009).

However, reinforced concrete structures (as is the case of most structures) are subjected to strong uncertainties, both related to the properties of the material and the applied loads. Consequently, the design of structures that will need to work under real conditions need to take into account these uncertainties to some degree.

Until about 1960 these uncertainties were considered by applying some safety factor during the design stage (Madsen et al., 1986). However, these safety factors were established only by means of "engineering judgment", and not by a rigorous scientific approach.

The next step was the design of structures according to limit states (Madsen et al., 1986), that is the approach recommended by most structural design standards nowadays. In this case, the properties of each material and the magnitude of each load is decreased/increased according to its respective factor. These factors are evaluated based on probabilistic analysis and presented as fixed values in design standards. For this reason design using limit states is also known as semi-probabilistic design. The factors were actually evaluated using probabilistic analysis, but the designer makes a deterministic analysis using reduced/increased resistances/loads.

It turns out that design standards are not able to cover the full range of application that engineers are able to conceive. Even if some kind of design standards are available for most kind of constructions (such as buildings, bridges and dams), sometimes the engineers need to design some structure that does not fit exactly in any standard due to its size, complexity or multidisciplinary nature. In these cases (or in cases that the engineer wants to) probabilistic analysis can be pursued.

Full probabilistic analysis, where one aims for a full probabilistic characterization of the behavior of the structure, needs in general much computational effort (Haldar and Mahadevan, 2000). Fortunately, in many cases it is enough to study the structure from the optics of "fail" versus "do not fail". In these cases one can substitute a full probabilistic analysis by a reliability analysis, where only the failure probability is evaluated. This takes much less computational effort, and can be successfully applied to several structural problems (Melchers, 1999; Haldar and Mahadevan, 2000; Ditlevsen and Madsen, 1996).

One common feature of most reliability analysis methods is that they need to evaluate the response of the system and its gradient according to the probabilistic variables several times (Melchers, 1999; Haldar and Mahadevan, 2000; Ditlevsen and Madsen, 1996). Consequently, the application of such methods to nonlinear problems must be made with care, since making a single nonlinear analysis can be a time consuming process. In this context, carrying out the nonlinear analysis and the gradient evaluation several times can lead, in some cases, to prohibitive computational efforts.

In the context of reinforced concrete frames and beams, an interesting strategy to tackle the reliability analysis problem considering collapse of the structure is the response surface approach, as presented by Soares et al. (2002). In this case, the response of the surface is approximated by an analytical function (response surface) and the reliability analysis problem is solved for this approximation. The approximation is then iteratively improved, and since the approximation is analytical, it can be evaluated efficiently. The main computational effort in this case lies in obtaining the approximation itself.

In this paper we propose an approach for the reliability analysis of reinforced concrete beams and frames using a First Order Reliability Method (FORM). Failure is assumed to occur when the displacements are bigger than some prescribed limit. Besides, the FORM algorithm is applied directly to the problem, and thus an efficient approach for carrying out sensitivity analysis is also presented. It is also proven that the reliability index given by this problem is a lower bound for the reliability index when collapse of the structure is considered. Finally, two examples are presented in order to validate the proposed approach.

2 FINITE ELEMENT MODEL

2.1 The iterative secant approach

The FEM formulation used here is that presented by Bontempi and Malerba (1998) and also described in details by Biondini et al. (2004c) and Biondini (2004). This formulation has been used for many applications such as bridge design (Biondini et al., 2004b), reliability analysis (Biondini et al., 2004c), lifetime analysis (Biondini et al., 2004a), dynamic analysis (Biondini, 2004) and analysis of structures exposed to fire (Biondini and Nero, 2006). In this approach, constitutive equations for concrete and steel in uniaxial compression/tension are assumed, based for example, on empirical data. Both constitutive models are better described by Bontempi and Malerba (1998), Biondini et al. (2004c), Biondini et al. (2004b) and Biondini (2004).

Based in these assumptions, the FEM is represented by

$$\mathbf{K}_s \mathbf{q} = \mathbf{F}, \tag{1}$$

where $\mathbf{K}_s$ is the secant stiffness matrix, $\mathbf{q}$ are the generalize displacements and $\mathbf{F}$ is the vector of applied forces. Note that the secant stiffness matrix depends on the displacements, that is a general rule from nonlinear analysis.

In this context, in order to evaluate the displacements $\mathbf{q}$ for a given set of applied forces $\mathbf{F}$, one takes an initial guess for the displacements $\mathbf{q}^{(0)}$ and apply the iterative relation

$$\mathbf{K}_s^{(k)} \mathbf{q}^{(k+1)} = \mathbf{F}, \tag{2}$$

where $\mathbf{K}_s^{(k)}$ is the secant stiffness matrix for the displacements $\mathbf{q}^{(k)}$ at iteration k. The initial guess $\mathbf{q}^{(0)}$ is generally taken as zeros vector.

The iterative procedure is then stopped when the change of displacements between two successive iterations is smaller than a given tolerance. When the structure is not able to support the applied loads (1) does not hold for any set of displacements and then the iterative procedure given by (2) does not converge. In practice, it is easy to know when the iterative procedure will not converge since for this case the displacements go to infinity.

2.2 Constitutive equations

A detailed review on constitutive modeling of reinforced concrete structures is presented by Chen (1982). When the structure is modeled using beam finite elements, the stress-strain relation for concrete under uniaxial compression can be accurately represented by a parabolic relation as the one shown in Fig. 1. The application of such constitutive models for the FEM formulation used here is presented by Biondini et al. (2004c), Biondini et al. (2004b) and Biondini (2004), and show that the results are accurate enough for most practical applications.

However, the reliability analysis algorithm needs to solve the nonlinear structural problem several times. Thus, in order to simplify the numerical model and reduce the computational effort involved, we assume here an approximated piecewise linear stress-strain relation as the one shown in Fig. 1. Note that a parabolic approximation can also be used, but this will eventually lead to an increase in the computational effort involved.

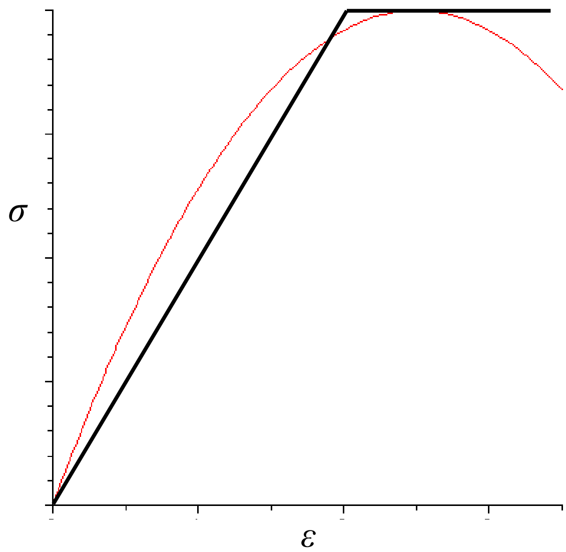

Figure 1: The parabolic approximation (red) for the stress-strain relation for concrete under uniaxial compression and the piecewise linear approximation (black).

Here concrete is modeled using a piecewise linear stress-strain relation as the one presented in Fig. 2. For an applied stress smaller than the yielding stress σ_y, the strain is given by

$$\sigma_y = E\varepsilon, \tag{3}$$

where E is the linear elastic modulus and ε is the strain.

For a strain bigger than ε_y the stress remains constant and equal to σ_y, until rupture occurs for an strain equal to ε_u and the material is not able support stress anymore.

In order to model this constitutive equation it is necessary to prescribe the ultimate strain ε_u and two other parameters. In most practical cases, the two other parameters prescribed are σ_y and E. However, here we choose to prescribe ε_y and σ_y, for reasons explained later. In this case, the elastic modulus is obtained from (3) by using ε_y instead of ε.

This constitutive model is used for concrete in both tension and compression. In tension, the yielding stress is called here σ^c_{yt}, the yielding strain ε^c_{yt} and the ultimate strain ε^c_{ut}, where c stands for concrete and t stands for tension. Using the same nomenclature, the parameters in compression are called here σ^c_{yc}, ε^c_{yc} and ε^c_{uc}.

The secant modulus, that is used in the iterative procedure described previously, is defined as (Chen, 1982)

$$E_s = \frac{\sigma}{\varepsilon}, \tag{4}$$

where σ and ε are obtained using the constitutive model described previously.

Steel is described using the same constitutive equations, but now the behavior in tension and in compression are assumed to be the same. Thus, the constitutive model of steel is fully described assuming a yielding stress σ_y^s, a yielding strain ε_y^s and an ultimate strain ε_u^s, where s stands for steel.

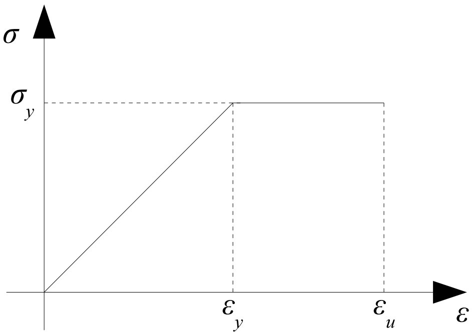

Figure 2: Constitutive model used.

The reason why we choose σ_y and ε_y as parameters of the constitutive models, and the elastic modulus E is defined implicitly from theses two parameters is pictured in Fig. 3 and Fig. 4. Note that if the elastic modulus is prescribed, instead of the yielding strain, changes to the yielding stress are not "felt" in the elastic range, as shown in Fig. 4. This behavior can lead to difficulties in the reliability problem preventing the algorithm to converge. When both the yielding stress and the yielding strain are prescribed, instead of the elastic modulus, modifications to the yielding stress are "felt" in the elastic range, as shown in Fig. 3. This leads to a better posed reliability problem.

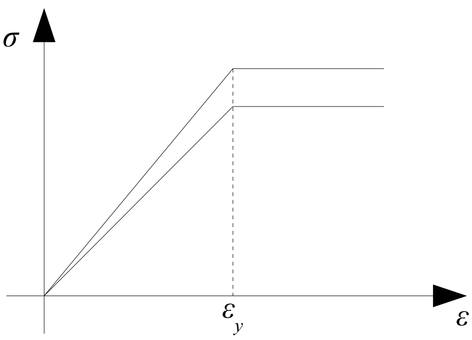

Figure 3: Two constitutive models for the same yielding strain but different yielding stresses.

3 STATEMENT OF THE RELIABILITY PROBLEM

3.1 The failure function

In the reliability problem failure is assumed to occur when the displacement of a given node of the structure is bigger than a prescribed limit. Thus, the failure function can be written as

$$g(\mathbf{x}, \mathbf{q}) = q_j(\mathbf{x}) - q_{max}, \tag{5}$$

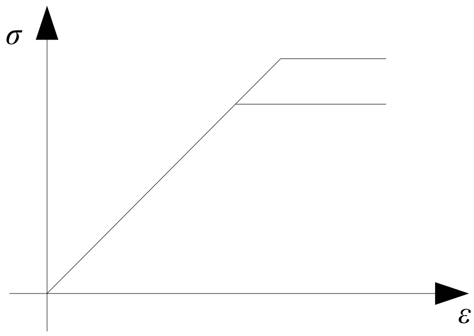

Figure 4: Two constitutive models for the same elastic modulus but different yielding stresses.

where q_j is the displacement in a given node, q_{max} is the maximum allowable displacement for this node and $\mathbf{x}$ is the vector of probabilistic variables, named from now on as parameters of the reliability problem. From (5) it can be seen that failure occurs when $g > 0$, that is, when the displacement is bigger than the allowable displacement.

The FORM also uses the gradient of the failure function according to the parameters of the problem in order to evaluate the reliability of the structure. The gradient of (5) is

$$\nabla_{\mathbf{x}} g = \nabla_{\mathbf{x}} q_j, \qquad (6)$$

since q_{max} is a fixed value. Thus, in order to evaluate the gradient of the failure function it is necessary to evaluate the gradient of the displacements according to the parameters of the problem. The evaluation of this information is described later.

Note that (5) and (6) are valid when the displacement q_j is expected to be positive. In cases when the displacement is expected to be negative, both the failure function and its gradient are multiplied by minus one.

Here the reliability analysis problem is solved using a First Order Reliability Method (FORM). The algorithm used is that developed by Rackwitz and Fissler, that is described by Haldar and Mahadevan (2000). Monte Carlo(MC)simulation is performed as described by Haldar and Mahadevan (2000).

3.2 The relation between the problem defined for maximum allowable displacements and the problem defined for collapse of the structure

Frequently the engineer needs to know the reliability of the structure considering collapse instead of some maximum allowable displacement. This is the case when the analysis is made for the ultimate limit state, for example (Soares et al., 2002). In this case, the reliability analysis problem will seek the combination of probabilistic variables that leads to the collapse of the structure. The reliability index will then be defined according to this point in the design space.

From the computational point of view, collapse of the structure happens when the tangent stiffness matrix becomes singular and thus any small load increment leads to very big displacements. That is, for a structure that collapses some displacements go to infinity. In order to makes things clear, we thus make the following definition.

Definition 1 *Collapse of the structure according to some degree of freedom is assumed when the displacement of this degree of freedom goes to infinity.*

Note that the definition of collapse presented above depends on the degree of freedom chosen. That is, the same structure may collapse according to some degree of freedom but do not collapse according to another. This kind of behavior is expected for structures as the one presented in Fig. 5. Collapse according to degree of freedom (d.o.f.) 2 do not necessarily mean collapse according to d.o.f. 1.

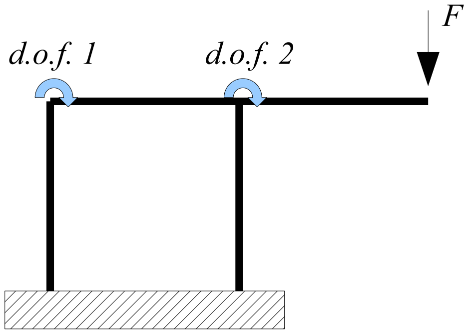

Figure 5: Two degrees of freedom of a plane frame.

Applying Definition 1 to (5), the reliability analysis problem for collapse of the structure can be defined as

$$g(\mathbf{x},\mathbf{q}) = \lim_{q_{max}\to\infty} \{q_j(\mathbf{x}) - q_{max}\}, \tag{7}$$

where q_j is the d.o.f. according to which collapse is defined.

According to (7), the structure would be safe for a finite displacement q_j and unsafe when this displacement goes to infinity.

For computational purposes (7) can be rewritten as

$$g(\mathbf{x},\mathbf{q}) = q_j(\mathbf{x}) - b, \tag{8}$$

where b is taken as a sufficiently big number, in order to take the role of the limit that appears in (7). Note that in most practical cases the displacements are of the order of 10^{-6}m to 10^{-2}m, and taking $b = 10$ would likely do the job.

In practice, considering such a big allowable displacement would lead to serious convergence difficulties. That's because the FORM would eventually move to a point in the design space that leads to collapse of the structure. However, in such points the displacements would be very big and the evaluation of the failure function and its gradient would surely fail. From this step onward, the FORM would start to move almost randomly, since the information given by the failure function and its gradient would not represent the problem appropriately. Consequently, solving the reliability analysis problem considering collapse of the structure using an approach similar to the ones given by (7) and (8) is not an efficient approach. Fortunately, it is possible to evaluate a lower bound for the reliability index of this problem.

The failure probability according to some failure function g_i is evaluated as (Haldar and Mahadevan, 2000)

$$P = \int_{\Omega} \phi(x_1, x_2, .., x_n) d\Omega, \tag{9}$$

where $\phi(x_1, x_2, .., x_n)$ is the joint probability density (PDF) of the design variables $x_1, x_2, .., x_n$ and Ω is the set of design vectors $\mathbf{x}$ for which $g_i(\mathbf{x}) > 0$, that is, the set of design vectors that lead to failure.

Now lets compare the failure probabilities for two different maximum allowable displacements. First we assume a maximum allowable displacement q_1, thus defining a failure function g_1. We then have a set Ω_1 of design vectors $\mathbf{x}$ for which $g_1 > 0$. Next we assume a maximum allowable displacement q_2, thus defining a failure function g_2. We then have a set Ω_2 of design vectors $\mathbf{x}$ for which $g_2 > 0$.

According to (5), the set Ω_1 is composed of design vectors $\mathbf{x}$ that respect

$$q_j(\mathbf{x}) > q_1, \tag{10}$$

while the set Ω_2 is composed of design vectors $\mathbf{x}$ that respect

$$q_j(\mathbf{x}) > q_2. \tag{11}$$

Evaluating the failure probability for both sets according to (9) we have

$$P_1 = \int_{\Omega_1} \phi(x_1, x_2, .., x_n) d\Omega_1 \tag{12}$$

and

$$P_2 = \int_{\Omega_2} \phi(x_1, x_2, .., x_n) d\Omega_2. \tag{13}$$

Suppose now that $q_2 > q_1$. Then we know that $\Omega_2 \subseteq \Omega_1$ since every displacement bigger than q_2 will also be bigger than q_1, but the opposite is not true. That is, the set Ω_2 is actually a subset of Ω_1 when $q_2 > q_1$. Since the PDF $\phi(\mathbf{x})$ is always non negative (there can be no negative probabilities) and Ω_2 is a subset of Ω_1, then the integrals from (12) and (13) give

$$P_2 \leq P_1 \tag{14}$$

when

$$q_2 > q_1. \tag{15}$$

This is true since (12) and (13) are evaluated for the same function ϕ. Since $\Omega_2 \subseteq \Omega_1$, (13) is evaluated in a subset of the domain used in (12). However, ϕ is always non negative and thus the integral from (13) must have a smaller value than that of (12).

If we rewrite the condition given by (14) for reliability indexes we have

$$\beta_2 \geq \beta_1. \tag{16}$$

Note that the conditions given by (14) and (16) just state that bigger displacements are less likely to be surpassed than smaller displacements, since in order to achieve bigger displacements its is necessary to surpass smaller displacements first.

Applying the definition of collapse from Definition 1 we have, from (15) and (16),

$$\beta_{\lim q_2 \to \infty} \geq \beta_1 \tag{17}$$

since

$$\lim q_2 \to \infty > q_1, \tag{18}$$

where $\beta_{\lim q_2 \to \infty}$ is the reliability index when the maximum allowable displacements goes to infinity, that is the definition of collapse.

That is, (17) states that every reliability index for problems considering finite maximum allowable displacements is a lower bound to the reliability index when collapse is considered. From the engineering point of view this make sense, since for structures that present finite displacements collapse as defined previously has not yet occurred. Thus, the reliability index for collapse must always be bigger than (or at least equal to) the reliability index given for maximum allowable displacements.

Note that this result holds when the reliability index is the true reliability index of the problem. In most cases, the FORM does not give the true reliability index of the problem (Haldar and Mahadevan, 2000), but an approximation. In cases when this approximation is not good enough, (17) may not hold anymore, because the reliability index given by the FORM is not the true reliability index of the problem. However, as presented in the following examples, the approximation given by the FORM is satisfactory in most cases.

Based on (17), one way of evaluating the reliability index for the problem considering collapse is by making successive reliability analysis for increasing allowable displacements. Every time the FORM converges, one obtains a reliability index for a structure that does not collapse, that is a lower bound for the reliability index when collapse is considered. This procedure is performed until the FORM does not achieve convergence anymore. When the procedure ends, one have a list of reliability indexes, that are all lower bounds for the reliability index when collapse is considered.

Using this procedure one is able to know that the reliability index for collapse is bigger than a given value, what is sufficient for most practical applications. However, when one needs to know accurately the reliability index for collapse this procedure is not appropriate.

Finally, note that (17) was obtained considering that the displacement q_j is positive. However, the whole discussion is also valid when this displacement is negative, since some equations would be multiplied by minus one without changing the main results.

4 SENSITIVITY ANALYSIS

4.1 Partial derivatives of the displacements

As discussed previously, FORM algorithms need to evaluate the gradient of the failure function according to the design variables in order to carry out the reliability analysis. The procedure involved in evaluating this information is known in literature as sensitivity analysis (Haftka and Gürdal, 1992).

Suppose that the structure being studied has n degrees of freedom and that the sensitivity analysis is made for m design variables (parameters of the reliability problem). Besides, the design variables are all grouped in a vector $\mathbf{x}$. In this case, sensitivity analysis of the displacements is accomplished when the following matrix is evaluated:

$$\nabla_{\mathbf{x}}\mathbf{q} = \begin{bmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} & \cdots & \frac{\partial q_1}{\partial x_m} \\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} & \cdots & \frac{\partial q_2}{\partial x_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial q_n}{\partial x_1} & \frac{\partial q_n}{\partial x_2} & \cdots & \frac{\partial q_n}{\partial x_m} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{q}}{\partial x_1} & \frac{\partial \mathbf{q}}{\partial x_2} & \cdots & \frac{\partial \mathbf{q}}{\partial x_m} \end{bmatrix}, \tag{19}$$

that is the gradient of the nodal displacements $\mathbf{q}$ according to the design variables $\mathbf{x}$. This is also the gradient of the failure function according to the design variables, as can be seen from (6).

The gradient from (19) can be evaluated using some finite difference scheme, by applying small changes to each design variable and then solving solving the entire structural problem again. However, this approach implies solving the nonlinear structural problem several times, at least once for each design variable. Since the computational cost needed for a single non-linear structural analysis is generally high (since several systems of linear equations must be solved for each analysis), evaluating (19) using finite differences can lead to almost prohibitive computational costs.

Another approach for evaluating (19) is by using some technique from sensitivity analysis. Some very efficient approaches are discussed by Haftka and Gürdal (1992), but here we use a slightly modified version of a technique that is generally applied to linear structural problems (Haftka and Gürdal, 1992).

First, we differentiate (1) according to some design variable x_j and rearrange to get

$$\mathbf{K_s}.\frac{\partial \mathbf{q}}{\partial x_j} = \frac{\partial \mathbf{F}}{\partial x_j} - \frac{\partial \mathbf{K_s}}{\partial x_j}.\mathbf{q}. \tag{20}$$

The partial derivative of the stiffness matrix $\mathbf{K_s}$ is given by

$$\frac{\partial \mathbf{K_s}}{\partial x_j} = \frac{d\mathbf{K_s}}{dx_j} + \sum_{i=1}^{n} \frac{d\mathbf{K_s}}{dq_i}\frac{\partial q_i}{\partial x_j} + \sum_{i=1}^{n} \frac{d\mathbf{K_s}}{dF_i}\frac{\partial F_i}{\partial x_j}, \tag{21}$$

where d stands for an ordinary derivative that does not take into account the implicit relation between $\mathbf{K}$ and x_j by means of $\mathbf{q}$ or $\mathbf{F}$.

According to (21), changes to a parameter x_j can lead to changes to the stiffness matrix $\mathbf{K_s}$ by three different ways, namely by its direct influence on the stiffness matrix, by its influence on the displacements $\mathbf{q}$ and by its influence on the loads $\mathbf{F}$. In general, the parameters x_j will exert small or no influence at all on the applied loads, and thus the last term of (21) can be neglected. This is not true when the parameter x_j is actually some applied load, but this case is discussed further on. Besides, it is expected that the indirect influence of x_j to the stiffness matrix, by means of the displacements will be small if compared to its direct influence. Thus, the second term from (21) is also neglected. Consequently, the following approximation is made to (21)

$$\frac{\partial \mathbf{K_s}}{\partial x_j} \approx \frac{d\mathbf{K_s}}{dx_j}, \tag{22}$$

that can be evaluated by finite differences without solving any system of linear equations since it is an explicit derivative. Note that in using (22) we are making a linear approximation for the derivative of the stiffness matrix (that is actually non linear), by neglecting the influence of the displacements and applied loads.

Substituting (22) into (20) we get

$$\mathbf{K_s}.\frac{\partial \mathbf{q}}{\partial x_j} \approx \frac{\partial \mathbf{F}}{\partial x_j} - \frac{d\mathbf{K_s}}{dx_j}.\mathbf{q}. \tag{23}$$

Sensitivity analysis is then made by solving (23) for the partial derivatives $\partial \mathbf{q}/\partial x_j$. Since the system of linear equations are solved for the same secant stiffness matrix $\mathbf{K_s}$, decomposition procedures (such as Cholesky or L.U. decomposition) can be used efficiently (Bathe, 1996). These techniques decompose the stiffness matrix into a product of two matrices that are lower triangular and upper triangular. Each system of linear equations involved in obtaining $\partial \mathbf{q}/\partial x_j$ is then solved using retro substitution. This is a very efficient procedure when one needs to solve several systems of linear equations for the same coefficients matrix, since the main computational effort lies in the decomposition procedure itself, that is made only once.

If sensitivity analysis is made for some applied load, then the parameter x_j is actually some nodal load F_i. In this case, (23) is not a good approximation since (22) was obtained by neglecting the influence of the applied loads. However, sensitivity analysis for nodal loads can be made using the following relation (Bathe, 1996):

$$\mathbf{K_t}.d\mathbf{q} = d\mathbf{F}, \tag{24}$$

where $\mathbf{K_t}$ is the tangent stiffness matrix and $d\mathbf{q}$ and $d\mathbf{F}$ are differentials related to displacements and applied loads. This relation is frequently used for non linear structural analysis using Newton's Method or Quasi Newton's methods (Bathe, 1996).

In order to obtain the partial derivatives $\partial\mathbf{q}/\partial F_j$ we make the following approximation, based on (24):

$$\mathbf{K_t}.\frac{\partial\mathbf{q}}{\partial F_j} \approx \frac{\partial\mathbf{F}}{\partial F_j}, \tag{25}$$

where the partial derivative $\partial\mathbf{F}/\partial F_j$ is the derivative of the vector of applied loads according to some applied load. As can be verified by the reader, this derivative is actually a vector with all components equal to zero but component j, that is equal to plus or minus one, depending on the sign of the force F_j.

Sensitivity analysis for applied loads is then made by solving (25) for the partial derivatives $\partial\mathbf{q}/\partial F_j$. Again, decomposition procedures can be used efficiently since all the systems of linear equations from (25) are solved for the same tangent stiffness matrix. Note that the tangent stiffness matrix can be evaluated by the same procedure used for the secant stiffness matrix, but using the tangent modulus of the materials instead of the secant one.

In order to carry the entire sensitivity analysis one applies (25) for design variables that are nodal loads and (23) for the other design variables. Note that one system of linear equations must be solved for each design variable, but that the systems of linear equations are all solved either for the secant stiffness matrix or for the tangent stiffness matrix. Consequently, decomposition procedures can be used efficiently, as discussed previously. If the sensitivity analysis is made using finite differences, instead, one needs to solve the entire non linear structural problem at least once for each design variable. This would need much more computational effort than the approach used here, since each non linear structural analysis needs to solve several systems of linear equations.

5 NUMERICAL RESULTS

5.1 First example

The first example is that from Fig. 6. The cross section has $h = 30$cm and $b = 20$cm. The beam is 4m long and there is an applied force at mid span. Three reinforcement bars of 12mm are used in the upper chord and other three bars of 12mm are used in the lower chord. The covering distance is $c = 2.5$cm for both reinforcements. The material properties are taken as $\sigma^c_{yc} = 25$MPa, $\sigma^c_{yt} = 2.5$MPa, $\varepsilon^c_{yc} = 2/1000$, $\varepsilon^c_{yt} = 0.2/1000$, $\varepsilon^c_{uc} = 3/1000$, $\varepsilon^c_{ut} = 0.3/1000$, $\sigma^s_y = 450$MPa, $\varepsilon^s_y = 2.25/1000$ and $\varepsilon^s_u = 35/1000$. The beam is divided into 8 finite elements of equal length.

The displacement at mid span for different magnitudes of the applied load are presented in Fig. 7. It can be seen that the behavior of the model is consistent. For a load of about 2.5 tons

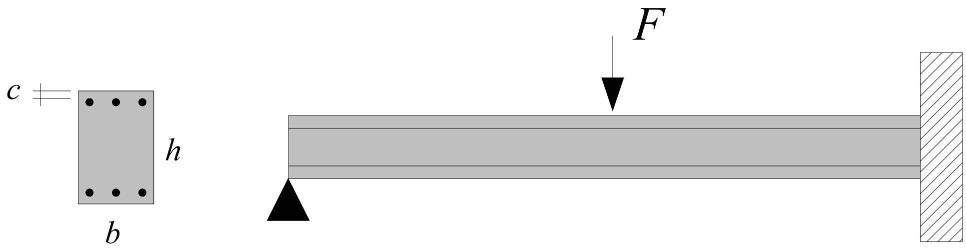

Figure 6: Reinforced concrete beam used in the first example.

the concrete is not able to resist tension anymore (it starts to crack), and so there is a change on the displacement evolution. For a load of about 8 tons the steel starts to yield finally leading to the collapse of the structure. This example was presented in order to show that even if the structural model is very simple it is able to capture the main nonlinear behavior of reinforced concrete structures.

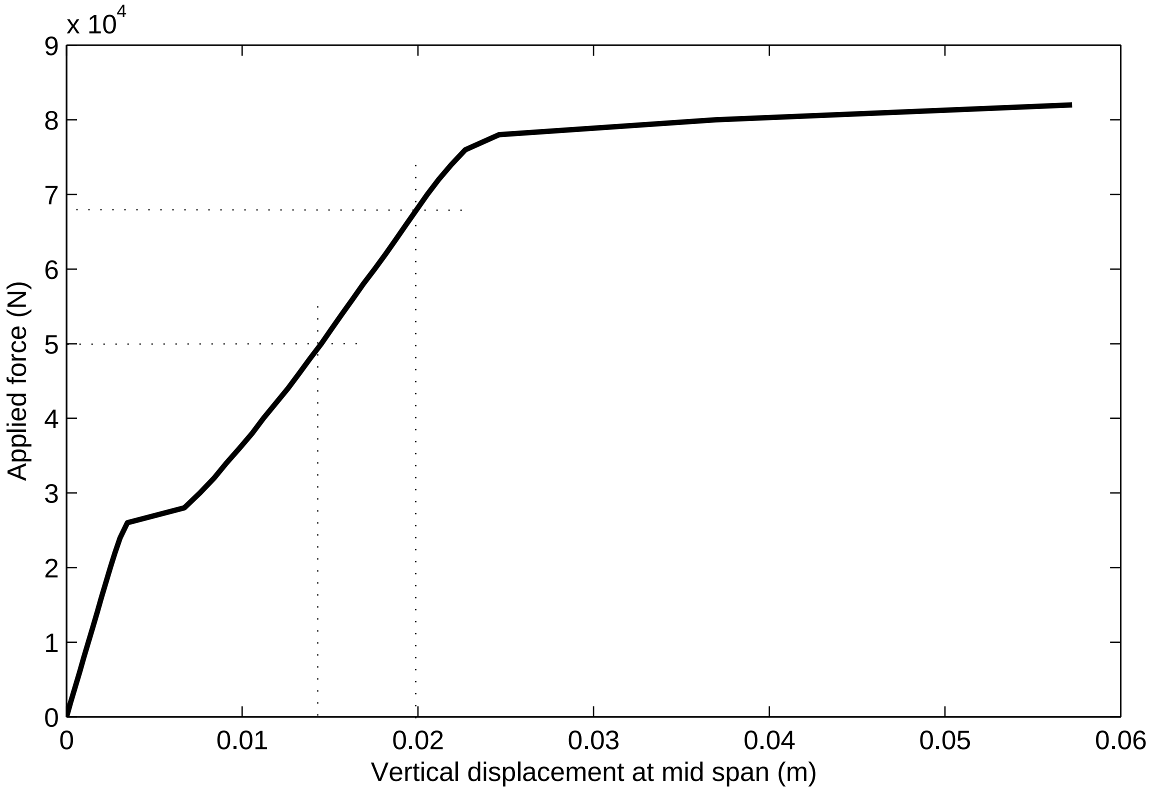

Figure 7: Displacement x Applied force for the beam used in the first example.

We now address the reliability analysis for this problem. The applied load is assumed to be a gaussian variable with mean equal to 5E4N(tons) and standard deviation equal to 0.5E4N(tons). This is called here parameter x_1. Besides, the standard deviation of the concrete yielding stress in compression σ_{yc}^c is assumed to have an standard deviation of 2.5MPa, while the standard deviation of the steel yielding stress σ_y^s is assumed to have an standard deviation of 10MPa. These variables are named here parameters x_2 and x_3, respectively. The other parameters are taken as deterministic variables with values as defined previously. The reliability analysis of this example is made assuming two allowable displacements at mid span, equal to 0.020m (Case 1) and equal to 0.023m (Case 2).

The results are presented in Tab. 1 and 2. The reliability index for Case 1 is 3.4850, while for Case 2 it is 5.0796. As expected, the reliability index increases when a larger displacement is allowed.

In order to study the influence of the probabilistic variables, we increase the standard deviation of the applied load to 1.0E4N(tons), the standard deviation of the concrete yielding stress in compression to 3.0MPa, and the standard deviation of the steel yielding stress to 20MPa. The allowable displacement is taken as 0.020m. This example is called here Case 3. The reliability index for this case is 1.7667 and the results are presented in Tab. 1 and 2. This results was also expected, since increasing the standard deviation of the random variables leads to a less reliable structure.

Case 3 was also solved using Monte Carlo Simulation (MCS) using 20,000 samples. The reliability index obtained was 1.7484, that is very close to the reliability index obtained with the FORM for this case (1.7667). However, the MCS took about 7h to run, while the FORM took about 2min on the same machine. Even if these times could be reduced by using more efficient programming techniques, it can be seen that the computational effort needed by the MCS is orders of magnitude bigger than that needed by the FORM. Besides, note that for Case 1, the number of samples used in the MCS would be even larger (about 100,000), since this structure is more reliable and consequently MCS needs more samples to give an accurate result.

$Case$	x_1	x_2	x_3	β
1	3.4317	-0.4387	-0.4201	3.4850
2	4.9815	-7.1527	-6.8900	5.0796
3	1.7487	-1.3104	-2.1509	1.7667

Table 1: Most probable failure point at the normalized space for the reliability analysis of the first example.

$Case$	F (E4N)	f_c (MPa)	f_s (MPa)
1	6.7158	23.9033	445.7988
2	7.4908	23.2118	443.1099
3	6.7488	24.6069	445.6983

Table 2: Most probable failure point at the real space for the reliability analysis of the first example.

5.2 Second example

The second example studied is that of the plane frame presented in Fig. 8, that is subjected to a lateral load of magnitude F. The cross section of each beam and the material properties are the same as used in the previous example. The base of the structure is $b = 4$m and each story is $h = 4$m height. Each beam is divided in two elements of equal length. Finally, the horizontal displacement u of the upper left node is measured, as pictured in Fig. 8.

The displacement u for different magnitudes of the load F is presented in Fig. 9. For the reliability analysis we assume that the allowable displacement u_{max} is equal to 0.050m. Besides, we take the applied load, the concrete yielding stress in compression and the steel yielding stress as the probabilistic variables. Two cases are studied. For Case 1, the applied load is assumed to be a gaussian variable with mean equal to 5E4N(tons) and standard deviation

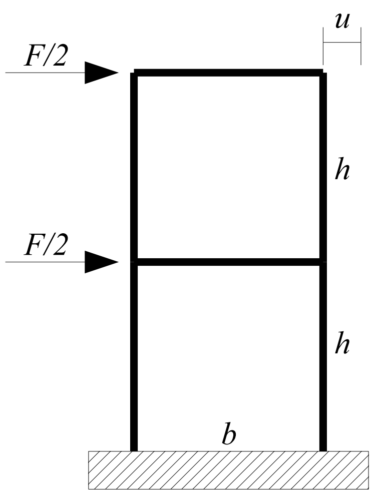

Figure 8: The plane frame used in the second example.

equal to 0.5E4N(tons). For Case 2, the load has the same mean value but the standard deviation is equal to 1.0E4N(tons). Besides, the standard deviation of the concrete yielding stress in compression is assumed to have an standard deviation of 2.5MPa, while the standard deviation of the steel yielding stress is assumed to have an standard deviation of 10MPa. The other parameters are taken as deterministic variables with values as defined for the previous example.

The reliability index obtained with the FORM was 2.2986 for Case 1 and 1.1780 for Case 2. The most probable failure points are presented in Tab. 3. As expected, increasing the standard deviation of the applied force leads to a less reliable structure.

Case 2 was also solved by MCS, using again 20,000 simulations. The reliability index found in this case was 1.1800, that is very similar to the one obtained with using FORM (1.1780). However, it is important to point out again that MCS needed much more computational effort than FORM, namely 3min against 10h.

$Case$	x_1	x_2	x_3	β
1	2.2279	-0.4593	-0.3300	2.2986
2	1.1689	-0.1173	-0.0872	1.1780

Table 3: Most probable failure point at the normalized space for the reliability analysis of the second example.

6 CONCLUSIONS

This paper presents an approach for the reliability analysis of nonlinear reinforced concrete beams and frames, assuming maximum allowable displacements. The reliability analysis problem is solved using a FORM algorithm, and sensitivity analysis is carried out in an efficient manner (without using finite differences). The numerical results obtained here agree with those obtained from MC simulation. However, MC needs much more computational effort than FORM, mainly for structures with a high level of reliability.

Reinforced concrete structures present significant nonlinear behavior, and thus reliability analysis should be done considering a nonlinear model. However, the reliability analysis prob-

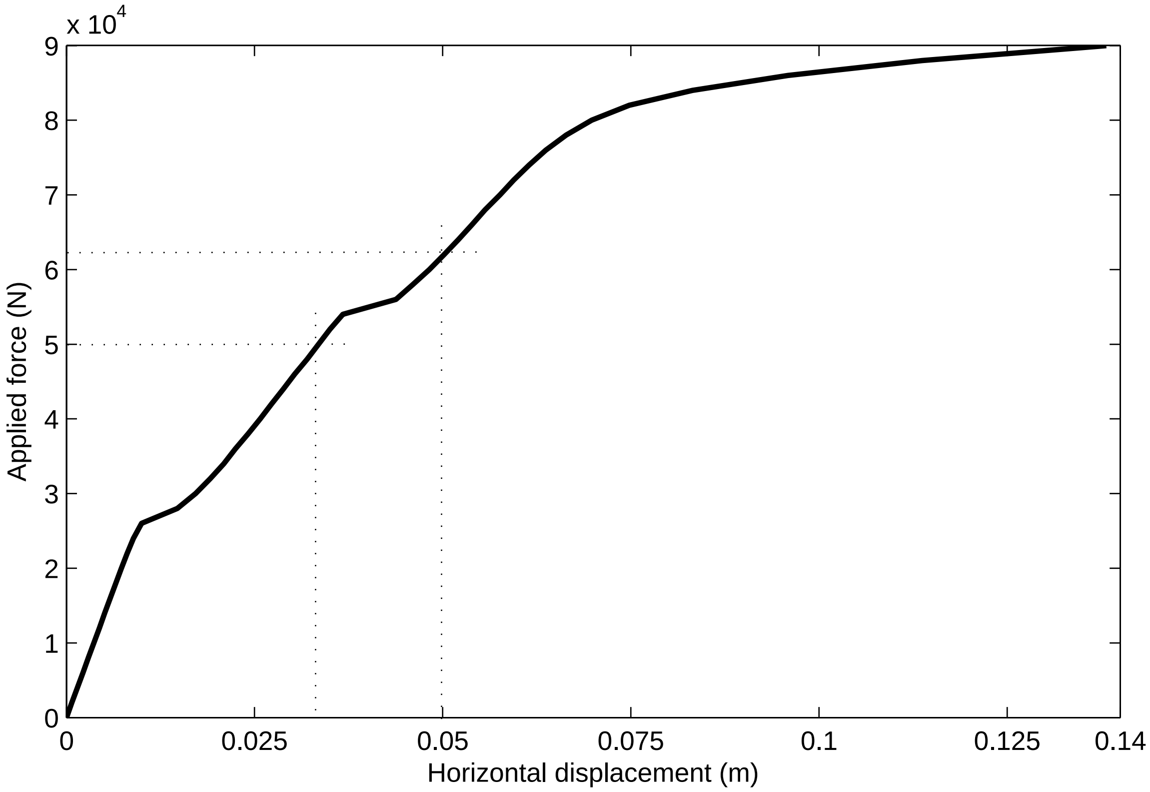

Figure 9: Displacement x Applied force for the beam used in the second example.

lem needs to carry out several deterministic analyses, thus leading to a high computational cost. In this context, it is interesting to have a structural model that is as simple as possible but that is able to represent the behavior of the structure appropriately. The structural model presented here is suitable for this task since it is efficient from the computational point of view. In a few words, using a very complex structural model would likely lead to prohibitive computational costs.

It has been proved that every reliability index considering some finite allowable displacement is a lower bound for the reliability index when collapse is considered. This allows one to estimate the reliability index indirectly by solving a few reliability analysis problems considering maximum allowable displacements. The importance of this result is that reliability analysis considering collapse can lead to computational difficulties, due to the very nature of collapse.

In most cases the designer does not need to know the exact value of the reliability index of the structure, but just need to know if the reliability index is bigger than a minimum value. In these cases, the reliability index considering collapse can be estimated as described here. Besides, in many current practical applications constraints on maximum displacements must be enforced in order to guarantee an appropriate use of the construction.

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