A MOVING BOILING-BOUNDARY MODEL OF AN ARBITRARY-POWERED TWO-PHASE FLOW LOOP

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Abstract. Usually, analysis of two-phase flow requires a substantial amount of mathematical effort, on the one hand by virtue of the intrinsic complexity of the physical phenomena involved and, on the other hand, because more often than not, the resulting equations are usually rather cumbersome. Several different approaches have been taken in order to solve particular cases by making some assumptions and applying approximations. One of these cases is the Clausse-Lahey model introduced twenty years ago, that proposes a system of differential-algebraic equations that approximately describes the transient behavior of a one-dimensional vertical two-phase flow channel. It uses a spatial scheme based on nodes that move with time, which reproduces experimental results better than traditional methods for the same number of nodes. In this paper, the Clausse-Lahey moving boiling-boundary model is extended to allow the inclusion of transient non-uniform power sources, to handle non-constant inlet enthalpy and arbitrary external pressure differences, so models of systems of industrial interest can be built. Some numerical results are shown as illustrations of the kind of problem the proposed extension to the original Clausse-Lahey model allows to solve.

1 INTRODUCTION

Flow involving phase changes occurs in several systems of industrial interest, being steam generators and nuclear reactor cores two emblematic examples. However, its mathematical description is, on the one hand, still not complete at microscopic levels; and, on the other hand, macroscopic formulations are usually complex and do not cover a wide variety of cases. There exists a particularly interesting simplified mathematical model that describes boiling flow in a heated vertical channel introduced by Clausse and Lahey in 1991, which has been used to study hydrodynamic instabilities (Delmastro (1993), Garea (1998), Garea et al. (1999), 2008), to model some systems of interest (for example Chang and Lahey (1997) and Theler (2008)) and even to study non-linear attractors caused due to non-linear nature of boiling phenomena (Clausse and Lahey (1991), Lahey (1992) and Theler (2008)). A detailed derivation of the model equations starting from the one-dimensional mass, energy and momentum conservation laws was given by the authors of the present article in 2010.

The mathematics contained in the original model—and developed step by step in the previous paper by Theler et al. (2010)—make some assumptions that may be too restrictive if the model is to be applied to systems of interest. Even though some extensions can be found in the literature, in most of the cases, the equations are just shown and neither derived nor justified, mainly because of length reasons. In this work we extend the original model to handle features that are mandatory when trying to apply it to closed-loop transient configurations. The present model allows for non-uniform non-steady power profiles, non-steady inlet enthalpy and non-steady external pressure drops.

Clausse and Lahey proposed a system of algebraic-differential equations that models the transient behavior of a uniformly heated vertical channel. As can be seen in the 2010 re-visit, the resulting expressions look rather cumbersome. Moreover, in the particular case of uniform heating the equations do have an explicit algebraic representation. When dealing with arbitrary space-and-time-dependent sources, the resulting equations are expressed in terms of integrals that may or may not have analytic solutions, depending on the arbitrariness of the power source. Therefore, as shown below in this paper, the equations themselves may need numerical evaluation even before performing the actual solution.

The mathematical steps proposed in this work are just one of many possible paths that an analyst may take to extend the original model. The particular development was mostly influenced by the objective of arriving at a set of equations suitable to be solved by the code MOCHIN, which is a software written by one of the authors of this article for his PhD thesis and freely released under the terms of the GNU General Public License. Incidentally, even though in the previous paper mochin was also used to solve the Clausse-Lahey model, the software was not mature enough and lacked many of the capabilities that are needed in order to deal with equations that are not themselves expressed as explicit algebraic expressions but rather as function of integrals of arbitrary functions.

Although the extensions of the original system of equations discussed by Theler et al. (2010) is explicitly emphasized, some of the equations are re-stated for completeness with further references to equations of the previous paper when appropriate. This work starts with the continuous space-and-time-dependent dimensionless problem formulation. The general continuous non-dimensinoal steady-state is then found. Explicit solutions for three particular axial distributions of power are shown, illustrating the differences with respect to the uniform-power case. Afterward, the Clausse-Lahey basis is applied to the continuous conservation equations arriving at a system of differential-algebraic equations, obtaining thus an extension to the original

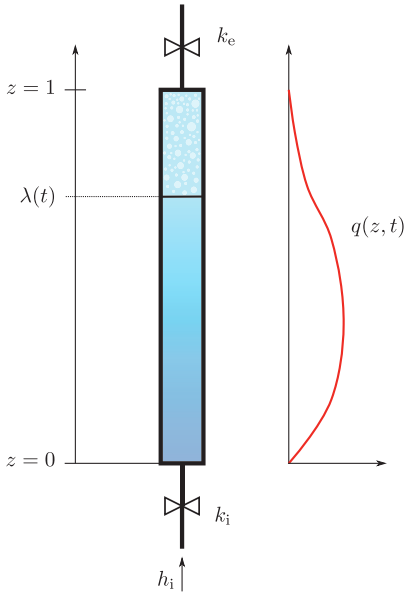


Figure 1: A non-dimensional vertical boiling channel with an arbitrary time-and-space dependent power source $q(z,t)$

model. Finally, some numerical results that show possible applications are provided, discussing and illustrating the differences the new model yields with respect to the original case. It is highly recommended to have a copy of the previous paper when reading the present one.

2 THE CONTINUOUS PROBLEM

The problem to be solved consists of a vertical heated channel subject to a certain external pressure drop in which subcooled fluid enters through the bottom and boils at some axial length $z = \lambda$ inside the channel. It is convenient to work with a dimensionless formulation, which is depicted in figure 1. The detailed steps that lead from the physical problem to the dimensionless equations, along with the introduction and definition of the different non-dimensional numbers that appear in the formulation are derived in sections 2.1 and 2.2 of the previous paper.

It is important to note that the axial power profile $q(z,t)$ is now allowed to vary both in space and in time. This variation can be a pre-defined function of space and time—for example when performing a laboratory experiment with a power ramp—or depend on the instantaneous state of the system—such as in a nuclear reactor where the fission rate depends on the temperatures of the materials and the void fraction distribution in the core, which in turn depend on the fission power. In any case, the function $q(z,t)$ is assumed to be computable for every axial position z and every time t.

Additionally, the inlet enthalpy $h_{\mathrm{i}}(t)$[1] is also allowed to vary with time. Analogously, this variation may be given as a pre-defined function of time or depend on the state of an autonomous system, like a closed loop in which the temperature at which the fluid enters into the heated length depends on the power history and on the conditions in which the fluid was cooled down and condensed back to the liquid state.

[1]Note that the subscript in h_{i} is written in roman style and refers to inlet conditions. This meaning should be differentiated from the subscript i which usually indicates an integer subscript as in the component x_i of a vector.

2.1 Dimensionless conservation equations

The non-dimensional homogeneous equilibrium mass, energy and momentum conservation equations for two-phase flow in one dimension are

$$\frac{\partial}{\partial t}\Big[\rho(z,t)\Big] + \frac{\partial}{\partial z}\Big[\rho(z,t)u(z,t)\Big] = 0 \tag{1}$$

$$\frac{\partial}{\partial t}\Big[\rho(z,t)h(z,t)\Big] + \frac{\partial}{\partial z}\Big[\rho(z,t)u(z,t)h(z,t)\Big] = \frac{N_{\text{sub}}}{N_{\text{pch}}} \cdot q(z,t) \tag{2}$$

$$\begin{aligned}\frac{\partial}{\partial t}\Big[\rho(z,t)u(z,t)\Big] + \frac{\partial}{\partial z}\Big[\rho(z,t)u^2(z,t)\Big] = &-\Big[\Lambda + k_{\text{i}}\delta(z) + k_{\text{e}}\delta(z-1)\Big] \cdot \rho(z,t)u^2(z,t) \\ &- \frac{\rho(z,t)}{\text{Fr}} - \frac{\partial p(z,t)}{\partial z}\end{aligned} \tag{3}$$

which correspond to equations (7), (10) and (11) of the previous paper. It is useful to integrate the momentum equation along the channel length to obtain

$$\begin{aligned}&\frac{d}{dt}\left[\int_0^1 \rho(z,t)\cdot u(z,t)\,dz\right] + \rho(1,t)u^2(1,t) - \rho(0,t)u^2(0,t) = \\ &\quad -\Lambda\int_0^1 \rho(z,t)u^2(z,t)\,dz - k_{\text{i}}\cdot\rho(0,t)\cdot u^2(0,t) - k_{\text{e}}\cdot\rho(1,t)\cdot u^2(1,t) \\ &\qquad -\frac{1}{\text{Fr}}\int_0^1 \rho(z,t)\,dz + \text{Eu}(t)\end{aligned} \tag{4}$$

that is analogous to equation (13) of the previous paper, with the exception that the Euler number is allowed to depend on time. Finally, the equation of state reads

$$\rho(z,t) = \begin{cases} 1 & \text{if } h(z,t) \le 0 \\ \dfrac{1}{1 + N_{\text{pch}}\cdot h(z,t)} & \text{if } h(z,t) > 0 \end{cases} \tag{5}$$

The dimensionless parameters that appear in the equations are briefly defined in table 2 and thoroughly described in the previous paper.

2.2 Steady state

In the steady state, the problem resides in finding the velocity $u^\star(z)$, enthalpy $h^\star(z)$ and density $\rho^\star(z)$ profiles that make all the time derivatives vanish. The subcooled length ranges from $z = 0$ up to $z = \lambda^\star$, which is the boiling boundary defined as the axial point at which the non-dimensional enthalpy is zero—that is to say the fluid starts boiling. Assuming incompressible flow for the subcooled region (i.e. that the dimensional ρ_f, v_g, h_{fg}, etc. are constant) the mass conservation equation (1) gives

$$\rho^\star(z) = 1 \quad \text{if } z < \lambda^\star$$

while by definition of the subcooling number N_{sub} and the phase change number N_{pch}, the steady-state velocity in the one-phase zone is

Dimensionless parameters

N_{pch}	$=$	$\dfrac{q^+}{\rho_f^+ A^+ u_0^+}\dfrac{v_{fg}^+}{h_{fg}^+ v_f^+}$	Phase-change number
N_{sub}	$=$	$\dfrac{h_f^+ - h_i^+}{h_{fg}^+}\dfrac{v_{fg}^+}{v_f^+}$	Subcooling number
Fr	$=$	$\dfrac{{u_{\text{ref}}^+}^2}{g^+ L^+}$	Froude number
Eu	$=$	$\dfrac{\Delta p_{\text{ext}}^+}{\rho_f^+ {u_{\text{ref}}^+}^2}$	Euler number
Λ	$=$	$\dfrac{1}{2}\dfrac{f L^+}{D_H^+}$	Distributed friction number
k_i	$=$	$\dfrac{\Delta p_i^+}{\rho_i^+ {u_i^+}^2}$	Inlet head loss coefficient
k_e	$=$	$\dfrac{\Delta p_e^+}{\rho_e^+ {u_e^+}^2}$	Outlet head loss coefficient

Figure 2: Dimensionless formulation of the vertical boiling channel problem

$$u^\star(z) = \frac{N_{\text{sub}}}{N_{\text{pch}}} \qquad \text{if } z < \lambda^\star$$

On the other hand, the energy conservation equation (2) can be written as

$$\frac{\partial}{\partial t}\Big[\rho(z,t)h(z,t)\Big] + \frac{\partial}{\partial z}\Big[\rho(z,t)u(z,t)h(z,t)\Big] = \frac{N_{\text{sub}}}{N_{\text{pch}}} \cdot q^\star(z)$$

where the steady-state power profile $q^\star(z) = q(z,0)$ should be normalized to one such that

$$\int_0^1 q^\star(z)\, dz = 1$$

In the steady-state subcooled zone, the energy equation states that

$$\frac{\partial h^\star(z)}{\partial z} = q^\star(z)$$

so with the boundary condition $h^\star(0) = h_{\text{i}}^\star = -N_{\text{sub}}/N_{\text{pch}}$ by definition, the enthalpy profile in the single-phase zone is

$$h^\star(z) = -\frac{N_{\text{sub}}}{N_{\text{pch}}} + \int_0^z q^\star(z')\,dz' \qquad \text{if } z < \lambda^\star$$

Note the difference between this expression and equation (15) of the previous paper for uniform power.

Now, the steady-state boiling boundary $\lambda^\star$ is the axial position at which the enthalpy is zero. In the previous paper, $\lambda^\star$ had the simple expression $N_{\text{sub}}/N_{\text{pch}}$, which was less than one for two-phase cases that imply that $N_{\text{sub}} > N_{\text{pch}}$. For an arbitrary power distribution, $\lambda^\star$ is the solution of the equation

$$\int_0^{\lambda^\star} q^\star(z')\,dz' - \frac{N_{\text{sub}}}{N_{\text{pch}}} = 0 \tag{6}$$

which as expected reduces to $\lambda^\star = N_{\text{sub}}/N_{\text{pch}}$ for uniform and normalized-to-one power. The integral of $q^\star(z')$ between zero and the boiling boundary is the fraction of the power dissipated in the single-phase length. For $\lambda^\star$ to be less than one, this fraction should also be less than one, which entails that $N_{\text{sub}} > N_{\text{pch}}$.

In the two-phase zone, the energy equation (2) reads

$$\frac{\partial}{\partial z}\Big[\rho^\star(z)\,u^\star(z)\,h^\star(z)\Big] = \frac{N_{\text{sub}}}{N_{\text{pch}}} \cdot q^\star(z)$$

but from the equation of state (5),

$$\rho^\star(z)\,h^\star(z) = \frac{1-\rho^\star(z)}{N_{\text{pch}}}$$

Therefore

$$\begin{aligned}
\frac{\partial}{\partial z}\left[\frac{1-\rho^\star(z)}{N_{\text{pch}}} \cdot u^\star(z)\right] &= \frac{N_{\text{sub}}}{N_{\text{pch}}} \cdot q^\star(z) \\
\frac{\partial}{\partial z}\Big[u^\star(z) - \rho^\star(z)\,u^\star(z)\Big] &= N_{\text{sub}} \cdot q^\star(z)
\end{aligned}$$

By virtue of the continuity equation (1), the spatial derivative of the product $\rho^\star(z)\,u^\star(z)$ vanishes and thus

$$\begin{aligned}
\frac{\partial u^\star(z)}{\partial z} &= N_{\text{sub}} \cdot q^\star(z) \\
u^\star(z) &= \frac{N_{\text{sub}}}{N_{\text{pch}}} + N_{\text{sub}} \cdot \int_{\lambda^\star}^z q^\star(z')\,dz' \qquad \text{for } z > \lambda^\star
\end{aligned} \tag{7}$$

Going back to the energy equation (2), we have

$$\frac{\partial}{\partial z}\Big[\rho^\star(z)\,u^\star(z)\,h^\star(z)\Big] = \frac{N_{\text{sub}}}{N_{\text{pch}}} \cdot q^\star(z)$$

which, after replacing the profiles found so far transforms into

$$\frac{\partial}{\partial z}\left[\frac{\left(\dfrac{N_{\text{sub}}}{N_{\text{pch}}}+N_{\text{sub}}\cdot\displaystyle\int_{\lambda^\star}^{z}q^\star(z')\,dz'\right)h^\star(z)}{1+N_{\text{pch}}\cdot h^\star(z)}\right]=\frac{N_{\text{sub}}}{N_{\text{pch}}}\cdot q^\star(z)$$

from where the steady-state enthalpy profile for the two-phase zone can be obtained by first integrating both members with respect to z

$$\frac{\left(\dfrac{N_{\text{sub}}}{N_{\text{pch}}}+N_{\text{sub}}\cdot\displaystyle\int_{\lambda^\star}^{z}q^\star(z')\,dz'\right)h^\star(z)}{1+N_{\text{pch}}\cdot h^\star(z)}=\frac{N_{\text{sub}}}{N_{\text{pch}}}\cdot\int_{\lambda^\star}^{z}q^\star(z')\,dz'$$

and then solving for $h^\star(z)$ to obtain

$$h^\star(z)=\int_{\lambda^\star}^{z}q^\star(z')\,dz' \qquad \text{for } z>\lambda^\star$$

Summing up, the steady-state enthalpy, density and velocity profiles are

$$h^\star(z)=-\frac{N_{\text{sub}}}{N_{\text{pch}}}+\int_{0}^{z}q^\star(z')\,dz' \tag{8}$$

$$u^\star(z)=\begin{cases}\dfrac{N_{\text{sub}}}{N_{\text{pch}}} & \text{if } z\le\lambda^\star\\[2ex] \dfrac{N_{\text{sub}}}{N_{\text{pch}}}+N_{\text{sub}}\cdot\displaystyle\int_{\lambda^\star}^{z}q^\star(z')\,dz' & \text{if } z>\lambda^\star\end{cases} \tag{9}$$

$$\rho^\star(z)=\begin{cases}1 & \text{if } z\le\lambda^\star\\[2ex] \dfrac{1}{1+N_{\text{pch}}\cdot\displaystyle\int_{\lambda^\star}^{z}q^\star(z')\,dz'} & \text{if } z>\lambda^\star\end{cases} \tag{10}$$

These expressions are to be compared with the corresponding equations (22), (23) and (24) of the previous paper, to which they reduce when $q^\star(z)$ is identically equal to one.

For arbitrary power profiles $q^\star(z)$—such as one defined by discrete abscissa-ordinate pairs—the steady-state solution as a function of the axial coordinate z depends on the integral of $q^\star(z)$ which in general does not have an algebraic expression. Moreover, out of the seven non-dimensional parameters defined in figure 2, only six are independent, even in the uniform-power case. In the previous paper, the relationship between the seven parameters is presented in equation (25). In the general, this expression can be generalized by replacing the steady-state profiles into the integrated momentum equation (4) as follows:

$$
\begin{aligned}
0 \quad = \quad & \frac{\left(\dfrac{N_{\text{sub}}}{N_{\text{pch}}} + N_{\text{sub}} \displaystyle\int_{\lambda^\star}^{1} q^\star(z')\, dz'\right)^2}{1 + N_{\text{pch}} \displaystyle\int_{\lambda^\star}^{1} q^\star(z')\, dz'} - \left(\frac{N_{\text{sub}}}{N_{\text{pch}}}\right)^2 + \Lambda \left(\frac{N_{\text{sub}}}{N_{\text{pch}}}\right)^2 \cdot \lambda^\star \\
& + \Lambda \cdot \left[\int_{\lambda^\star}^{1} \frac{\left(\dfrac{N_{\text{sub}}}{N_{\text{pch}}} + N_{\text{sub}} \displaystyle\int_{\lambda^\star}^{z} q^\star(z')\, dz'\right)^2}{1 + N_{\text{pch}} \displaystyle\int_{\lambda^\star}^{z} q^\star(z')\, dz'} dz \right] \\
& + k_{\text{i}} \cdot \left(\frac{N_{\text{sub}}}{N_{\text{pch}}}\right)^2 + k_{\text{e}} \cdot \frac{\left(\dfrac{N_{\text{sub}}}{N_{\text{pch}}} + N_{\text{sub}} \cdot \displaystyle\int_{\lambda^\star}^{1} q^\star(z')\, dz'\right)^2}{1 + N_{\text{pch}} \cdot \displaystyle\int_{\lambda^\star}^{1} q^\star(z')\, dz'} \\
& + \frac{1}{\text{Fr}} \cdot \left[\lambda^\star + \int_{\lambda^\star}^{1} \frac{1}{1 + N_{\text{pch}} \displaystyle\int_{\lambda^\star}^{z} q^\star(z')\, dz'} \right] - \text{Eu}^\star
\end{aligned} \tag{11}
$$

where, recalling equation (6), the steady-state boiling boundary $\lambda^\star$ is the solution of

$$
\int_{0}^{\lambda^\star} q^\star(z')\, dz' - \frac{N_{\text{sub}}}{N_{\text{pch}}} = 0 \tag{6}
$$

If the steady-state power profile $q^\star(z)$ is independent from the problem parameters (i.e. it comes from an external electric power source), then equations (11) and (6) should be well-defined and have one unique solution. However, if the steady-state power profile depends on the problem parameters (for example it comes from a nuclear source, in which the local fission rate depends on the temperature and density distributions of the materials in the reactor core), then an outer-inner iterative scheme will be needed. In this case, the mathematical formulation may be ill-posed and/or have convergence issues. A general result cannot be obtained and specific analyses for each particular problem are needed.

Because of its definition, the phase-change number N_{pch} cannot be directly computed from the problem dimensional parameters because it depends on the steady-state inlet velocity u_0 (see the previous paper for details on how this parameter enters into the conservation equations). Therefore, it is usually this parameter that is calculated as a function of the rest, which are directly computed from the definition and geometry of the dimensional problem.

Equations (11) and (6) look far more complicated to solve than the enthalpy, velocity and density profiles given by equations (8), (9) and (10). Indeed, it is in the computation of the phase-change number and the boiling boundary where the difficulty of the vertical boiling channel resides. However, as dazzling as might seem, equations (11) and (6) are perfectly suitable to be tackled with the general mathematical tools provided in the wasora framework (Theler, 2012b), which the DAE-solving software mochin (Theler, 2012a) together with the library SUNDIALS (Hindmarsh et al., 2005) use.[2] In effect, the following input instructs the wa-

[2] Actually, this fact is less a coincidence than a desired result. If an engineer working in the nuclear industry—

sora code to solve for the phase-change number and to obtain the steady-state conditions for $N_{\text{sub}} = 5$, $\text{Eu}^\star = 10$, $\text{Fr} = 5$, $k_\text{i} = 3$, $k_\text{e} = 2$ and $\Lambda = 3$:

```
# steady-state computation of the continuous boiling
# channel problem
VAR z z     # some dummy variables

# the problem parameters (remember, the difficulty
# relies in computing Npch as a function of the rest)
Nsub = 5
Eu = 10
Fr = 5
Lambda = 3
ki = 6
ke = 2

#uniform power profile
#FUNCTION qstar(z) = 1

# sinusoidal power profile
#FUNCTION qstar(z) = pi/2 * sin(z*pi)

#arbitray normalized spline-interpolated power profile
FUNCTION potencia(z) INTERPOLATION splines DATA {
0       0
0.2     2.5
0.5     3
0.6     2.5
0.7     1.4
0.85    0.3
1       0
}
norm = integral(potencia(z), z, 0, 1)
FUNCTION qstar(z) = 1/norm * potencia(z)


# define the boiling boundary as a function of Npch
FUNCTION lambdastar(Npch) = root(integral(qstar(z), z, 0, z) -
    Nsub/Npch, z, 0, 1)

# this expression appears several times, so it is handy
# to have it as a function
FUNCTION q2phistar(z,Npch) = integral(qstar(z), z, lambdastar(
    Npch), z)

# define the function that has to be zero when the
# phase-change number is correctly defined
FUNCTION F(Npch) = {
    (Nsub/Npch + Nsub*q2phistar(1,Npch))^2/(1 + Npch * q2phistar(1,
        Npch))
 - (Nsub/Npch)^2
 + Lambda*(Nsub/Npch)^2*lambdastar(Npch)
 + Lambda*integral((Nsub/Npch + Nsub*q2phistar(z,Npch))^2/(1 +
```

which at the same time is a PhD student—develops scientific codes to solve the problems he encounters in his daily activities, sooner or later he will find out that his tools finally fit his needs.

```
    Npch*q2phistar(z,Npch)), z, lambdastar(Npch), 1)
 + ki*(Nsub/Npch)^2
 + ke*(Nsub/Npch + Nsub*q2phistar(1,Npch))^2 / (1 + Npch*q2phistar
    (1,Npch))
 + 1/Fr * (lambdastar(Npch) + integral(1/(1 + Npch*q2phistar(z,
    Npch)), z, lambdastar(Npch), 1))
 - Eu
}

# the steady-state solutions
FUNCTION ustar(z) = if(less(z, lambda), Nsub/Npch, Nsub/Npch +
   Nsub * integral(qstar(z ), z , lambda, z))
FUNCTION hstar(z) = -Nsub/Npch + integral(qstar(z ), z , 0, z)
FUNCTION rhostar(z) = if(less(z, lambda), 1, 1/(1 + Npch *
   integral(qstar(z ), z , lambda, z)))

# compute the phase-change number
Npch = root(F(Npch), Npch, Nsub+1e-3, 50)

# print the results to the standard output for plotting
PRINT TEXT "\# Npch = " Npch
PRINT TEXT "\# lambda* = " lambdastar(Npch)
PRINT_FUNCTION qstar ustar hstar rhostar MIN 0 MAX 1 STEP 1e-3
```

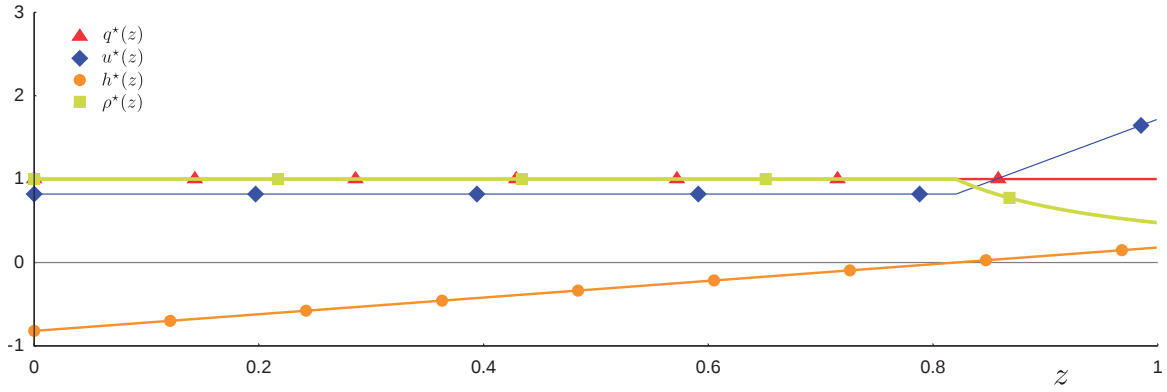
Figure 3 shows the steady-state solutions for three different power distributions. The first one, figure 3a, corresponds to the uniform power case—commented out in the above input. The second—also commented out—in figure 3b represents an algebraic sinusoidal power distribution. The third one in figure 3c—the only uncommented in the input—shows the solution for an arbitrarily point-wise-defined spline-interpolated power profile. It can be seen how N_{pch} increases and $\lambda^\star$ decreases as more power is shifted to the one-phase region and thus forcing the fluid to boil earlier than in the uniform-power case.

These results illustrate how equation (11) could be implemented just be entering it into a plain text file using some predefined rules (explained in wasora s documentation (Theler, 2012b)) and then letting the code taking care of the rest. A similar approach is used when dealing with the transient DAE equations of the extended Clausse-Lahey model in the following sections.

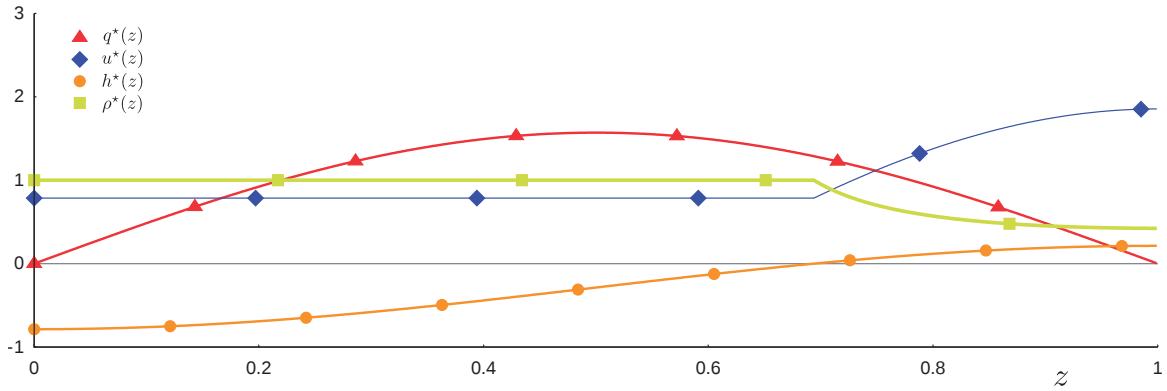
3 THE EXTENDED CLAUSSE-LAHEY MODEL

Solving the time-dependent conservation equations for the vertical boiling channel involves far more mathematical effort than the needed to obtain the steady-state conditions. Indeed, there exist many different ways of tackling the transient case. One of them is the interesting approach proposed by Clausse and Lahey (1991) in which, instead of dividing the channel into fixed axial nodes and solving for the enthalpy at those locations, the authors allow the position of the nodes to move along the channel following fixed-enthalpy values (figure 4)). Some of the advantages of this method is that few nodes are enough to replicate experimental results, there is no explicit Courant limit on the time step and, most importantly—at least from this paper s point of view—that it leads to a differential-algebraic system of equations of the form

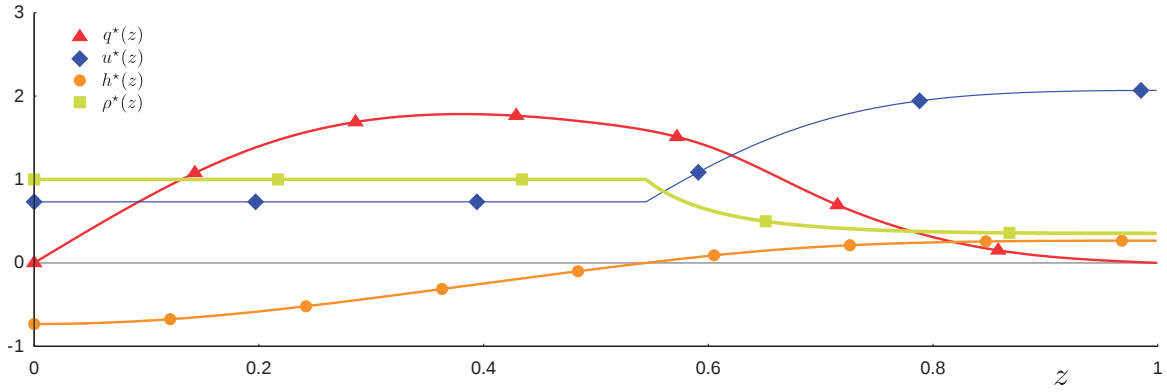
$$\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t) = 0 \tag{12}$$



(a) Steady-state profiles for $N_{\text{sub}} = 5$, $\text{Eu}^\star = 10$, $\text{Fr} = 5$, $k_{\text{i}} = 3$, $k_{\text{e}} = 2$ and $\Lambda = 3$. The resulting phase change number is $N_{\text{pch}} = 6.095254\ldots$ and the steady-state boiling boundary is $\lambda^\star = 8.203103\ldots$.



(b) Idem as above with $N_{\text{pch}} = 6.359455\ldots$ and $\lambda^\star = 6.940115\ldots$.



(c) Idem as above with $N_{\text{pch}} = 6.822077\ldots$ and $\lambda^\star = 5.445669\ldots$.

Figure 3: Steady state solutions for three cases of power distributions

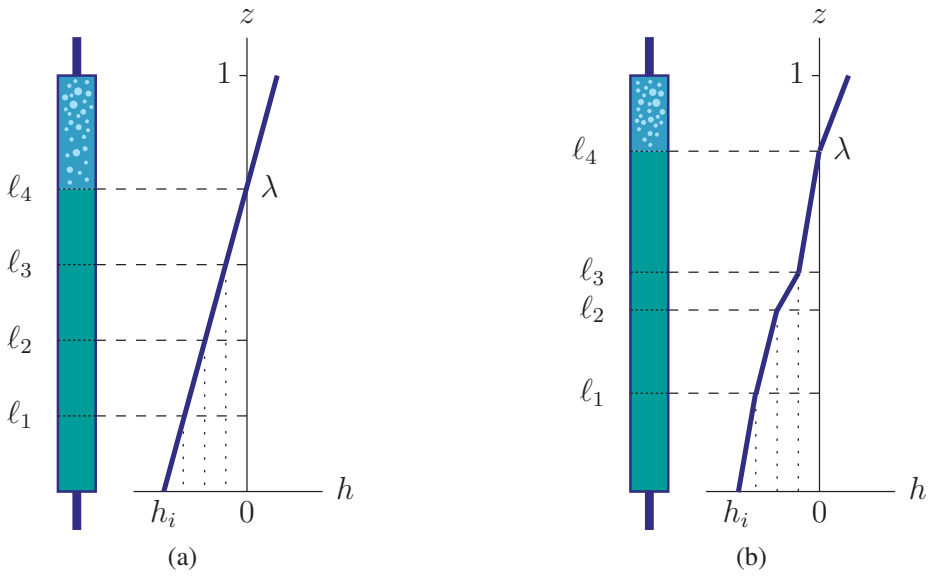


Figure 4: Basis of the moving boiling-boundary model. The one-phase zone is divided into $N_1 = 4$ cells whose boundaries are allowed to move following points of fixed enthalpy. (a) Node positions in the steady state for an uniform power profile. (b) Non-steady instantaneous state. Figure reproduced from Theler et al. (2010)

where $\mathbf{x} \in \mathbb{R}^n$ is a state vector in an n-dimensional phase space, which can then be solved and analyzed using the full baggage of non-linear dynamical systems theory (Solari et al., 1996; Alligood et al., 1997; Ott, 2002).

As explained in the previous paper, the basic idea is to divide the single-phase zone into N_1 cells but with the particularity that their $N_1 + 1$ boundaries—herein after called nodes—are allowed to move along the channel with time. The model includes, amongst others, differential equations that describe the dynamics of these node positions. The original paper by Clausse and Lahey (1991) just presents the model with no detailed derivation. Some other works such as Lahey (1992) and Chang and Lahey (1997) do provide some more insight, but they still skip some mathematical steps that may be interesting for the reader to look at in order to understand the rationale behind the equations of the model. Therefore, the previous paper (Theler et al., 2010) was prepared, in which many—but not all—of the intermediate steps of the derivation of the final model were explicitly shown, and—whenever possible—a brief explanation of why certain approximation or decision was—or not—taken. We now proceed to re-derive the equations for solving the vertical boiling channel equation in the transient case. Nevertheless, we just go back to the original approach of skipping mathematical steps and just stating the main idea and the final equation we arrive at, otherwise this paper would be a mere repetition of the previous one. The rationale of each of the equations may be inferred by comparing the two papers.

We define the positions $\ell_n(t)$ of the $n-$th single-phase nodes for $n = 0, \ldots, N_1$ as the axial coordinates at which

$$h\left[\ell_n(t)\right] = h_\mathrm{i}(t)\left[1 - \frac{n}{N_1}\right] \tag{13}$$

where it must be remembered that the non-dimensional inlet enthalpy h_i is negative, and may depend on time either as a known function or according to the instantaneous state of the channel. In any case, this equation ought to be compared to equation (28) of the previous paper, for the constant inlet enthalpy case. Note that $\ell_0(t) = 0$ and $\ell_{N_1}(t) = \lambda(t)$.

A set of differential equations for the positions of the nodes $\ell_n(t)$ can be obtained by integrating the energy equation (2) between ℓ_{n-1} and ℓ_n

$$\int_{\ell_{n-1}(t)}^{\ell n(t)} \frac{\partial}{\partial t}\Big[\rho(z,t)h(z,t)\Big]\,dz + \int_{\ell_{n-1}(t)}^{\ell n(t)} \frac{\partial}{\partial z}\Big[\rho(z,t)u(z,t)h(z,t)\Big]\,dz = \frac{N_\mathrm{sub}}{N_\mathrm{pch}} \int_{\ell_{n-1}(t)}^{\ell n(t)} q(z,t)\,dz$$

As the integration limits depend on time, the Leibnitz rule

$$\int_{a(t)}^{b(t)} \frac{\partial \mathcal{F}(x)}{\partial t}\,dx = \frac{d}{dt}\left[\int_{a(t)}^{b(t)} \mathcal{F}(x)\,dx\right] - \mathcal{F}\left[b(t)\right]\frac{db}{dt} + \mathcal{F}\left[a(t)\right]\frac{da}{dt}$$

has to be used in order to transform the partial time derivative into a total derivative, so the energy equation reads

$$\frac{d}{dt}\left[\int_{\ell_{n-1}(t)}^{\ell n(t)} h(z,t)\,dz\right] - h\big(\ell_n(t)\big)\frac{d\ell_n}{dt} + h\big(\ell_{n-1}(t)\big)\frac{d\ell_{n-1}}{dt} + u_\mathrm{i}(t)\big[h\big(\ell_n(t)\big) - h\big(\ell_{n-1}(t)\big)\big] = \frac{N_\mathrm{sub}}{N_\mathrm{pch}} \int_{\ell_{n-1}(t)}^{\ell n(t)} q(z,t)\,dz$$

Assuming a linear enthalpy profile between two adjacent nodes, the integral in the first term can be easily obtained using the trapezoid rule as

$$\int_{\ell_{n-1}(t)}^{\ell n(t)} h(z,t)\,dz = \frac{1}{2}\cdot\Big[h\big(\ell_n(t)\big) + h\big(\ell_{n-1}(t)\big)\Big]\cdot\big[\ell_n(t) - \ell_{n-1}(t)\big]$$

Replacing this equation, taking into account equation (13) and rearranging terms in such a way that the problem variables appear explicitly in the equation, we arrive at

$$-\frac{1}{h_\mathrm{i}(t)}\cdot\frac{dh_\mathrm{i}}{dt}\left(N_1 - n - \frac{1}{2}\right)\Big[\ell_n(t) - \ell_{n-1}(t)\Big] + \frac{1}{2}\left(\frac{d\ell_n}{dt} + \frac{d\ell_{n-1}}{dt}\right) - u_\mathrm{i}(t) - \frac{N_1}{h_\mathrm{i}(t)}\cdot\frac{N_\mathrm{sub}}{N_\mathrm{pch}}\int_{\ell_{n-1}(t)}^{\ell n(t)} q(z,t)\,dz = 0 \qquad (14)$$

Note that if the inlet enthalpy depends on the state vector $\mathbf{x}$ of an hypothetical dynamical system of the form (12), then $h_\mathrm{i}(t)$ will be one of the components of $\mathbf{x}$. In this case, it is better to leave the first term as it is instead of re-writing it as the derivative of the logarithm, because both h_i and $\dot{h}_\mathrm{i}$ will be already available for evaluation the differential equation, whilst the derivative of the logarithm would require additional computation.

Equations (14) depend on the power $q(z,t)$, on the inlet enthalpy $h_\mathrm{i}(t)$ and on the inlet velocity $u_\mathrm{i}(t)$. How these three variables change with time depend on the actual problem being solved. That is to say, the power distribution may change with time in a pre-defined manner or

depend on the problem state vector. Analogously, the inlet enthalpy may change in a certain way when dealing with an open or closed loop. However, the case of the inlet velocity is different. Once the Euler number—which is the dimensionless version of the pressure difference across the channel—is fixed by the problem conditions such as closed or open loop, area changes, etc., then the velocity in the channel can be computed only having into account $\mathrm{Eu}(t)$. In effect, the momentum equation (4) reads

$$\begin{aligned}\frac{d}{dt}\left[\int_0^1 \rho(z,t)\cdot u(z,t)\,dz\right] + \Big(\rho_{\mathrm{e}}(t)\,u_{\mathrm{e}}^2(t) - \rho_{\mathrm{i}}(t)\,u_{\mathrm{i}}^2(t)\Big) = \\ -\Lambda\int_0^1 \rho(z,t)\,u^2(z,t)\,dz - k_{\mathrm{i}}\cdot\rho_{\mathrm{i}}(t)\,u_{\mathrm{i}}^2(t) - k_{\mathrm{e}}\cdot\rho_{\mathrm{e}}(t)\,u_{\mathrm{e}}^2(t) - \frac{1}{\mathrm{Fr}}\int_0^1 \rho(z,t)\,dz + \mathrm{Eu}(t)\end{aligned} \tag{15}$$

To evaluate the term in parenthesis, we first note that in the single-phase zone, the speed profile equals the inlet velocity

$$u(z,t) = u_{\mathrm{i}}(t) \qquad \text{for } z < \lambda(t)$$

Notably, a similar result to equation (7) can be obtained in the general transient case. The transient energy equation (2) is

$$\frac{\partial}{\partial t}\Big[\rho(z,t)h(z,t)\Big] + \frac{\partial}{\partial z}\Big[\rho(z,t)u(z,t)h(z,t)\Big] = \frac{N_{\mathrm{sub}}}{N_{\mathrm{pch}}}\cdot q(z,t) \tag{2}$$

and replacing the following result that holds from the equation of state (5),

$$\rho(z,t)\,h(z,t) = \frac{1-\rho(z,t)}{N_{\mathrm{pch}}}$$

we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1-\rho(z,t)}{N_{\mathrm{pch}}}\right) + \frac{\partial}{\partial z}\left[\frac{1-\rho(z,t)}{N_{\mathrm{pch}}}\cdot u(z,t)\right] &= \frac{N_{\mathrm{sub}}}{N_{\mathrm{pch}}}\cdot q(z,t) \\ -\frac{\partial\rho(z,t)}{\partial t} - \frac{\partial}{\partial z}\Big[\rho(z,t)\cdot u(z,t)\Big] + \frac{\partial u(z,t)}{\partial z} &= N_{\mathrm{sub}}\cdot q(z,t) \\ \frac{\partial u(z,t)}{\partial z} &= N_{\mathrm{sub}}\cdot q(z,t)\end{aligned}$$

For the two-phase zone,

$$u(z,t) = u_{\mathrm{i}}(t) + N_{\mathrm{sub}}\int_{\lambda(t)}^{z} q(z',t)\,dz' \qquad \text{for } z > \lambda(t) \tag{16}$$

In particular,

$$u_{\mathrm{e}}(t) = u_{\mathrm{i}}(t) + N_{\mathrm{sub}}\int_{\lambda(t)}^{1} q(z',t)\,dz' \tag{17}$$

We now need an expression for the exit density $\rho_{\text{e}}(t)$. We assume that in the two-phase region, the fluid enthalpy is, in the same way as in the steady-state, the integral of the power profile but modified by a certain slope $\eta(t)$ as

$$h(z,t) = \eta(t) \cdot \int_{\lambda(t)}^{z} q(z',t)\, dz' \qquad \text{for } z > \lambda(t) \tag{18}$$

where $\eta(0) = 1$ to recover the steady-state result (8). The density profile is thus

$$\rho(z,t) = \frac{1}{1 + N_{\text{pch}} \cdot \eta(t) \cdot \displaystyle\int_{\lambda(t)}^{z} q(z',t)\, dz'} \qquad \text{for } z > \lambda(t) \tag{19}$$

and the exit density is

$$\rho_{\text{e}}(t) = \frac{1}{1 + N_{\text{pch}} \cdot \eta(t) \cdot \displaystyle\int_{\lambda(t)}^{1} q(z',t)\, dz'} \tag{20}$$

It is useful to state what we have obtained so far. We tried to derive one (or more) equations that will allow us to compute the inlet velocity $u_{\text{i}}(t)$ as a function of time, that is what we need to solve the positions of the $N_1 + 1$ nodes $\ell_n(t)$ given by equation (14). In the previous paper, all of the equations derived in this stage of the development have explicit analytic expressions—including those that involve integrals of axial distributions—which involve the problem variables that are the elements of the state vector $\mathbf{x}$ in phase space. In the general power-profile case we are now dealing with, there appear integrals in which the axial power profile cannot be explicitly solved because $q(z,t)$ is not known beforehand. Nevertheless, as long as they are expressed in terms of elements of the state vector—such as $u_{\text{i}}(t)$, $\rho_{\text{e}}(t)$, $\eta(t)$, etc.—the instantaneous value of the integrals can be numerically assessed and then, the time-integration of the DAE system may proceed. In order to compute $u_{\text{i}}(t)$, we then payed attention to the integrated momentum equation (15), and we obtained equations (17) and (20) which allow us to evaluate the spatial acceleration and the concentrated head losses terms, at the expense of introducing a new unknown—namely $\eta(t)$. Additionally, the instantaneous mass $m(t)$ may be used to compute the Froude term. The Euler term now has the same character as the inlet enthalpy $h_{\text{i}}(t)$ and of the power profile $q(z,t)$. It is either taken as a known function of time or as dependent on other known problem variables.

The distributed head loss cannot be explicitly evaluated when the power profile depends both on time and space. Replacing equations (16) and (19), this term is equal to

$$\Lambda \int_0^1 \rho(z,t)\, u^2(z,t)\, dz = \Lambda \cdot \left[u_{\text{i}}^2(t) \cdot \lambda(t) + \int_{\lambda(t)}^{1} \frac{\left(u_{\text{i}}(t) + N_{\text{sub}} \displaystyle\int_{\lambda(t)}^{z} q(z',t)\, dz' \right)^2}{1 + N_{\text{pch}} \cdot \eta(t) \cdot \displaystyle\int_{\lambda(t)}^{z} q(z',t)\, dz'} \, dz \right]$$

which does not have an analytical expression. However, for a certain time t, if $u_{\text{i}}(t)$, $\lambda(t)$, $\eta(t)$ and $q(z,t)$ are known, the expression can be numerically integrated to evaluate the instantaneous distributed head losses in the momentum equation (15). In the same way, the integral

in the Froude term is the instantaneous mass contained in the channel, which with the current assumptions, is

$$m(t) = \int_0^1 \rho(z,t)\,dz = \lambda(t) + \int_{\lambda(t)}^1 \frac{dz}{1 + N_{\text{pch}} \cdot \eta(t) \cdot \int_{\lambda(t)}^z q(z',t)\,dz'} \tag{21}$$

Of course, if for a particular problem, the power profile $q(z,t)$ does have a certain form that makes it possible to analytically integrate these expressions—as with uniform power—then the the resulting system will be significantly easier. Nevertheless, it is still possible to solve the generic equations even with a high-level code, as illustrated below.

Let us now turn our attention to the total derivative of equation (15)

$$\frac{d}{dt}\left[\int_0^1 \rho(z,t)\cdot u(z,t)\,dz\right] = \frac{d}{dt}\left[u_{\text{i}}(t)\lambda(t) + \int_{\lambda(t)}^1 \frac{u_{\text{i}}(t) + N_{\text{sub}}\int_{\lambda(t)}^z q(z',t)\,dz'}{1 + N_{\text{pch}} \cdot \eta(t) \cdot \int_{\lambda(t)}^z q(z',t)\,dz'}\,dz\right]$$

To obtain a DAE equation of the form (12), all the equations have to be expressed either as functions of the elements of the vector state $\mathbf{x}$ or their time derivatives. So far, all the integrals that have appeared satisfy this condition. But the total derivative with time of the integral above cannot be expressed as a function of either $\mathbf{x}$ or $\dot{\mathbf{x}}$. To overcome this inconvenient, we just add a new variable $\varphi(t)$ to the phase space, defined by the cumbersome integral

$$\varphi(t) = \int_{\lambda(t)}^1 \frac{u_{\text{i}}(t) + N_{\text{sub}}\int_{\lambda(t)}^z q(z',t)\,dz'}{1 + N_{\text{pch}} \cdot \eta(t) \cdot \int_{\lambda(t)}^z q(z',t)\,dz'}\,dz \tag{22}$$

that can be computed only from the state vector components. This definition allows us to write the time derivative as neatly as

$$\frac{d}{dt}\left[\int_0^1 \rho(z,t)\cdot u(z,t)\,dz\right] = \frac{du_{\text{i}}(t)}{dt}\cdot\lambda(t) + u_{\text{i}}(t)\cdot\frac{d\lambda(t)}{dt} + \frac{d\varphi(t)}{dt}$$

at the expense of increasing the dimension of the phase space even further.

To obtain a self-contained set of variables that constitute the state vector $\mathbf{x}$ and the associated system of equations that describes its temporal evolution in phase space, we need one more independent equation because we included an arbitrary variable $\eta(t)$ to modulate the transient enthalpy profile $h(z,t)$ in equation (18).

The equations (14) for the node positions $\ell_n(t)$ were derived from the energy conservation. The big equation (15) from which the inlet velocity $u_{\text{i}}(t)$ is computed, is the momentum conservation. We are left with only the mass conservation to close our model. In effect, by integrating equation (1) along the channel

$$\int_0^1 \frac{\partial}{\partial t}\Big[\rho(z,t)\Big]\,dz + \int_0^1 \frac{\partial}{\partial t}\Big[\rho(z,t)u(z,t)\Big]\,dz = 0$$

$$\frac{dm(t)}{dt} + \rho_{\mathrm{e}}(t)\cdot u_{\mathrm{e}}(t) - u_{\mathrm{i}}(t) = 0 \tag{23}$$

we obtain another independent equation that involves $\eta(t)$—although indirectly through $m(t)$ and $\rho_{\mathrm{e}}(t)$—and gives us an autonomous system of differential-algebraic equations that models the transient behavior of a vertical boiling channel with an arbitrary transient power profile and non-steady inlet enthalpy.

Summing up, equations (14), (15), (17), (20), (21), (22) and (23) plus proper equations (either differential or algebraic) for the inlet enthalpy, the Euler number, give a system of N_1+8 differential-algebraic equations of the form (12) for the state vector

$$\mathbf{x} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_{N_1-1} \\ \lambda \\ u_{\mathrm{i}} \\ u_{\mathrm{e}} \\ \eta \\ \rho_{\mathrm{e}} \\ m \\ \varphi \\ h_{\mathrm{i}} \\ \mathrm{Eu} \end{bmatrix}$$

provided a proper definition of the power source $q(z,t)$ is given. The equations are summarized in figure 5, which constitutes our extension to the original Clausse-Lahey model.

Taking into account the results obtained in section 2.2, the steady state $\mathbf{x}^\star$ of the dynamical system is

$$\ell_n^\star \quad \text{such that} \qquad \int_0^{\ell_n^\star} q^\star(z)\,dz - \frac{N_{\mathrm{sub}}}{N_{\mathrm{pch}}}\frac{n}{N_1} = 0$$

$$\lambda^\star = \ell_{N_1}$$

$$h_{\mathrm{i}}^\star = -\frac{N_{\mathrm{sub}}}{N_{\mathrm{pch}}}$$

$$u_{\mathrm{i}}^\star = \frac{N_{\mathrm{sub}}}{N_{\mathrm{pch}}}$$

$$u_{\mathrm{e}}^\star = \frac{N_{\mathrm{sub}}}{N_{\mathrm{pch}}} + N_{\mathrm{sub}}\cdot\int_{\lambda^\star}^1 q^\star(z)\,dz$$

The extended Clausse-Lahey model

$$0 = -\frac{1}{h_{\mathrm{i}}(t)} \cdot \frac{dh_{\mathrm{i}}}{dt} \left(N_1 - n - \frac{1}{2} \right) \Big[\ell_n(t) - \ell_{n-1}(t) \Big] + \frac{1}{2} \left(\frac{d\ell_n}{dt} + \frac{d\ell_{n-1}}{dt} \right)$$
$$- u_{\mathrm{i}}(t) - \frac{N_1}{h_{\mathrm{i}}(t)} \cdot \frac{N_{\mathrm{sub}}}{N_{\mathrm{pch}}} \int_{\ell_{n-1}(t)}^{\ell n(t)} q(z,t)\, dz \qquad \text{for } n = 1, \ldots, N_1$$

$$0 = u_{\mathrm{i}}(t) - u_{\mathrm{e}}(t) + N_{\mathrm{sub}} \int_{\lambda(t)}^{1} q(z',t)\, dz'$$

$$0 = \rho_{\mathrm{e}}(t) - \frac{1}{1 + N_{\mathrm{pch}} \cdot \eta(t) \cdot \displaystyle\int_{\lambda(t)}^{1} q(z',t)\, dz'}$$

$$0 = \lambda(t) - m(t) + \int_{\lambda(t)}^{1} \frac{dz}{1 + N_{\mathrm{pch}} \cdot \eta(t) \cdot \displaystyle\int_{\lambda(t)}^{z} q(z',t)\, dz'}$$

$$0 = \varphi(t) - \int_{\lambda(t)}^{1} \frac{u_{\mathrm{i}}(t) + N_{\mathrm{sub}} \displaystyle\int_{\lambda(t)}^{z} q(z',t)\, dz'}{1 + N_{\mathrm{pch}} \cdot \eta(t) \cdot \displaystyle\int_{\lambda(t)}^{z} q(z',t)\, dz'}\, dz$$

$$0 = \frac{dm(t)}{dt} + \rho_{\mathrm{e}}(t) \cdot u_{\mathrm{e}}(t) - u_{\mathrm{i}}(t)$$

$$0 = \frac{du_{\mathrm{i}}(t)}{dt} \cdot \lambda(t) + u_{\mathrm{i}}(t) \cdot \frac{d\lambda(t)}{dt} + \frac{d\varphi(t)}{dt} + \rho_{\mathrm{e}}(t)\, u_{\mathrm{e}}^2(t) - \rho_{\mathrm{i}}(t)\, u_{\mathrm{i}}^2(t)$$
$$+ \Lambda \cdot \left[u_{\mathrm{i}}^2(t) \cdot \lambda(t) + \int_{\lambda(t)}^{1} \frac{\left(u_{\mathrm{i}}(t) + N_{\mathrm{sub}} \displaystyle\int_{\lambda(t)}^{z} q(z',t)\, dz' \right)^2}{1 + N_{\mathrm{pch}} \cdot \eta(t) \cdot \displaystyle\int_{\lambda(t)}^{z} q(z',t)\, dz'}\, dz \right]$$
$$+ k_{\mathrm{i}} \cdot \rho_{\mathrm{i}}(t)\, u_{\mathrm{i}}^2(t) + k_{\mathrm{e}} \cdot \rho_{\mathrm{e}}(t)\, u_{\mathrm{e}}^2(t) + \frac{m(t)}{\mathrm{Fr}} - \mathrm{Eu}(t)$$

$$0 = h_{\mathrm{i}}(t) + f(\mathbf{x}, \dot{\mathbf{x}}, t)$$

$$0 = \mathrm{Eu}(t) + g(\mathbf{x}, \dot{\mathbf{x}}, t)$$

Figure 5: The extended Lahey-Clausse model for arbitrary transient power sources, inlet enthalpies and pressure differences. All $q(z,t)$, $h_{\mathrm{i}}(t)$ and $\mathrm{Eu}(t)$ are assumed to be either known functions of time or to depend on state vector $\mathbf{x}$ in a way that depends on the particular problem being solved.

$$\rho_e^\star = \frac{1}{1 + N_{\text{pch}} \cdot \displaystyle\int_{\lambda^\star}^{1} q^\star(z)\, dz}$$

$$\eta^\star = 1$$

$$m^\star = \lambda^\star + \int_{\lambda^\star}^{1} \frac{1}{1 + N_{\text{pch}} \cdot \displaystyle\int_{\lambda^\star}^{1} q^\star(z')\, dz'}$$

$$\varphi^\star = \int_{\lambda^\star}^{1} \frac{\dfrac{N_{\text{sub}}}{N_{\text{pch}}} + N_{\text{sub}} \displaystyle\int_{\lambda^\star}^{z} q^\star(z')\, dz'}{1 + N_{\text{pch}} \cdot \displaystyle\int_{\lambda^\star}^{z} q^\star(z)\, dz'} dz$$

where equation (11) should also hold.

4 NUMERICAL RESULTS

As when computing the steady-state of the continuous problem, again the equations in figure 5 seem rather cumbersome. They may be solved by writing an ad-hoc low-level computer code to numerically calculate the integrals that appear in the equations and afterward solve the system of differential-algebraic equations itself. However, even though there are no explicit algebraic expressions in many of the terms, their formulation is suitable to be solved by the free code mochin (Theler, 2012a), which is essentially a high-level interface where the user enters the equations to be solved—plus the initial conditions and other additional information—into a text file, then runs the program and collects the requested output. The code has a robust algebraic parser, which uses many of the features provided by the GNU Scientific Library (Galassi et al., 2012) such as numerical integration and root-finding (amongst others) and then interfaces with the SUNDIALS non-linear DAE-solving library (Hindmarsh et al., 2005).

The main idea behind this paper is the extension of the original model in order to cope with more general conditions, and thus to apply it to cases of industrial interest rather than leaving it just as an academic curiosity. Anyhow, because of the intrinsic complexity these cases present, the development of particular models might require a dedicated paper on their own.

Just as an illustration, the following input solves a transient vertical channel with a normalized power profile

$$q(z,t) = q^\star(z) = \frac{2}{\pi} \cdot \sin\left(z \cdot \pi\right)$$

and $\text{Eu} = 10$, $N_{\text{sub}} = 6$, $\Lambda = 3$, $\text{Fr} = 1$, $k_{\text{i}} = 6$ and $k_{\text{e}} = 2$. The resulting phase-change number is computed from the implicit relationship given in equation (11), which for the current case gives $N_{\text{pch}} = 10.444$. The inlet enthalpy is considered constant and the external pressure difference is fixed at $\text{Eu} = 10$.

A mochin input that solves the extended Clausse-Lahey model starting from a slightly disturbed steady-state is the following:

```
VAR z z
```

```
CONST N1 Npch

N1 = 6
VECTOR l 6
ALIAS lambda l_6

# problem parameters
# Npch is computed below as a function of these six
Eu = 10
Nsub = 6
Lambda = 3
Fr = 1
ki = 6
ke = 2

# integration parameters
error_bound = 1e-4    # relative accepted error
dt_0 = 1e-4           # initial time step size
end_time = 60         # final integration time

# the elements of the state vector
PHASE_SPACE l ui ue rhoe eta m phi hi Eu

# the power profile(s)
#FUNCTION qstar(z) = 1
FUNCTION qstar(z) = pi/2 * sin(z*pi)
#arbitray normalized spline-interpolated power profile
# FUNCTION potencia(z) INTERPOLATION splines DATA {
# 0       0
# 0.2     2.5
# 0.5     3
# 0.6     2.5
# 0.7     1.4
# 0.85    0.3
# 1       0 }
# norm = integral(potencia(z'), z', 0, 1)
# FUNCTION qstar(z) = 1/norm * potencia(z)

FUNCTION q(z,t) = qstar(z)


# functions needed for the steady-state computation
FUNCTION lambdastar(Npch) = root(integral(qstar(z ), z , 0, z) - ↩
    Nsub/Npch, z, 0, 1)
FUNCTION q2phistar(z,Npch) = integral(qstar(z ), z , lambdastar( ↩
    Npch), z)
FUNCTION F(Npch) = {
    (Nsub/Npch + Nsub*q2phistar(1,Npch))^2/(1 + Npch * q2phistar(1, ↩
        Npch))
 - (Nsub/Npch)^2
 + Lambda*(Nsub/Npch)^2*lambdastar(Npch)
 + Lambda*integral((Nsub/Npch + Nsub*q2phistar(z,Npch))^2/(1 + ↩
     Npch*q2phistar(z,Npch)), z, lambdastar(Npch), 1)
 + ki*(Nsub/Npch)^2
 + ke*(Nsub/Npch + Nsub*q2phistar(1,Npch))^2 / (1 + Npch*q2phistar ↩
     (1,Npch))
```

```
 + 1/Fr * (lambdastar(Npch) + integral(1/(1 + Npch*q2phistar(z, ↩
     Npch)), z, lambdastar(Npch), 1))
 - Eu
}
Npch = root(F(Npch), Npch, Nsub+1e-3, 50)

# initial conditions as provided here
INITIAL_CONDITIONS FROM_VARIABLES

ui_0 = 0.9*Nsub/Npch  # disturbance

hi_0 = -Nsub/Npch
ue_0 = Nsub/Npch + Nsub * integral(qstar(z ), z , lambdastar(Npch) ↩
    , 1)
rhoe_0 = 1/(1 + Npch * integral(qstar(z ), z , lambdastar(Npch), ↩
    1))
eta_0 = 1
m_0 = lambdastar(Npch) + integral(1/(1 + Npch * eta_0 * integral( ↩
    qstar(z ),z ,lambdastar(Npch),z)), z, lambdastar(Npch), 1)
l_i_0 = root(hi_0 * i/N1 + integral(qstar(z ), z , 0, z), z, 0, 1)
phi_0 = integral((Nsub/Npch + Nsub*integral(qstar(z ), z , ↩
    lambdastar(Npch), z))/(1 + Npch*eta*integral(qstar(z ), z , ↩
    lambdastar(Npch), z)), z, lambdastar(Npch), 1)

# the equations
0 = -1/hi*hi_dot * (N1 - 1-0.5)*(l_1 - 0)    + 0.5*(l_1_dot + 0) ↩
        - ui - Nsub/Npch * N1/hi * integral(q(z,t), z, 0,   l_1)
0 = -1/hi*hi_dot * (N1 - 2-0.5)*(l_2 - l_1) + 0.5*(l_2_dot + ↩
    l_1_dot) - ui - Nsub/Npch * N1/hi * integral(q(z,t), z, l_1, ↩
    l_2)
0 = -1/hi*hi_dot * (N1 - 3-0.5)*(l_3 - l_2) + 0.5*(l_3_dot + ↩
    l_2_dot) - ui - Nsub/Npch * N1/hi * integral(q(z,t), z, l_2, ↩
    l_3)
0 = -1/hi*hi_dot * (N1 - 4-0.5)*(l_4 - l_3) + 0.5*(l_4_dot + ↩
    l_3_dot) - ui - Nsub/Npch * N1/hi * integral(q(z,t), z, l_3, ↩
    l_4)
0 = -1/hi*hi_dot * (N1 - 5-0.5)*(l_5 - l_4) + 0.5*(l_5_dot + ↩
    l_4_dot) - ui - Nsub/Npch * N1/hi * integral(q(z,t), z, l_4, ↩
    l_5)
0 = -1/hi*hi_dot * (N1 - 6-0.5)*(l_6 - l_5) + 0.5*(l_6_dot + ↩
    l_5_dot) - ui - Nsub/Npch * N1/hi * integral(q(z,t), z, l_5, ↩
    l_6)

0 = ui - ue + Nsub*integral(q(z,t), z, lambda, 1)
0 = rhoe - 1/(1 + Npch * eta * (1 - lambda))
0 = lambda - m + integral(1/(1 + Npch*eta*integral(q(z ,t),z , ↩
    lambda,z)), z, lambda, 1)
0 = m_dot + rhoe*ue - ui
0 = phi - integral((ui + Nsub*integral(q(z ,t), z , lambda, z))/(1 ↩
    + Npch*eta*integral(q(z ,t), z , lambda, z)), z, lambda, 1)
0 ={  ui_dot*lambda + ui*lambda_dot + phi_dot + rhoe*ue^2 - ui^2
  + Lambda*(ui^2*lambda +
      integral((ui + Nsub  *  integral(q(z ,t), z , lambda, z))^2/
               ( 1 + Npch*eta*integral(q(z ,t), z , lambda, z)),
                z, lambda, 1) )
  + ki*ui^2 + ke*rhoe*ue^2 + 1/Fr*m - Eu
```

```
}

# constant inlet enthalpy and pressure drop
0 = hi_dot
0 = Eu_dot


# write information (commented out) in the ouput header
PRINT HEADER TEXT "\# vertical boiling channel"
PRINT HEADER TEXT "\# N_pch = "  Npch
PRINT HEADER TEXT "\# N_sub = "  Nsub
PRINT HEADER TEXT "\# Fr = "     Fr
PRINT HEADER TEXT "\# Lambda = " Lambda
PRINT HEADER TEXT "\# k_i = "    ki
PRINT HEADER TEXT "\# k_e = "    ke
PRINT HEADER TEXT "\# Eu = "     Eu

# print lambda and ui vs time
PRINT t lambda ui
```

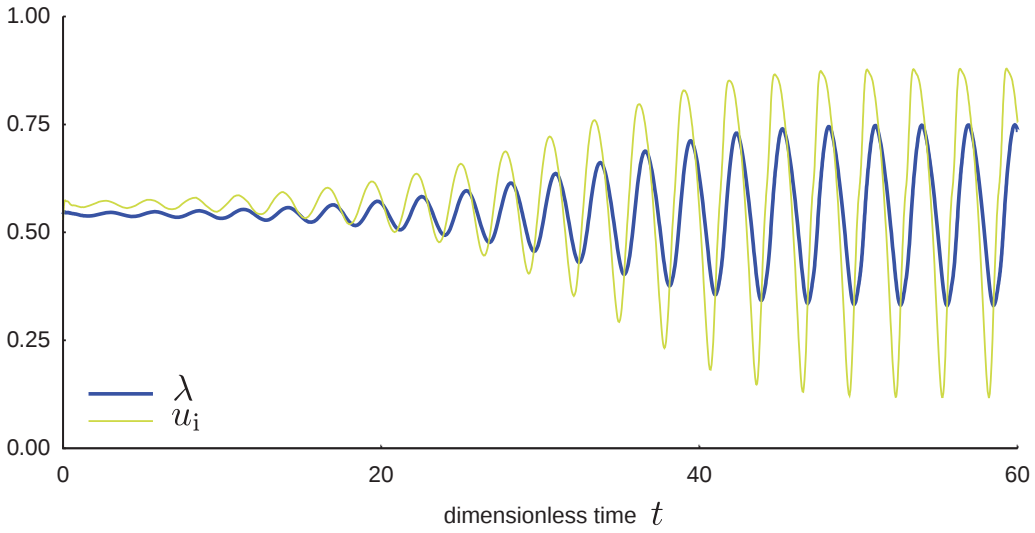


Figure 6: The position of the boiling boundary $\lambda(t)$ and the inlet velocity u_i versus time for a sine-shaped power profile and $N_\mathrm{sub} = 6$, $\mathrm{Fr} = 1$, $\Lambda = 3$, $k_i = 6$, $k_e = 2$, $\mathrm{Eu}(t) = 10$ (constant) and $h_\mathrm{i} = h^\star$ (constant). The resulting phase-change number computed with equation (11) is $N_\mathrm{pch} = 10.444$. At $t = 0$ the steady-state is perturbed with a small disturbance on the inlet velocity and the channel evolves into a stable limit cycle.

Figure 6 shows the output of the above input. For the chosen parameters, a small disturbance of the steady state develops a periodic limit cycle. Depending on the combination of parameters, the system may present either stable, periodic or unstable behaviors, as shown in figures 7 and 8, which show a projection of the phase-space into the λ-u_i plane, for both the uniform power (figure 7) and the sine-shaped power (figure 8) cases. A wide variety of other non-linear effects, including chaotic attractors, when considering also particular dynamics for the power—such as in a nuclear reactor (Chang and Lahey, 1997; Theler, 2008)—and for the rest of non-heated

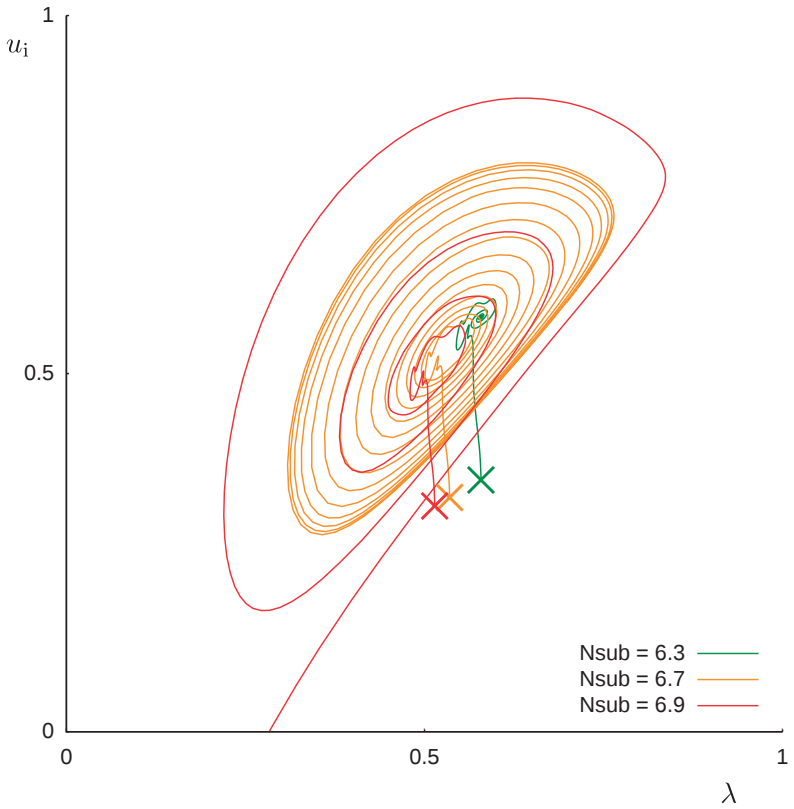


Figure 7: Projection of the phase-space into the λ-u_i plane for the uniform-power case. Stable, periodic and an unstable trajectories are shown. The initial conditions are marked with crosses.

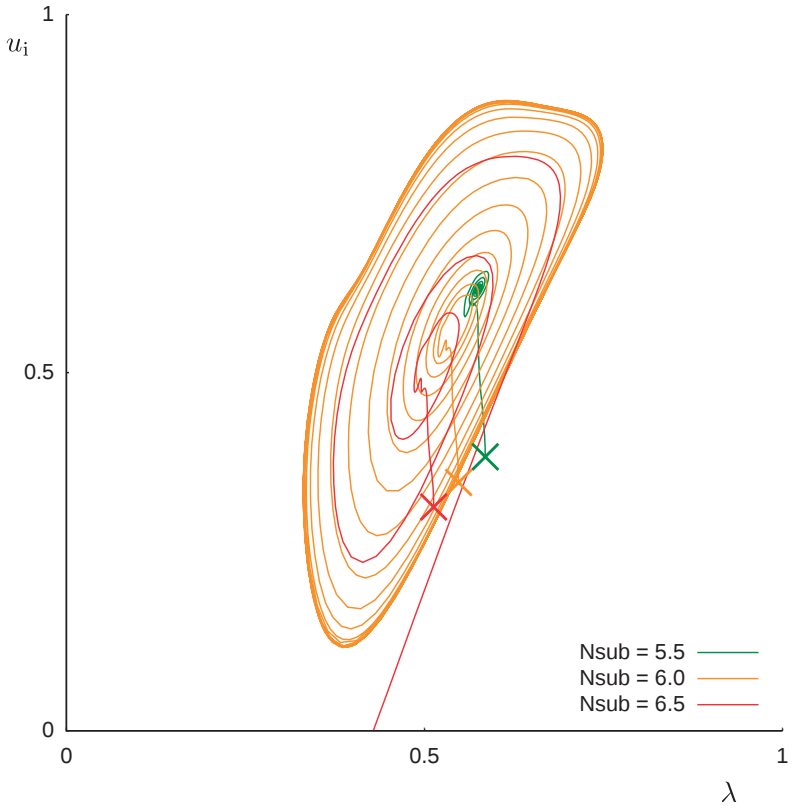


Figure 8: Idem as figure 7 for the sine-shaped power profile.

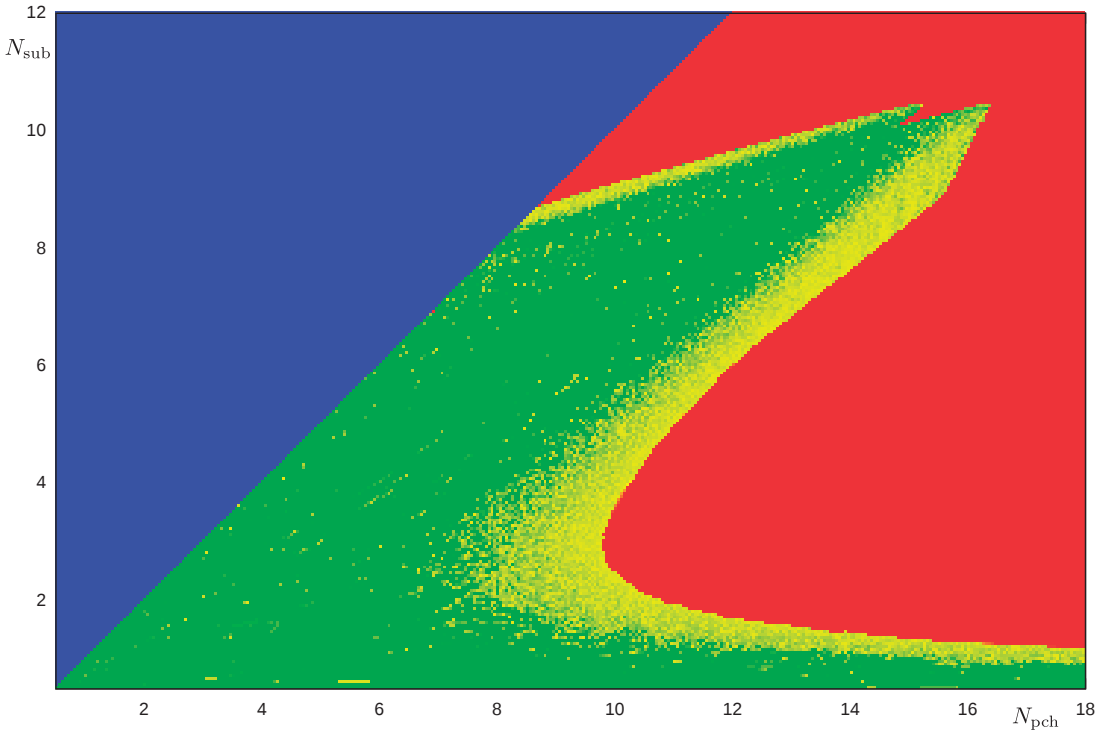


Figure 9: Simplified stability map in the N_{sub}-N_{pch} parameter space for the uniform-power case. The blue zone implies $N_{\text{sub}} > N_{\text{pch}}$ and no two-phase flow is involved. The green zone is the stability region, the red zone defines the unstable subspace, and yellow points represent the zone where periodic behavior may be found.

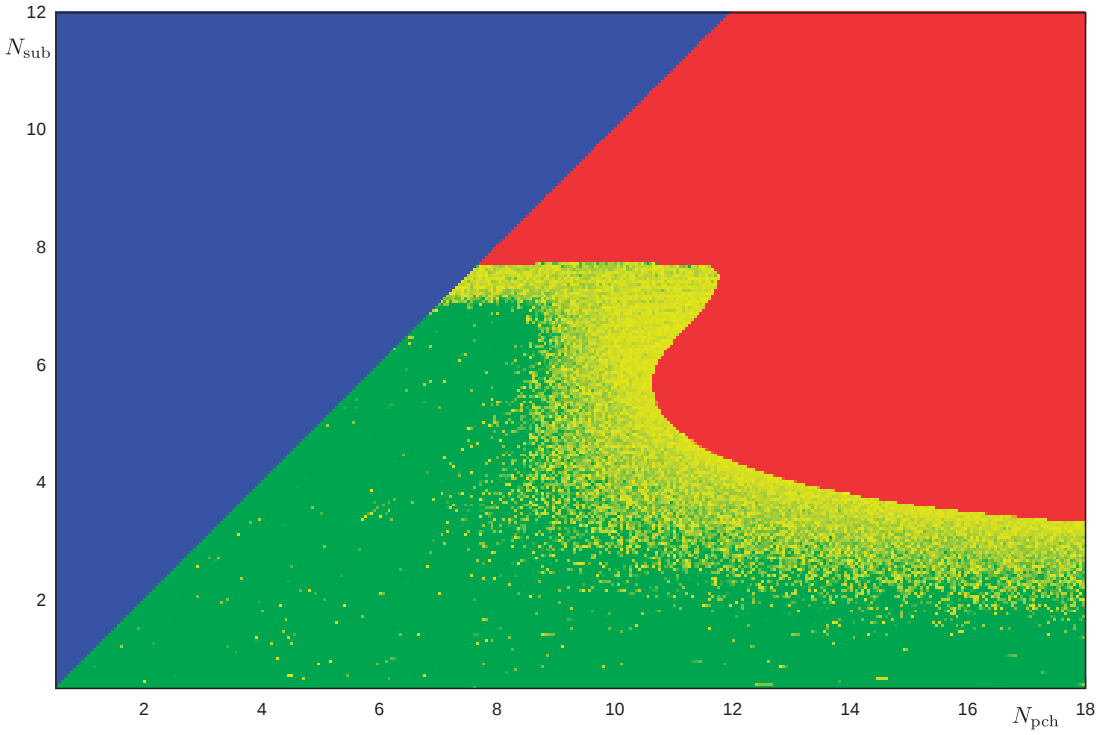


Figure 10: Idem as figure 9 for the sine-shaped power profile.

sections—such as adiabatic risers (Lahey, 1992) or steam generators (Theler, 2008).

The constant-enthalpy, constant Euler and uniform power case was already shown in the previous paper. Indeed, stability maps in the N_{sub}-N_{pch} parameter space were constructed by parametrically solving the Clausse-Lahey model in the previous paper. Figure 9 shows a basic stability map for the uniform case, and how it compares to the sine-shaped power case—shown in figure 10—which could not be computed with the original model.

In order to compute these rudimentary stability maps, the phase-change number is taken as an input parameter and the needed Euler number is taken as the dependent parameter by solving equation (11). The code mochin was run parametrically[3] over the range $[0.5, 12] \times [0.5 + \epsilon, 18]$ in the $N_{\text{sub}} \times N_{\text{pch}}$ space. Each pixel color represents the actual final integration time, which for stable cases is usually larger than the stipulated end time because once the steady-state solution attracts the orbits, the time step is automatically increased by the solver. Unstable cases forces either $\lambda(t)$ or $u_{\text{i}}(t)$ to leave the allowed $[0, 1]$ range for times less than the desired end time, and thus are marked as red. Actual final times that are close to the desired one probably correspond to periodic solutions, because the time step is kept more or less constant throughout the integration. Of course, in order to thoroughly analyze the different behaviors and compute the stability of a certain problem some kind of further refinement of the output data ought to be performed. The maps shown in figures 9 and 10 are just quick results that illustrate the applicability of the extended model, whose derivation was the core of this work.

5 CONCLUSIONS

An extension of the Clausse-Lahey model for boiling channel dynamics was presented to deal with more realistic situations, like non-uniform transient power distributions and variable inlet conditions. A rigorous mathematical derivation was followed reaching at a closed set of integro-differential equations, which were solved using available numerical tools. A case of a sine power profile was solved showing that in effect the influence of this departure from the uniform power-profile case is not negligible. The proposed extension allows to model general boiling channels and is particularly suited for analyzing nuclear reactors, either for full power or start-up conditions, both in normal operation or in accident plant conditions. The inclusion of closed-loop conditions, heat sinks and fission power feedback equations is expected to increase the non-linearity of the model and to give rise to a new variety of dynamical behaviors.

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[3] Actually one of the features of the codes in the wasora suite is that they can be run in parametric mode spanning a certain subset of the parameter space as easily as inserting one extra keyword in the input. See wasora s documentation in Theler (2012b) for further information.

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