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THE FORCED INVISCID BURGERS EQUATION

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Abstract. *The response to resonant and near-resonant forcing is studied in a simple model using the forced inviscid Burgers equation* $u_t + \left(\frac{u^2}{2}\right)_x = f$. *The nonlinear term represents simultaneously energy transfer and dissipation mechanisms in more complex systems.*

In this work we show the energetic interplay between the dissipation done by the solution of the forced Burgers equation and the work done by the forcing function.

1 INTRODUCTION

The forced inviscid Burgers equation is studied as a model for the non-linear interaction of dispersive waves:

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = f(x,\,t) \quad (1)$$

where the dependent variable $u(x,t)$ represents a mode with linear frecuency $\omega = 0$, and u 2π is periodic in space with vanishing mean $\int_0^{2\pi} u \quad dx = 0$ and the force $f(x,t)$ is tuned to mimic the effects of the other modes, near or far from resonance with u. Moreover f smooth 2π periodic in space with vanishing mean $\int_0^{2\pi} f dx = 0$.

A resonant force f is one that does not depend on time, and a near resonant force is one that evolves slowly on time.

The model equation (1) above is a simplified version of the equations describing the interaction among resonat triads involving a non-dispersive wave[1] .

The simplification consists in freezing the two dispersive members of the triad, thus making them act as a prescribed force on $u(x,t)$.

The tone of this paper is mostly descriptive, with the emphasis placed on the relevance of the striking behavior of our simple model to more general systems. For the proofs of many of its results, as well as for a more detailed account of the numerics, we refer the reader to[2] .

It is well known that the inviscid Burgers equation develops shocks. The equation has an energy:

$$E(t) = \int_0^{2\pi} \frac{1}{2}\,u^2(x,\,t)dx$$

which satisfies the equation:

$$\frac{dE}{dt} + Ed = Wf$$

$$\frac{dE}{dt} - \sum\left(\frac{1}{12}[u]^3\right) = \int_0^{2\pi} u(x,t)\,f(x,t)\,dx.$$

Brackets stand for jump across of the enclosed expression $[u] = u^+ - u^-$. $Ed > 0$ accounts for the dissipation of energy at the shocks. Wf represents the work done by the forcing term f.

For the case of a single forcing mode the equation (1) has the form:

$$u_t + \left(\frac{1}{2}u^2\right)_x = f(x - \omega t) \quad (2)$$

where f is 'nice' and that the initial conditions are such that the solution is at all times piecewise smooth with a finite number of shocks. The equation (2) admits exact travelling waves solutions of the form:

$u(x,t) = G(x - \omega\, t) \quad z = x - \omega t$

Then the equation becomes the ode

$$\frac{d}{dz}\left(\frac{1}{2}(G-\omega)^2\right) = f(z).$$

This has solution

$$G(z) = \omega \pm \sqrt{2F(z)} \quad (3)$$

where $F = \int f(s)ds$, with the integration constant selected so that $F(z) \geq 0$. We define F_{cr} such that $min_{z\in[0,2\pi)}F_{cr} = 0$.

These solutions not only can be written exactly in closed form but they describe the long behavior for the solution of (2).

Generically, three distinct cases can arise with the solution determined uniquely by ω if F_{cr} has a single minimum per period[2] .

2 TWO FORCING MODES

Here we study the effects on the solutions of (1) of a forcing term consisting of the sum of two travelling waves of different speeds.

Our interest in this situation arises from the general question of the effects of the superposition of many near resonats interactions in general systems.

$$u_t + \left(\frac{1}{2}u^2\right)_x = g_1(x) + g_2(x - \Omega\, t). \quad (4)$$

We distinguish two distinct extreme regimes.

1) $\Omega >> 1$, the leading order effects of g_2 on u cancel due to averaging, see[2] .

2) Quasi-steady forcing:

$\Omega \ll 1$, g_1 and g_2 combine into a single quasisteady force yielding a quasi- steady solution $u(x, \Omega t)$ punctuated by intermitent events.

We will only clarify the second case.

When $0 < \Omega << 1$ we can think of the solution of (4) as frozen in time near each $t = t_0$.

This yields a quasi-steady leading order solution $u = u(x, \Omega t)$ where $u(x, \Omega t_0)$ is given by the steady state solution (case $\Omega = 0$) to the case with a single forcing mode with $f = g_1(x) + g_2(x - \Omega t_0)$.

We consider $\tau = \Omega\, t$ slow time variable and the following asymptotic expansion:

$$u(x,\, t) = u_0(x,\, \tau) + \Omega\, u_1(x,\, \tau) + O(\Omega^2), \quad (5)$$

then at the leading order we obtain:

$u_0(x,\, \tau) = \pm\sqrt{2\, G(x,\, \tau)} \quad G = \int g_1(x) + g_2(x - \tau)dx,\, minG = 0.$

These solution works as long G has a single minimum per period in which the position of the shock $s(\tau)$ and the position for the smootly crossing $x_m(\tau)$ (where $minG = 0$), are well defined and depend smoothly on τ.

However there will generally be some special times $\tau = \tau_{cr}$ at which fails, generically G will have several local minimus evolving in time with one of then smaller than the other. The generic special times occur when two local minimums become the global minimum, see figure 1.

At this times s and x_m cease to be smooth, jump discontinuily from one position to the other.

There is a set of discrete times when the solution u needs to adjust rapidly from one quasi-steady to another. The existence of these adjustment process which we call ”storms” raises some questions:

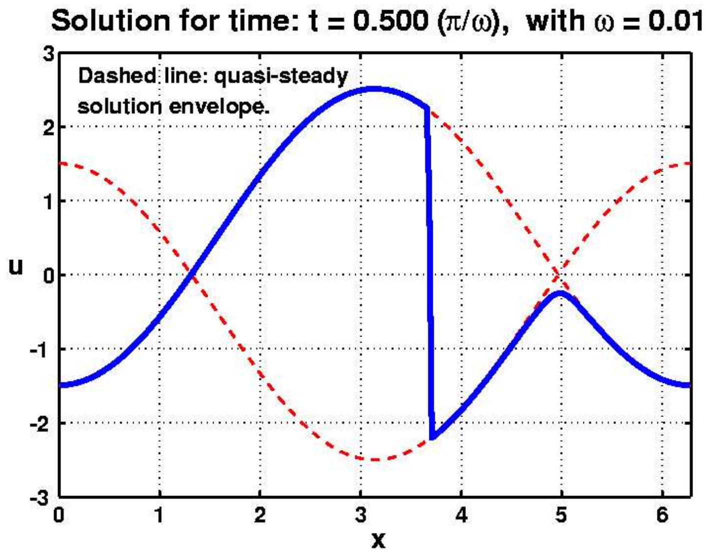

Figure 1: Asymptotic $t \to \infty$ solution to the equation

What is the time scale (ie. duration) of a storm?

During a storm are there significant effects in the energy exchange between the forcing f and the solution u?

Away from the storm there is leading order balance between W_f and E_d, $E_d - W_f = O(\Omega)$.

How is the balance affected by the storms?

The energy $E \sim \int_0^{2\pi} \frac{1}{2} u_0^2 dx = \int_0^{2\pi} G dx$, is continuous in τ even though u_0 itself is not. The any extra energy exchange between u and f during the storm will need to be matched by extra dissipation.

By using an asymptotic expansion we can compute that the scale of the time during the storm is: $T = \sqrt{\Omega}\, t\,.$

From figure 2 we can observe that the energy dissipation rate E_d (the same is valid for the work done by the forcing W_f) shows a marked spike during the storm, these doubling of the energy dissipation rate is easily explained as arising from the apparearance of an extra shock during the storm of a size comparable to the regular one.

The close agreement between the energy dissipated and the work performed by the forcing can be explained by the slow evolution of the storms faster than the regular $O(\Omega t)$ but clearly slower than $O(t)$ rate. Hence at any particular time the energy input and output need to be balance to leading order, then even during the storm the solution is quasi-steady, $E_d - W_f = 0(\sqrt{(}\Omega))$.

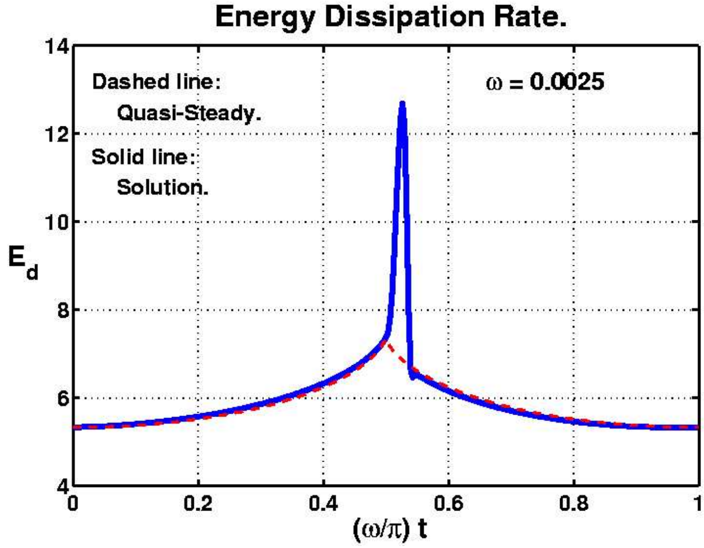

Figure 2: Energy dissipation rate $E_d = E_d(t)$

3 CONCLUSIONS

Non linear dissipation by breaking waves introduces a rich scenario. Moreover we obseve a sharp transition between near-resonant and far from resonant behavior and an intermitent behavior with multiple near resonances.

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